The Isocohomological Property

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Group Cohomology

Suppose $G$ is a finitely generated discrete group.

- $C^n(G) = \{ \phi : G^n \to \mathbb{C} \}$
- $d : C^n(G) \to C^{n+1}(G)$

\[
(d\phi)(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \phi(g_0, g_1, \ldots, \widehat{g_i}, \ldots, g_n)
\]

$0 \to C^0(G) \to C^1(G) \to \ldots$

A usual cochain complex for calculating group cohomology, $H^*(G)$. 
Accounting for Growth

Endow $G$ with a word-length function $\ell_G$.

$\phi \in PC^n(G) \subset C^n(G)$ if there is a polynomial $P$ such that

$$|\phi(g_1, \ldots, g_n)| \leq P(\ell_G(g_1) + \ell_G(g_2) + \ldots + \ell_G(g_n))$$

$PC^*(G)$ forms a subcomplex of $C^*(G)$.

$HP^n(G)$, the polynomial cohomology of $G$.

$PC^*(G) \to C^*(G)$ induces a comparison map $HP^*(G) \to H^*(G)$.

For many groups this map is an isomorphism.
With Coefficients

For a $\mathbb{C}G$-module $V$:

- $H^*(G; V) = \text{Ext}^*_G(\mathbb{C}, V)$.
- $0 \leftarrow \mathbb{C} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots$
- $0 \rightarrow \text{Hom}_{\mathbb{C}G}(P_0, V) \rightarrow \text{Hom}_{\mathbb{C}G}(P_1, V) \rightarrow \text{Hom}_{\mathbb{C}G}(P_2, V) \rightarrow \ldots$
- $\text{Ext}^*_G(\mathbb{C}, V)$ is cohomology of this complex.
With Coefficients

\[ SG = \left\{ \phi : G \to \mathbb{C} \mid \forall_k \sum_{g \in G} |\phi(g)| (1 + \ell_G(g))^k < \infty \right\} \]

Suppose \( V \) is a bornological \( SG \)-module.

- \( HP^* (G; V) = b\text{Ext}^*_S(G, V) \).

- \[
0 \leftarrow \mathbb{C} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots
\]

- \[
0 \to b\text{Hom}_S(P_0, V) \to b\text{Hom}_S(P_1, V) \to b\text{Hom}_S(P_2, V) \to \ldots
\]

- \( \mathbb{C}G \hookrightarrow SG \) induces \( HP^* (G; V) \to H^* (G; V) \) for all bornological \( SG \)-modules \( V \).
The Isocohomological Property

Definition

$G$ has the (strong) isocohomological property if for all bornological $SG$-modules $V$, the comparison map $HP^*(G; V) \rightarrow H^*(G; V)$ is an isomorphism. $G$ is isocohomological for a particular $SG$-module $V$ if the particular comparison map is an isomorphism.

- Nilpotent groups (Ron Ji, Ralf Meyer)
- Combable groups (Crichton Ogle, Ralf Meyer)
Other Bounding Classes

\[ \mathcal{B} \subset \{ \phi : [0, \infty) \rightarrow (0, \infty) | \phi \text{ is nondecreasing} \} \]

- \( 1 \in \mathcal{B} \).
- If \( \phi \) and \( \phi' \in \mathcal{B} \), there is \( \varphi \in \mathcal{B} \) such that \( \lambda \phi + \mu \phi' \leq \varphi \), for nonnegative real \( \lambda, \mu \).
- If \( \phi \in \mathcal{B} \) and \( g \) is a linear function, there is \( \psi \in \mathcal{B} \) such that \( \phi \circ g \leq \psi \).

Examples: \( \mathbb{R}^+ \), \( \{ e^f \mid f \text{ is linear} \} \).
Theorem (Connes-Moscovici, 90)

Suppose $G$ is a finitely generated discrete group endowed with word-length function $\ell_G$. If $G$ has the Rapid Decay property, and has cohomology of polynomial growth, then $G$ satisfies the Strong Novikov Conjecture.

\[ \sum_{g \in G} |f(g)|^2 (1 + \ell_G(g))^{2k} \]

- $HP^*(G) \to H^*(G)$ surjective.

- Chatterji-Ruane: Groups hyperbolic relative to polynomial growth subgroups are RD.

- Those groups are also (strongly) isocohomological.
Bass Conjecture

Due to Burghelea, \( HC_*(\mathbb{C}G) = \bigoplus_{x \in <G>} HC_*(\mathbb{C}G)_x \).

**Conjecture**

**Strong Bass Conjecture** *For each non-elliptic class \( x \), the image of the composition \( \pi_x \circ ch_* : K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G)_x \) is zero.*

\( x \in <G> \) satisfies ‘nilpotency condition’ if

\( S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x \) is nilpotent.

**Observation (Eckmann, Ji)**

Let \( x \) be a non-elliptic conjugacy class satisfying the nilpotency condition. The composition \( K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G) \xrightarrow{\pi_x} HC_*(\mathbb{C}G)_x \) is zero. In particular the Strong Bass Conjecture holds for \( G \) if each non-elliptic conjugacy class satisfies the nilpotency condition.
Bass Conjecture

- For a non-elliptic \( x \in \langle G \rangle \), take \( h \in x \).
- \( G_h \) centralizer with \( N_h = G_h/(h) \).
- Burghelea: \( HC_\ast(\mathbb{C}G)_x \cong H_\ast(N_h) \).
- \( S_x : HC_\ast(\mathbb{C}G)_x \rightarrow HC_{\ast-2}(\mathbb{C}G)_x \) acts as
  
  \[
  0 \rightarrow (h) \rightarrow G_h \rightarrow N_h \rightarrow 0
  \]
- If \( N_h \) has finite virtual cohomological dimension, \( G \) satisfies nilpotency condition.
\(\ell^1\) Bass Conjecture

- \(K_*((\ell^1 G)) \cong K_*(SG)\).
- \(ch_* : K_*(SG) \to HC_*(\ell^1 G)\) factors through \(HC_*(SG)\).

**Conjecture**

**Strong \(\ell^1\) - Bass Conjecture** *For each non-elliptic conjugacy class, the image of the composition \(\pi_x \circ ch_* : K_*(SG) \to HC_*(SG)_x\) is zero.*

Is true whenever, for each non-elliptic conjugacy class \(x\), \(S^t_x : HC_*(SG)_x \to HC_{*-2}(SG)_x\) is nilpotent.

**Question**

“If \(S_x\) is nilpotent, when is \(S^t_x\) nilpotent?”
Bass Conjecture

Definition

$G$ satisfies a polynomial conjugacy problem if for each non-elliptic $x \in < G >$ there is $P_x$ such that: $u, v \in x$ then there is $g \in G$ with $g^{-1}ug = v$ such that $\ell_G(g) \leq P_x(\ell_G(u) + \ell_G(v))$.

- Hyperbolic groups
- Pseudo-Anosov classes in Mapping class groups
- Mapping class groups

If $G$ satisfies a polynomial conjugacy bound for a non-elliptic class $x$, $HC_*(SG)_x \cong HP^\ell_G(N_h)$.

If in addition $N_h$ isocohomological $S^{t}_{x} : HC_*(SG)_x \rightarrow HC_{*-2}(SG)_x$ is nilpotent, too.
**HF$^\infty$ Groups**

**Definition**

A group is of type $HF^\infty$ if it has a classifying space the type of a “simplicial complex” with finitely many cells in each dimension.
Dehn Functions
Weighted Dehn Functions

Suppose that $X$ is a weakly contractible complex with fixed basepoint $x_0$.

- Define the weight of a vertex $v$ to be $\ell_X(v) = d_X^{(1)}(v, x_0)$.
- Define the weight of a higher dimensional simplex to be the sum of the weights of its vertices.
- The weighted volume of an $n$-dimensional subcomplex is the sum of the weights of its $n$-dimensional cells.
- Get ‘Weighted Dehn Functions’ rather than just ‘Dehn Functions’.
Theorem (Ji-R, 2009)

For an HF\( \infty \) group G, the following are equivalent.

1. All higher Dehn functions of G are polynomially bounded.
2. \( HP^\ast (G; V) \to H^\ast (G; V) \) is an isomorphism for all coefficients V. (i.e. G is strongly isocohomological)
3. \( HP^\ast (G; V) \to H^\ast (G; V) \) is surjective for all coefficients V.
(1) implies (2)

- Denote by $X$ is the universal cover of the $HF^\infty$ classifying space.
- $C_\ast(X)$ is a projective resolution of $\mathbb{C}$ over $\mathbb{C}G$.
- Length function on the vertices of $X$: $\ell_X(x) = d_X(x, \ast)$.
- Length function on $X^{(n)}$: $\ell_X(\sigma) = \sum_{v \in \sigma} \ell_X(v)$.
- $S_n(X)$ the completion of $C_n(X)$ under the family of norms given by
  \[ \| \phi \|_k = \sum_{\sigma \in X^{(n)}} |\phi(\sigma)| (1 + \ell_X(\sigma))^k \]
- $S_\ast(X)$ a projective resolution of $\mathbb{C}$ over $SG$. (The Dehn function bounds ensure a bounded contracting homotopy of $S_\ast(X)$.)
(1) implies (2)

For each $n$ there is finite dimensional $W_n$ with

$S_n(X) \implies S G \hat{\otimes} W_n$
$C_n(X) \implies C G \otimes W_n$

$b\text{Hom}_{SG}(S_n(X), V) \implies b\text{Hom}_{SG}(SG \hat{\otimes} W_n, V)$
$\implies \text{Hom}(W_n, V)$
$\implies \text{Hom}_{CG}(CG \otimes W_n, V)$
$\implies \text{Hom}_{CG}(C_n(X), V)$

After applying $b\text{Hom}_{SG}(\cdot, V)$ to $S_*(X)$ and $\text{Hom}_{CG}(\cdot, V)$ to $C_*(X)$ we obtain isomorphic cochain complexes.
the rest

- (2) implies (3) is obvious.
- (3) implies (1): This implication is similar to Mineyev’s corresponding result on hyperbolic group and bounded cohomology.
- $S$ free abelian group with free generating set $\{s_1, s_2\}$.
- $A$ free abelian group with free generating set $\{a_1, a_2, a_3\}$.
- $\beta : S \rightarrow SL(3, \mathbb{Z})$ an injection such that $\beta(s_i)$ is semi-simple with real spectrum.
- $P = A \rtimes_{\beta} S$.
- For all $a \in A$, $\ell_P(a) \leq C \log(1 + \ell_A(a)) + \epsilon$
- A solvable group with quadratic first Dehn function.
- Higher Dehn functions?
Hochschild-Serre Spectral Sequence

1. $0 \to H \to G \to Q \to 0$
2. Let $H$ be isocohomological for $\mathbb{C}$ with respect to the restricted length from $G$.
3. Equip $Q$ with the quotient length.

**Theorem (Ogle, R)**

*There is a spectral sequence with $E_2^{p,q} \cong HP^p(Q; HP^q(H))$ which converges to $HP^*(G)$.*
Hochschild-Serre Spectral Sequence

- \[ C^{p,q} = \text{bHom}_{SQ}(SQ^\otimes p+1, \text{bHom}_{SH}(SG^\otimes q+1, \mathbb{C})) \]
- Rowwise filtration collapses to \( HP^*(G) \).
- To identify \( E_2 \) term, need the isocohomological property of \( H \) and the ‘bounded mapping theorem’ of Hogbe-Nlend.
Comparing this Spectral Sequence with the usual Hochschild-Serre Spectral Sequence we get the following.

**Corollary**

*If Q is isocohomological for the twisted coefficients $\text{HP}^*(H)$, in the quotient length, then G is isocohomological for $\mathbb{C}$.*
Polynomial extensions

Definition
An extension $0 \to H \to G \to Q \to 0$ is a polynomial extension if there is a cross section yielding a cocycle of polynomial growth and inducing a polynomial action of $Q$ on $H$.

These extensions were first studied by Noskov in relation to the RD property.

Theorem (Noskov, 92)

Let $G$ be a polynomial extension of the finitely generated group $Q$ by the finitely generated group $H$. If $H$ and $Q$ have the Rapid Decay property, so does $G$. 
Lemma (Ji-Ogle-R)

The comparison map \( \Phi : HP^3(P) \to H^3(P) \) is not surjective.

Use the commutative diagram below and the fact that the map \( HP^3(A) \to H^3(A) \) is zero.

\[
\begin{array}{c}
HP^3(P) \xrightarrow{\Phi} HP^3(A) \\
\downarrow \quad \downarrow \\
H^3(P) \xrightarrow{} H^3(A)
\end{array}
\]

Corollary

The second Dehn function \( d_P^2 \) of \( P \) satisfies \( e^n \leq d_P^2(n) \leq e^{n^2} \)
Bass-Serre Theory

- $G$ acts cocompactly and without inversion on a tree $T$.
- $V$, $E$ representatives of orbits of vertices and edges under $G$.
- For $v \in V$, $G_v$ the stabilizer of that vertex. $G_e$ similarly.

Theorem (Serre, 77)

*For each $G$-module $M$, there is a long-exact sequence*

$$
\ldots \to H^i(G; M) \to \prod_{v \in V} H^i(G_v; M) \to \prod_{e \in V} H^i(G_e; M) \to H^{i+1}(G; M) \to \ldots
$$
Bass-Serre Theory

- Equip $G_v$, $G_e$ with restricted length, $\ell_G$.

**Lemma (R)**

*For each bornological $SG$-module $M$, there is a long exact sequence*

$$
\ldots \to HP^i(G; M) \to \prod_{v \in V} HP^i(G_v; M) \to \prod_{e \in V} HP^i(G_e; M) \to HP^{i+1}(G; M) \to \ldots
$$

**Corollary**

*Let $G$, $G_v$, and $G_e$ be as above. If each $G_v$ and $G_e$ are isocohomological in $\ell_G$, then $G$ is isocohomological.*
Complexes of Groups

- Group $G$ acting cocompactly on contractible simplicial complex $X$ without inversion.
- ‘Complexes of Groups’ instead of ‘Graphs of Groups’
- $\Sigma$ a set of representatives of the orbits of simplices of $X$, under $G$.
- For $\sigma \in \Sigma$, $G_\sigma$ the stabilizer of $\sigma$.

**Theorem (Serre, 71)**

*For each $G$-module $M$ there is a spectral sequence with $E_1$ term the product*

$$E_1^{p,q} \cong \prod_{\sigma \in \Sigma_p} H^q(G_\sigma; M)$$

*and which converges to $H^*(G; M)$.*
Complexes of Groups

- Dehn functions of $X$ polynomially bounded.
- Equip each $G_\sigma$ with $\ell_G$.

**Theorem (Ji-Ogle-R)**

For each bornological $SG$-module $M$ there is a spectral sequence with

$$E_1^{p,q} \cong \prod_{\sigma \in \Sigma_p} HP^q(G_\sigma; M)$$

and which converges to $HP^\ast(G; M)$. 
Complexes of Groups

- Finite edge stabilizers ensure $G$ finitely relatively presented.
- Polynomial Dehn function of $X$ gives polynomially bounded relative Dehn function of $G$.
- These ensure that the $G_\sigma$ are only polynomially distorted in $G$.

**Corollary**

*If each $G_\sigma$ is isocohomological, so is $G$.*

This generalizes our earlier result.

**Theorem (Ji-R, 2009)**

*Suppose that the group $G$ is relatively hyperbolic with respect to the $HF^\infty$ subgroups $H_1, \ldots, H_n$. If each $H_i$ is isocohomological, so is $G$.***