The Isocohomological Property

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Group Cohomology

Suppose G is a finitely generated discrete group.

- $C^n(G) = \{\phi : G^n \to \mathbb{C}\}$
- $d : C^n(G) \to C^{n+1}(G)$

$$(d\phi)(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \phi(g_0, g_1, \ldots, \widehat{g_i}, \ldots, g_n)$$

$$0 \to C^0(G) \to C^1(G) \to \ldots$$

A usual cochain complex for calculating group cohomology, $H^*(G)$.

Accounting for Growth

Endow G with a word-length function ℓ_G.

- $\phi \in PC^n(G) \subset C^n(G)$ if there is a polynomial P such that

$$|\phi(g_1, \ldots, g_n)| \leq P(\ell_G(g_1) + \ell_G(g_2) + \ldots + \ell_G(g_n))$$

- $PC^*(G)$ forms a subcomplex of $C^*(G)$.
- $HP^n(G)$, the polynomial cohomology of G.
- $PC^*(G) \to C^*(G)$ induces a comparison map $HP^*(G) \to H^*(G)$.
- For many groups this map is an isomorphism.

With Coefficients

For a $\mathbb{C}G$-module V:

- $H^*(G; V) = \mathrm{Ext}^*_{\mathbb{C}G}(\mathbb{C}, V)$.
- $$0 \leftarrow \mathbb{C} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$
- $$0 \to \mathrm{Hom}_{\mathbb{C}G}(P_0, V) \to \mathrm{Hom}_{\mathbb{C}G}(P_1, V) \to \mathrm{Hom}_{\mathbb{C}G}(P_2, V) \to \dots$$
- $\mathrm{Ext}^*_{\mathbb{C}G}(\mathbb{C}, V)$ is cohomology of this complex.

With Coefficients

$$\mathcal{S}G = \left\{ \phi : G \to \mathbb{C} \mid \ \forall_k \sum_{g \in G} |\phi(g)| \left(1 + \ell_G(g)\right)^k < \infty \right\}$$

Suppose V is a bornological $\mathcal{S}G$-module.

- $HP^*(G; V) = \mathrm{bExt}^*_{\mathcal{S}G}(\mathbb{C}, V)$.
- $$0 \leftarrow \mathbb{C} \rightleftarrows P_0 \rightleftarrows P_1 \rightleftarrows P_2 \rightleftarrows \dots$$
- $$0 \to \mathrm{bHom}_{\mathcal{S}G}(P_0, V) \to \mathrm{bHom}_{\mathcal{S}G}(P_1, V) \to \mathrm{bHom}_{\mathcal{S}G}(P_2, V) \to \dots$$
- $\mathbb{C}G \hookrightarrow \mathcal{S}G$ induces $HP^*(G; V) \to H^*(G; V)$ for all bornological $\mathcal{S}G$-modules V.

The Isocohomological Property

Definition

G has the (strong) isocohomological property if for all bornological $\mathcal{S}G$-modules V, the comparison map $HP^*(G;V) \to H^*(G;V)$ is an isomorphism. G is isocohomological for a particular $\mathcal{S}G$-module V if the particular comparison map is an isomorphism.

- Nilpotent groups (Ron Ji, Ralf Meyer)
- Combable groups (Crichton Ogle, Ralf Meyer)

Other Bounding Classes

$\mathcal{B} \subset \{\phi : [0,\infty) \to (0,\infty) | \phi \text{ is nondecreasing } \}$

- $1 \in \mathcal{B}$.
- If ϕ and $\phi' \in \mathcal{B}$, there is $\varphi \in \mathcal{B}$ such that $\lambda\phi + \mu\phi' \leq \varphi$, for nonnegative real λ, μ.
- If $\phi \in \mathcal{B}$ and g is a linear function, there is $\psi \in \mathcal{B}$ such that $\phi \circ g \leq \psi$.

Examples: $\mathbb{R}^+$, $\left\{ e^f \mid f \text{ is linear } \right\}$.

Connes-Moscovici

Theorem (Connes-Moscovici, 90)

Suppose G is a finitely generated discrete group endowed with word-length function ℓ_G. If G has the Rapid Decay property, and has cohomology of polynomial growth, then G satisfies the Strong Novikov Conjecture.

- $$\sum_{g \in G} |f(g)|^2 \left(1 + \ell_G(g)\right)^{2k}$$
- $HP^*(G) \to H^*(G)$ surjective.
- Chatterji-Ruane: Groups hyperbolic relative to polynomial growth subgroups are RD.
- Those groups are also (strongly) isocohomological.

Bass Conjecture

Due to Burghelea, $HC_*(\mathbb{C}G) = \oplus_{x \in <G>} HC_*(\mathbb{C}G)_x$.

Conjecture

Strong Bass Conjecture *For each non-elliptic class x, the image of the composition $\pi_x \circ ch_* : K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G)_x$ is zero.*

$x \in < G >$ satisfies 'nilpotency condition' if
$S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x$ is nilpotent.

Observation (Eckmann, Ji)

Let x be a non-elliptic conjugacy class satisfying the nilpotency condition. The composition $K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G) \xrightarrow{\pi_x} HC_*(\mathbb{C}G)_x$ is zero. In particular the Strong Bass Conjecture holds for G if each non-elliptic conjugacy class satisfies the nilpotency condition.

Bass Conjecture

- For a non-elliptic $x \in < G >$, take $h \in x$.
- G_h centralizer with $N_h = G_h/(h)$.
- Burghelea: $HC_*(\mathbb{C}G)_x \cong H_*(N_h)$.
- $S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x$ acts as

$$0 \to (h) \to G_h \to N_h \to 0$$

- If N_h has finite virtual cohomological dimension, G satisfies nilpotency condition.

ℓ^1 Bass Conjecture

- $K_*(\ell^1 G) \cong K_*(\mathcal{S}G)$.
- $ch_* : K_*(\mathcal{S}G) \to HC_*(\ell^1 G)$ factors through $HC_*(\mathcal{S}G)$.

Conjecture

Strong ℓ^1 - Bass Conjecture *For each non-elliptic conjugacy class, the image of the composition $\pi_x \circ ch_* : K_*(\mathcal{S}G) \to HC_*(\mathcal{S}G)_x$ is zero.*

Is true whenever, for each non-elliptic conjugacy class x, $S_x^t : HC_*(\mathcal{S}G)_x \to HC_{*-2}(\mathcal{S}G)_x$ is nilpotent.

Question

"If S_x is nilpotent, when is S_x^t nilpotent?"

ℓ^1 Bass Conjecture

Definition

G satisfies a polynomial conjugacy problem if for each non-elliptic $x \in < G >$ there is P_x such that: $u, v \in x$ then there is $g \in G$ with $g^{-1}ug = v$ such that $\ell_G(g) \leq P_x(\ell_G(u) + \ell_G(v))$.

- Hyperbolic groups
- Pseudo-Anosov classes in Mapping class groups
- Mapping class groups

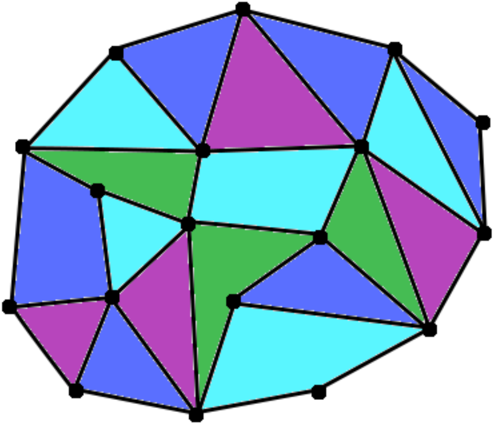
If G satisfies a polynomial conjugacy bound for a non-elliptic class x, $HC_*(\mathcal{S}G)_x \cong HP_*^{\ell_G}(N_h)$.
If in addition N_h isocohomological $S_x^t : HC_*(\mathcal{S}G)_x \to HC_{*-2}(\mathcal{S}G)_x$ is nilpotent, too.

HF^∞ Groups

Definition

A group is of type HF^∞ if it has a classifying space the type of a "simplicial complex" with finitely many cells in each dimension.

Dehn Functions



Weighted Dehn Functions

Suppose that X is a weakly contractible complex with fixed basepoint x_0.

- Define the weight of a vertex v to be $\ell_X(v) = d_{X^{(1)}}(v, x_0)$.
- Define the weight of a higher dimensional simplex to be the sum of the weights of its vertices.
- The weighted volume of an n-dimensional subcomplex is the sum of the weights of its n-dimensional cells.
- Get 'Weighted Dehn Functions' rather than just 'Dehn Functions'.

A Geometric Characterization

Theorem (Ji-R,2009)

For an HF^∞ group G, the following are equivalent.

(1) *All higher Dehn functions of G are polynomially bounded.*

(2) $HP^*(G;V) \to H^*(G;V)$ *is an isomorphism for all coefficients* V. *(i.e. G is strongly isocohomological)*

(3) $HP^*(G;V) \to H^*(G;V)$ *is surjective for all coefficients* V.

(1) implies (2)

- Denote by X is the universal cover of the HF^∞ classifying space.
- $C_*(X)$ is a projective resolution of $\mathbb{C}$ over $\mathbb{C}G$.
- Length function on the vertices of X: $\ell_X(x) = d_X(x, *)$.
- Length function on $X^{(n)}$: $\ell_X(\sigma) = \sum_{v\in\sigma} \ell_X(v)$.
- $S_n(X)$ the completion of $C_n(X)$ under the family of norms given by
$$\|\phi\|_k = \sum_{\sigma\in X^{(n)}} |\phi(\sigma)| \left(1 + \ell_X(\sigma)\right)^k$$
- $S_*(X)$ a projective resolution of $\mathbb{C}$ over $\mathcal{S}G$. (The Dehn function bounds ensure a bounded contracting homotopy of $S_*(X)$)

(1) implies (2)

For each n there is finite dimensional W_n with

$$\begin{aligned} S_n(X) &\cong \mathcal{S}G \hat{\otimes} W_n \\ C_n(X) &\cong \mathbb{C}G \otimes W_n \end{aligned}$$

$$\begin{aligned} \mathrm{bHom}_{\mathcal{S}G}(S_n(X), V) &\cong \mathrm{bHom}_{\mathcal{S}G}(\mathcal{S}G \hat{\otimes} W_n, V) \\ &\cong \mathrm{Hom}(W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(C_n(X), V) \end{aligned}$$

After applying $\mathrm{bHom}_{\mathcal{S}G}(\cdot, V)$ to $S_*(X)$ and $\mathrm{Hom}_{\mathbb{C}G}(\cdot, V)$ to $C_*(X)$ we obtain isomorphic cochain complexes.

the rest

- (2) implies (3) is obvious.
- (3) implies (1): This implication is similar to Mineyev's corresponding result on hyperbolic group and bounded cohomology.

Arzhantseva-Osin

- S free abelian group with free generating set $\{s_1, s_2\}$.
- A free abelian group with free generating set $\{a_1, a_2, a_3\}$.
- $\beta : S \to SL(3, \mathbb{Z})$ an injection such that $\beta(s_i)$ is semi-simple with real spectrum.
- $P = A \rtimes_\beta S$.
- For all $a \in A$, $\ell_P(a) \leq C \log(1 + \ell_A(a)) + \epsilon$
- A solvable group with quadratic first Dehn function.
- Higher Dehn functions?

Hochschild-Serre Spectral Sequence

- $0 \to H \to G \to Q \to 0$
- Let H be isocohomological for $\mathbb{C}$ with respect to the restricted length from G.
- Equip Q with the quotient length.

Theorem (Ogle, R)

There is a spectral sequence with $E_2^{p,q} \cong HP^p(Q; HP^q(H))$ which converges to $HP^*(G)$.

Hochschild-Serre Spectral Sequence

- $$C^{p,q} = \mathrm{bHom}_{\mathcal{S}Q}(\mathcal{S}Q^{\hat{\otimes}p+1}, \mathrm{bHom}_{\mathcal{S}H}(\mathcal{S}G^{\hat{\otimes}q+1}, \mathbb{C}))$$
- Rowwise filtration collapses to $HP^*(G)$.
- To identify E_2 term, need the isocohomological property of H and the 'bounded mapping theorem' of Hogbe-Nlend.

Hochschild-Serre Spectral Sequence

Comparing this Spectral Sequence with the usual Hochschild-Serre Spectral Sequence we get the following.

Corollary

If Q is isocohomological for the twisted coefficients $HP^*(H)$, in the quotient length, then G is isocohomological for $\mathbb{C}$.

Polynomial extensions

Definition

An extension $0 \to H \to G \to Q \to 0$ is a polynomial extension if there is a cross section yielding a cocycle of polynomial growth and inducing a polynomial action of Q on H.

These extensions were first studied by Noskov in relation to the RD property.

Theorem (Noskov, 92)

Let G be a polynomial extension of the finitely generated group Q by the finitely generated group H. If H and Q have the Rapid Decay property, so does G.

Back to Arzhantseva-Osin

Lemma (Ji-Ogle-R)

The comparison map $\Phi : HP^3(P) \to H^3(P)$ is not surjective.

Use the commutative diagram below and the fact that the map $HP^3(A) \to H^3(A)$ is zero.

$$\begin{array}{ccc} HP^3(P) & \longrightarrow & HP^3(A) \\ \Phi\downarrow & & \downarrow \\ H^3(P) & \longrightarrow & H^3(A) \end{array}$$

Corollary

The second Dehn function d_P^2 of P satisfies $e^n \leq d_P^2(n) \leq e^{n^2}$

Bass-Serre Theory

- G acts cocompactly and without inversion on a tree T.
- V, E representatives of orbits of vertices and edges under G.
- For $v \in V$, G_v the stabilizer of that vertex. G_e similarly.

Theorem (Serre, 77)

For each G-module M, there is a long-exact sequence

$$\ldots \to H^i(G; M) \to \prod_{v \in V} H^i(G_v; M) \to \prod_{e \in V} H^i(G_e; M) \to H^{i+1}(G; M) \to \ldots$$

Bass-Serre Theory

- Equip G_v, G_e with restricted length, ℓ_G.

Lemma (R)

For each bornological $\mathcal{S}G$-module M, there is a long exact sequence

$$\ldots \to HP^i(G;M) \to \prod_{v\in V} HP^i(G_v;M) \to \prod_{e\in V} HP^i(G_e;M) \to HP^{i+1}(G;M$$

Corollary

Let G, G_v, and G_e be as above. If each G_v and G_e are isocohomological in ℓ_G, then G is isocohomological.

Complexes of Groups

- Group G acting cocompactly on contractible simplicial complex X without inversion.
- ‘Complexes of Groups’ instead of ‘Graphs of Groups’
- Σ a set of representatives of the orbits of simplices of X, under G.
- For $\sigma \in \Sigma$, G_σ the stabilizer of σ.

Theorem (Serre, 71)

For each G-module M there is a spectral sequence with E_1 term the product

$$E_1^{p,q} \cong \prod_{\sigma \in \Sigma_p} H^q(G_\sigma; M)$$

and which converges to $H^*(G; M)$.

Complexes of Groups

- Dehn functions of X polynomially bounded.
- Equip each G_σ with ℓ_G.

Theorem (Ji-Ogle-R)

For each bornological $\mathcal{S}G$-module M there is a spectral sequence with

$$E_1^{p,q} \cong \prod_{\sigma \in \Sigma_p} HP^q(G_\sigma; M)$$

and which converges to $HP^*(G; M)$.

Complexes of Groups

- Finite edge stabilizers ensure G finitely relatively presented.
- Polynomial Dehn function of X gives polynomially bounded relative Dehn function of G.
- These ensure that the G_σ are only polynomially distorted in G.

Corollary

If each G_σ is isocohomological, so is G.

This generalizes our earlier result.

Theorem (Ji-R, 2009)

Suppose that the group G is relatively hyperbolic with respect to the HF^∞ subgroups $H_1, \ldots, H_n$. If each H_i is isocohomological, so is G.