The Isocohomological Property, Higher Dehn Functions, and Relatively Hyperbolic Groups

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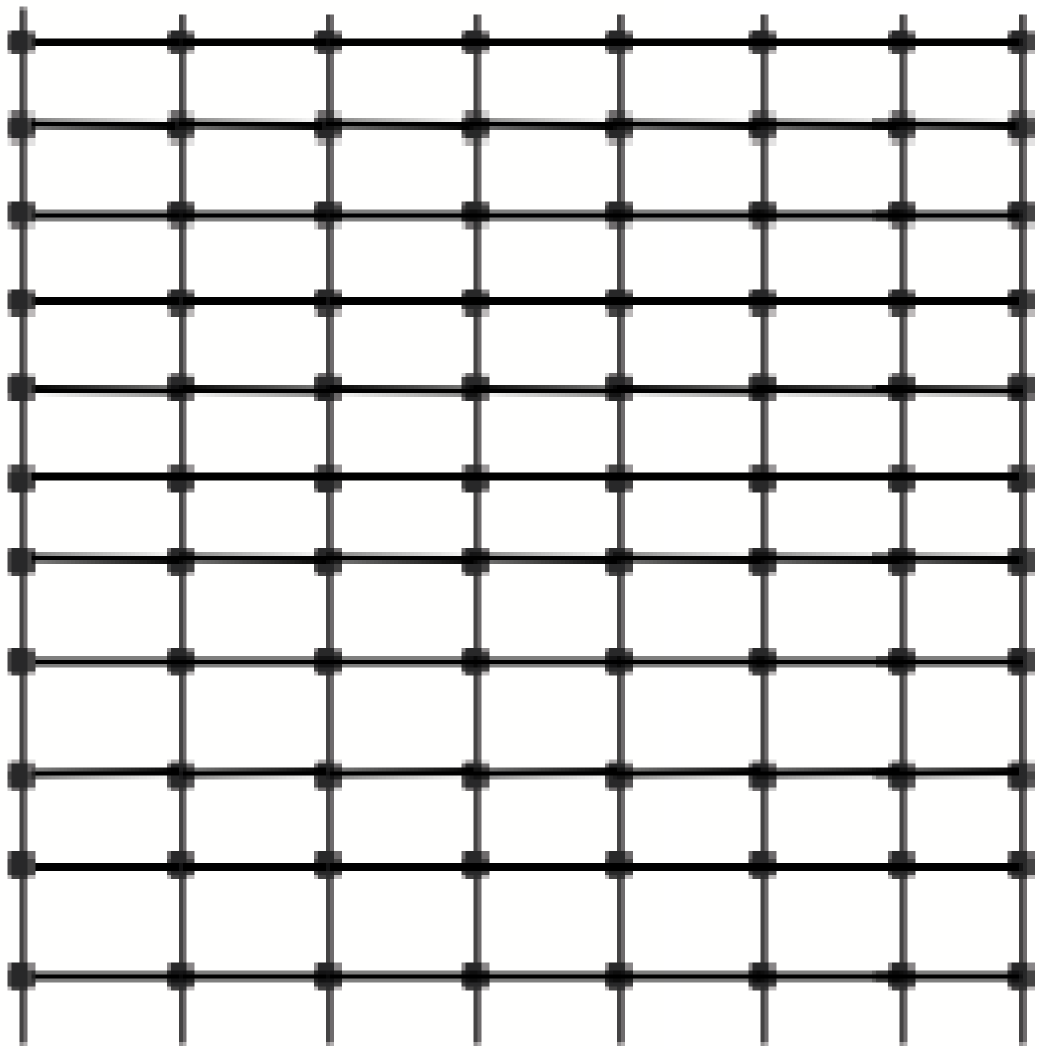
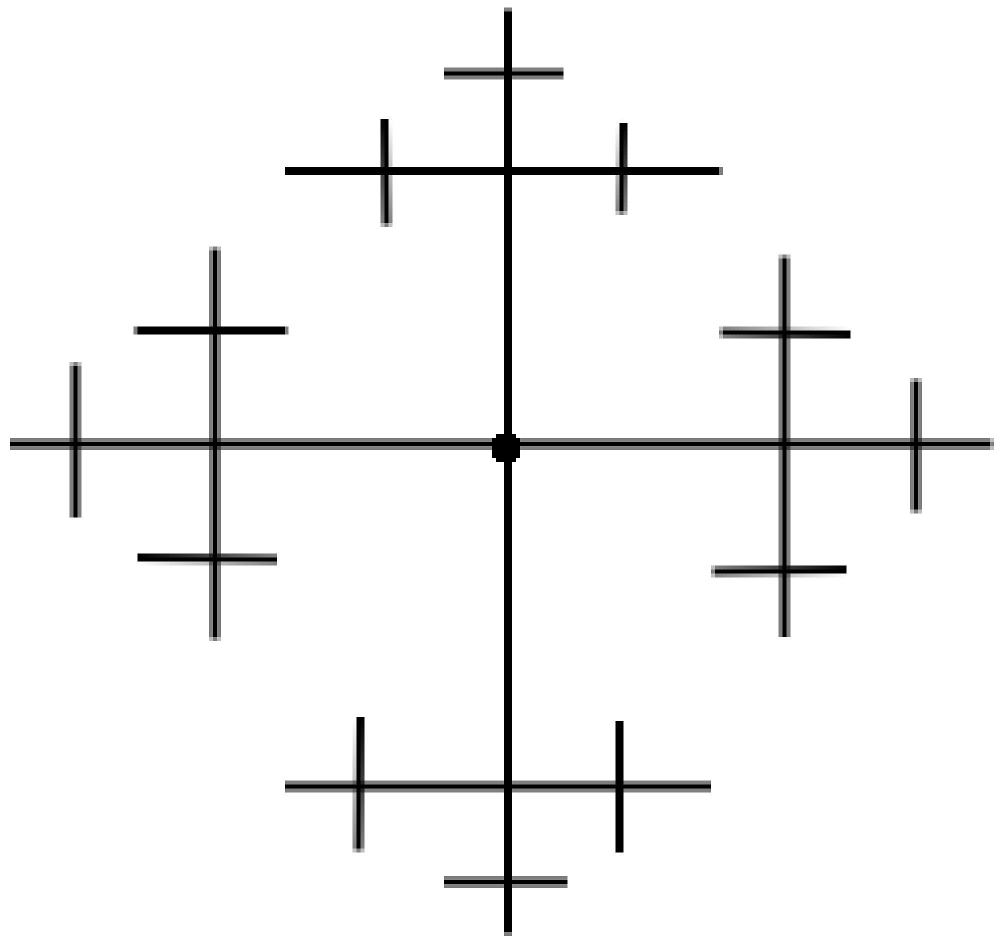
- Given a finite generating set, a word-length function on G, $\ell(g)$, is the least number of generators needed to multiply to g. The length function gives G the structure of a metric space.
- The group is usually treated as a graph, by treating the group elements as vertices and attaching edges between two group elements if they are at a distance of 1 apart. As such we can talk about geodesics in the group, ... etc.

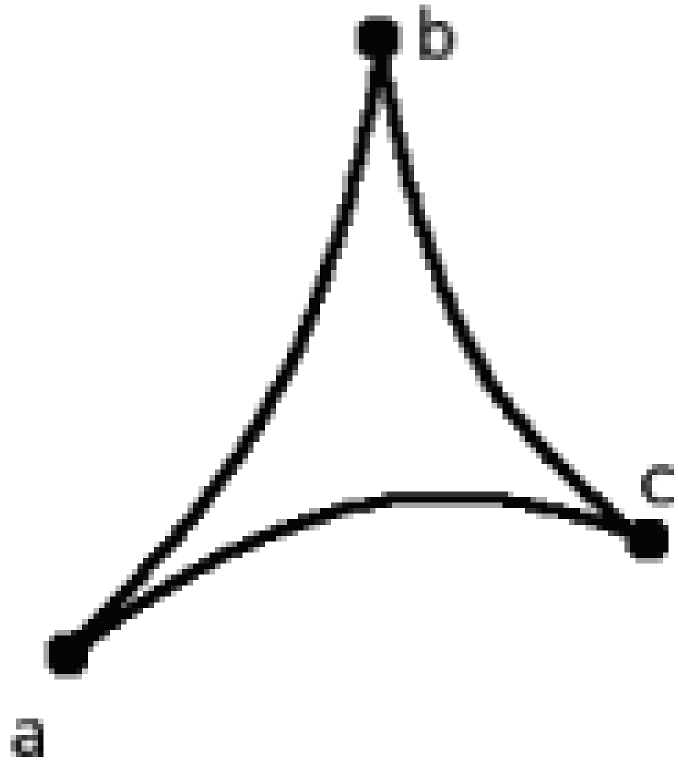




Isometry is too strict when trying to classify these metric spaces. We use quasi-isometry.
A function $\phi : (X, d_X) \to (Y, d_Y)$ is a quasi-isometry if there are positive constants λ, C, and D such that

- $\frac{1}{\lambda} d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq \lambda d_X(x, x') + C$
- For all $y \in Y$ there is $x \in X$ such that $d_Y(\phi(x), y) \leq D$

X and Y are quasi-isometric if there is some quasi-isometry from X to Y.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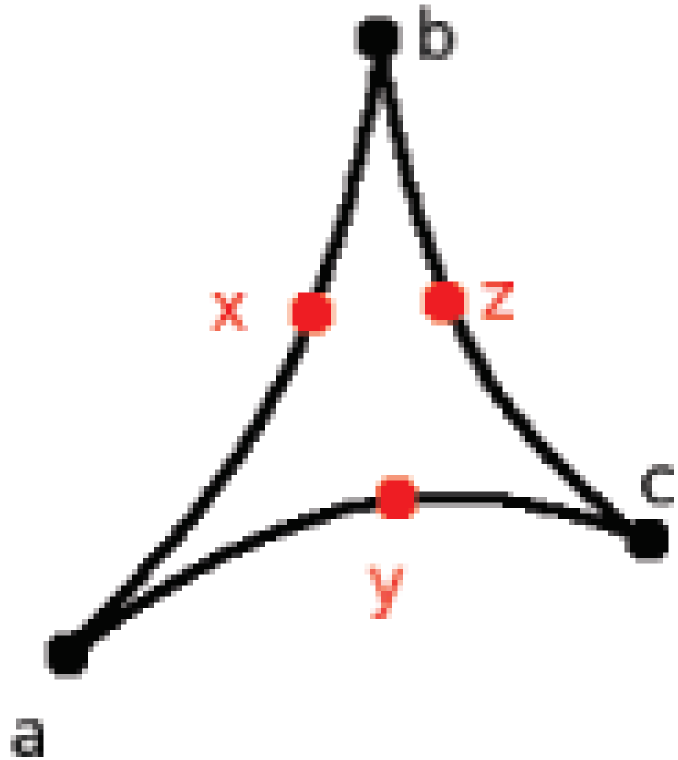
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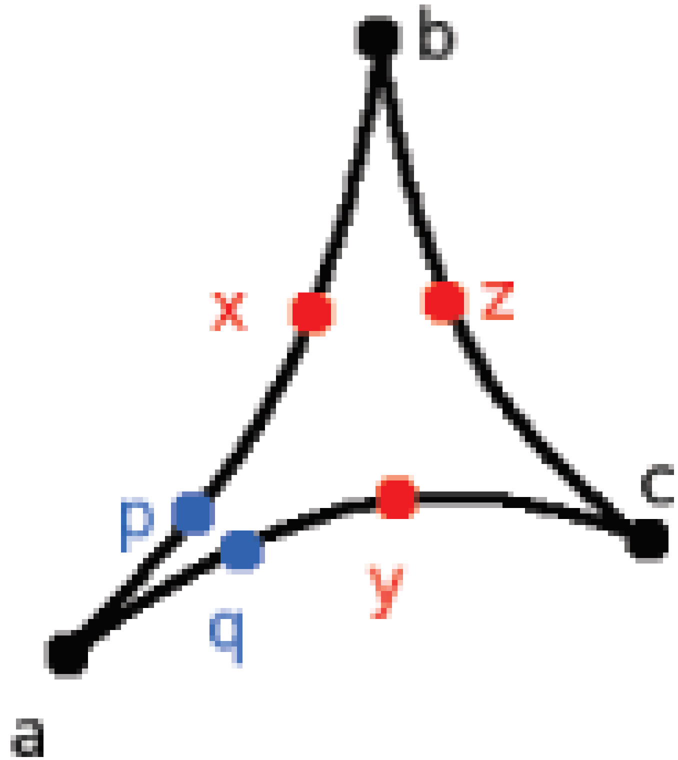
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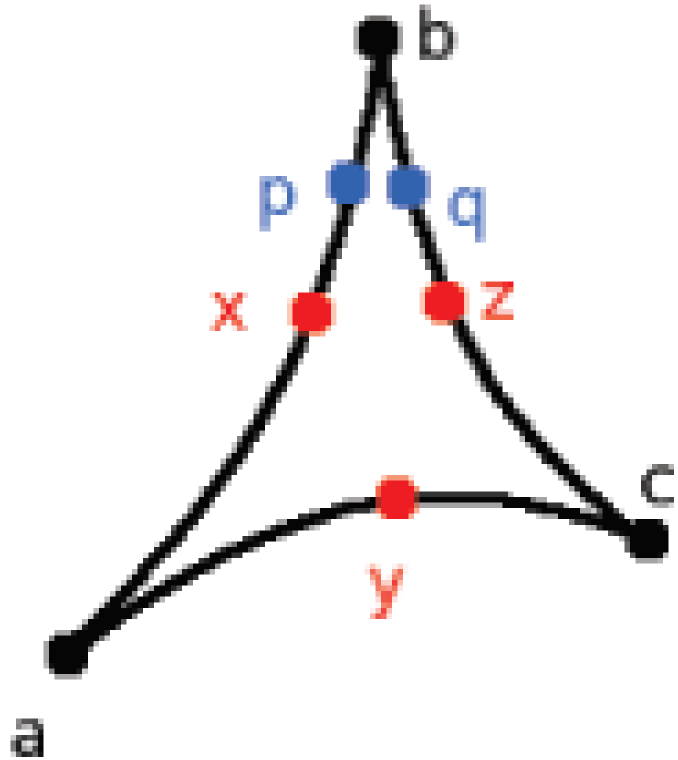
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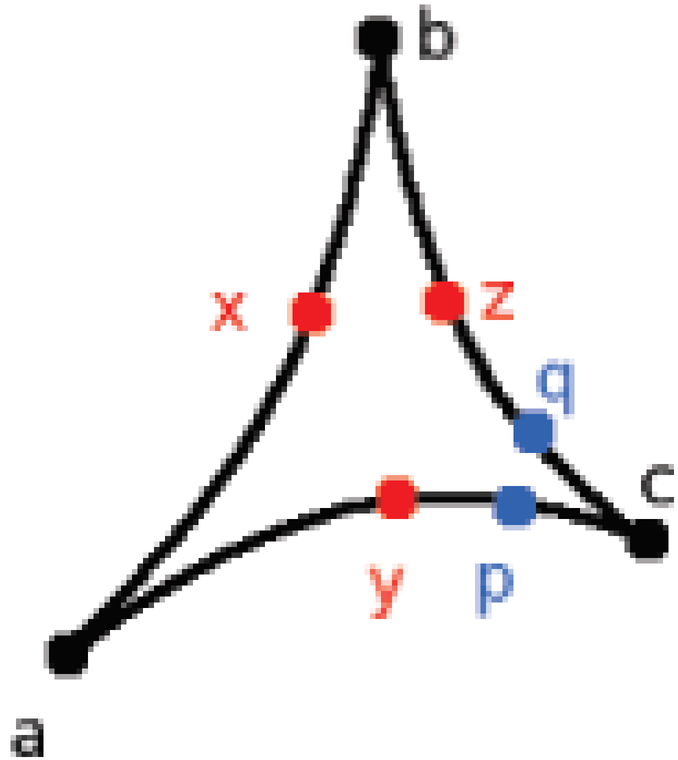
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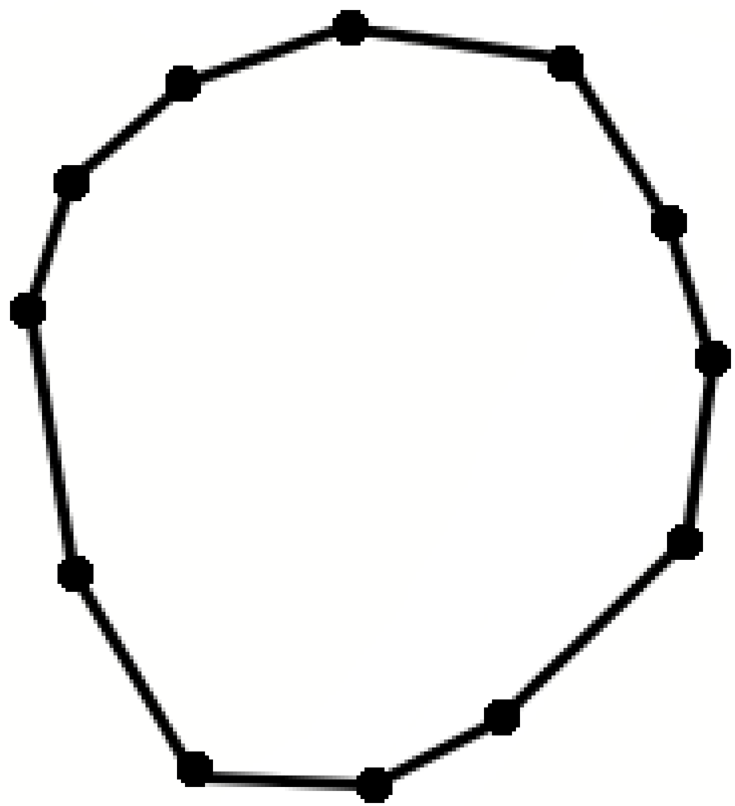
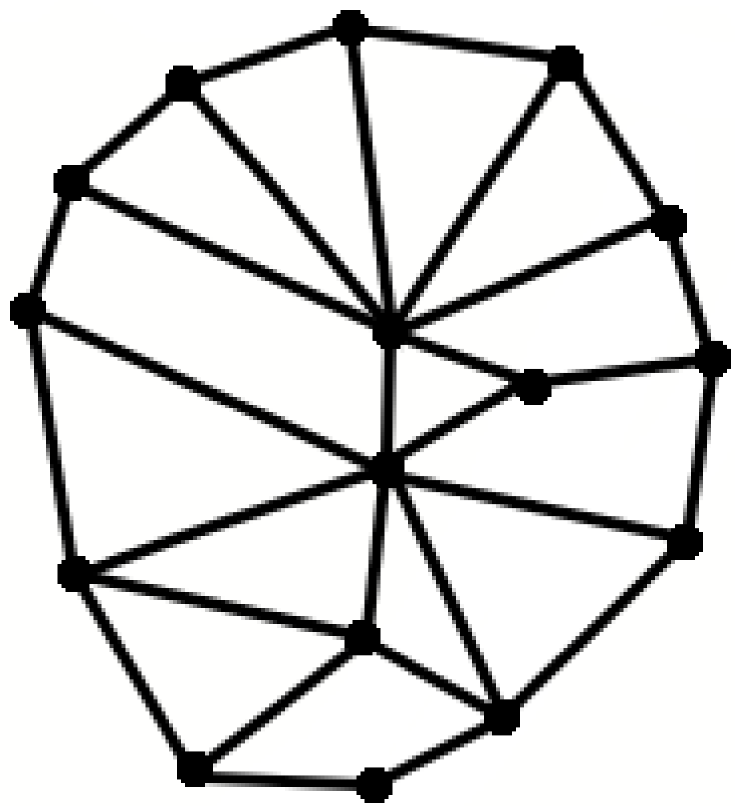
- Finite groups
- Free groups
- Fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature
- Virtually all finitely presented groups

- Suppose G admits free action on a topological space X with all homotopy groups trivial.
- Quotient BG called a classifying space for G.
- Unique up to homotopy-equivalence.
- Simplicial complex.

- Torsion free hyperbolic groups: Rips complex
- $\mathbb{Z}^n$: n-torus
- F_n, F_∞





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- Abelian groups have quadratic first Dehn function. All Dehn functions are polynomially bounded.
- Nilpotent groups have first Dehn function polynomially bounded.

$C^n(G) = \{\phi : G^n \to \mathbb{C}\}$
$d : C^n(G) \to C^{n+1}(G)$ given by

$$(d\phi)(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \phi(g_0, g_1, \ldots, \widehat{g_i}, \ldots, g_n)$$

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$H^n(G)$ is the quotient $\frac{\ker d}{\operatorname{im} d}$

$\phi \in C^n(G)$ is polynomially bounded if there is a polynomial P such that

$$|\phi(g_1, \ldots, g_n)| \leq P(\ell(g_1) + \ell(g_2) + \ldots + \ell(g_n))$$

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$HP^n(G) \to H^n(G)$

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- G is isocohomological if $HP^n(G;A) \to H^n(G;A)$ is isomorphism for all A

Let M be a closed oriented manifold, BG the classifying space of a group G, and $f : M \to BG$. For $x \in H^{n-4i}(BG; \mathbb{Q})$ and $L_i(M)$ the Hirzebruch class of M, the value of the pairing $< f^*(x) \cup L_i(M), [M] > \in \mathbb{Q}$ is called a *higher signature* of M.

Conjecture. *Novikov Conjecture The higher signatures are homotopy invariants of* (M, f).

Theorem. *Let G be a finitely generated group. Suppose that G has the rapid decay property, and that the homomorphism $HP^*(G) \to H^*(G)$ is surjective. Then the Novikov conjecture holds for all M with fundamental group G.*

G has the rapid decay property if every function $\phi : G \to \mathbb{C}$ with $\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2k} < \infty$ for each k, acts as a bounded operator on $\ell^2(G)$.

It was known that hyperbolic groups had both these properties, so Connes and Moscovici were able to verify the Novikov conjecture for them.

- Connes-Moscovici
- Ji
- Meyer, Ogle


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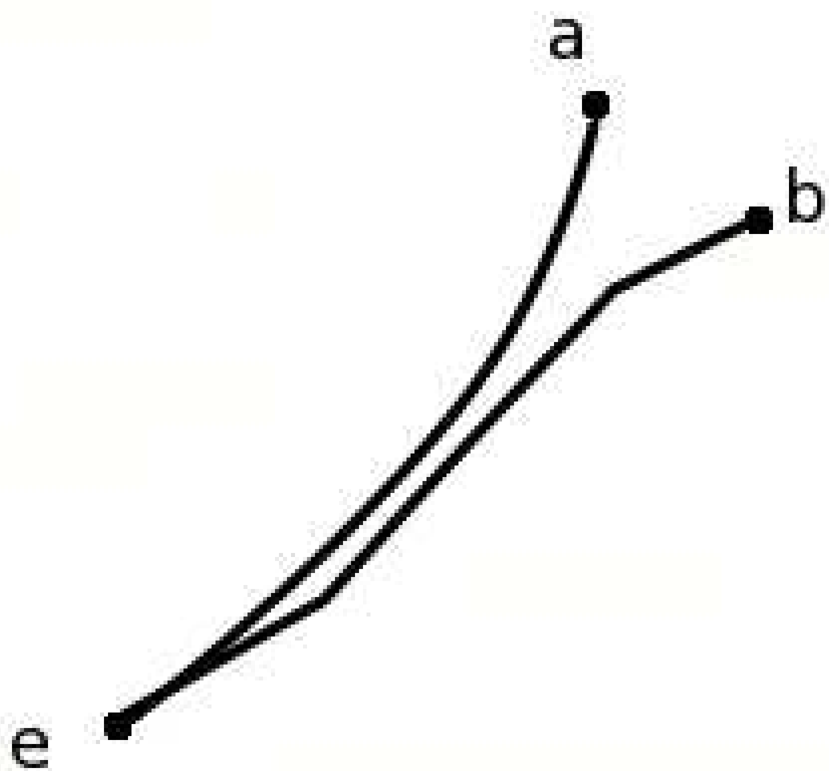
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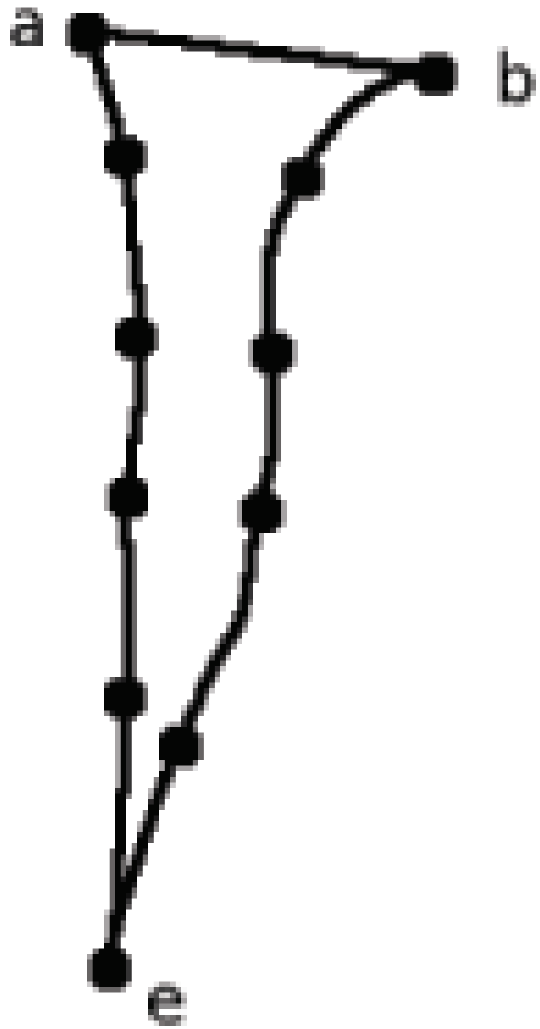

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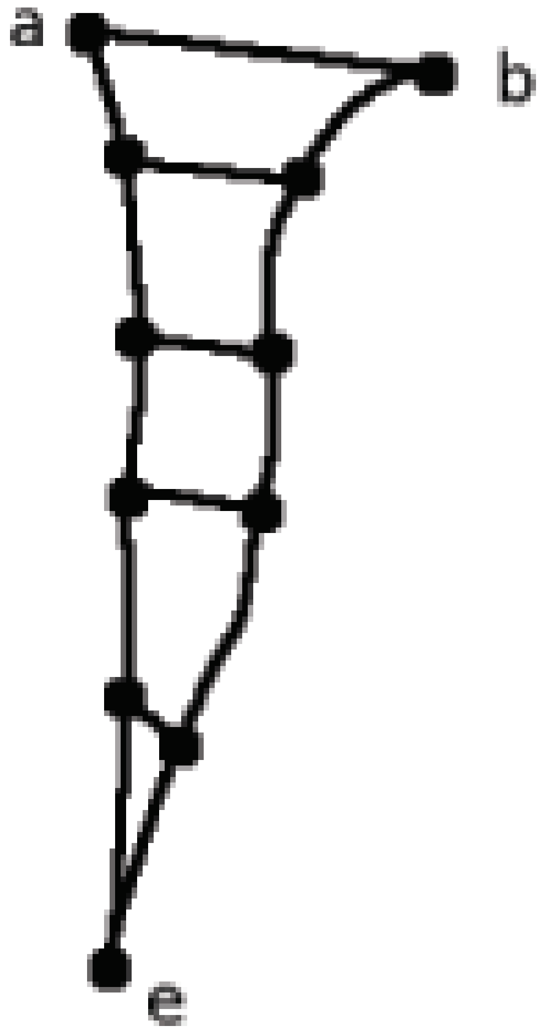
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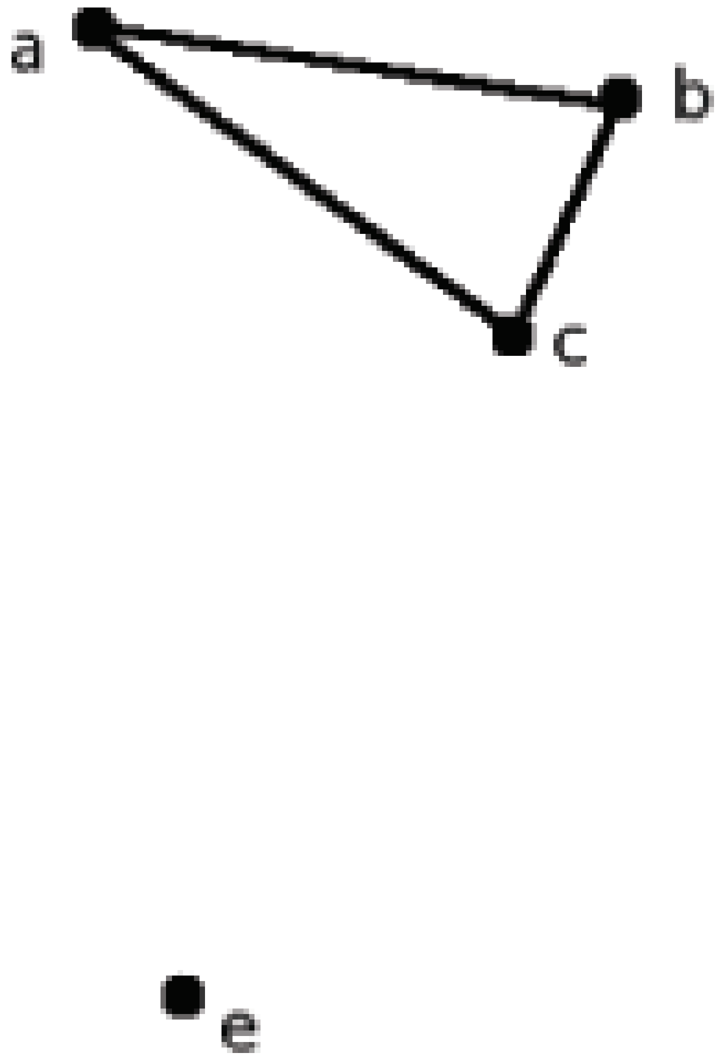
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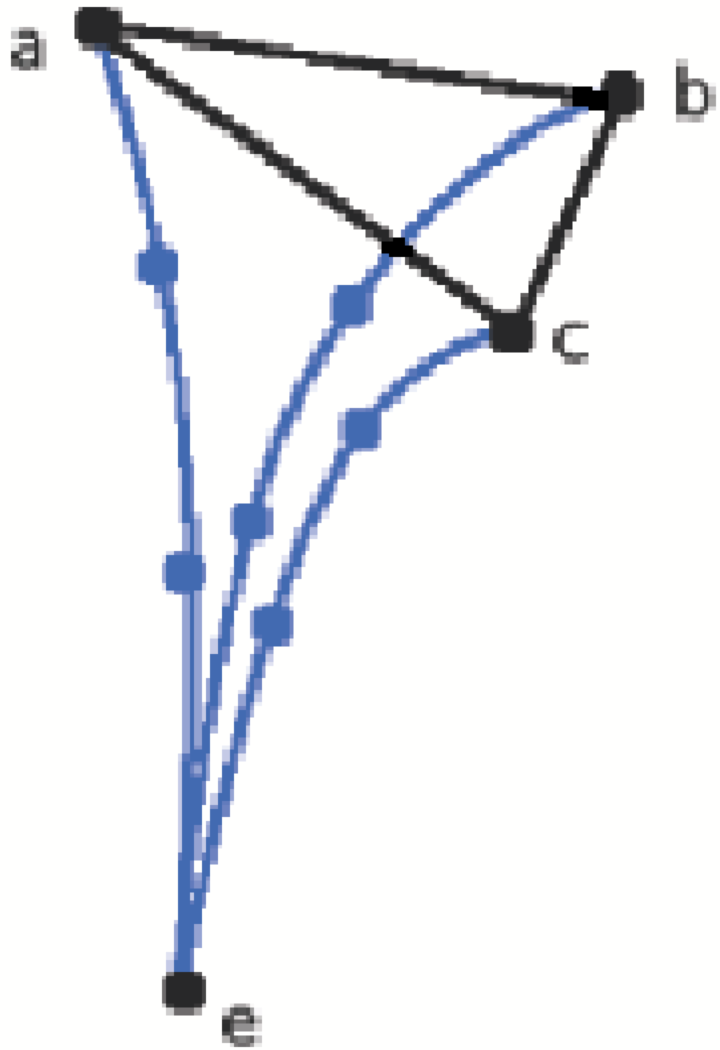
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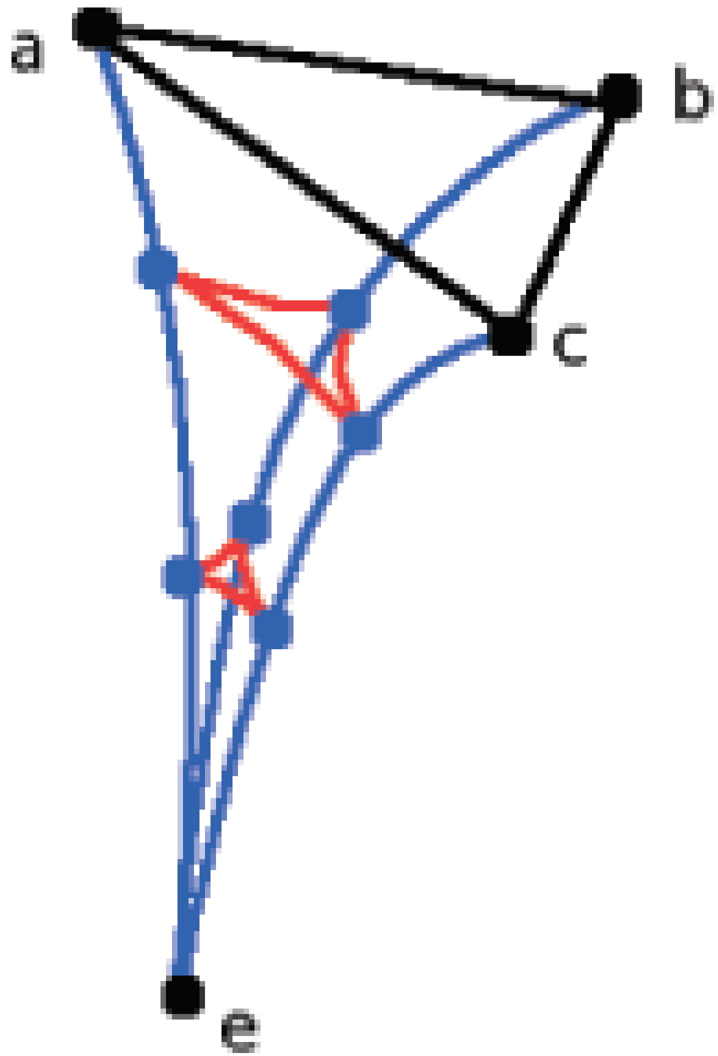
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- Combable groups are F_∞.
- Polynomially combable groups have all Dehn functions polynomially bounded.
- These include abelian groups, hyperbolic groups, automatic groups, ...

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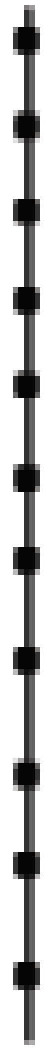
If G is F_∞ with all Dehn functions polynomially bounded, we can use BG to compare $HP^*(G;A)$ and $H^*(G;A)$.

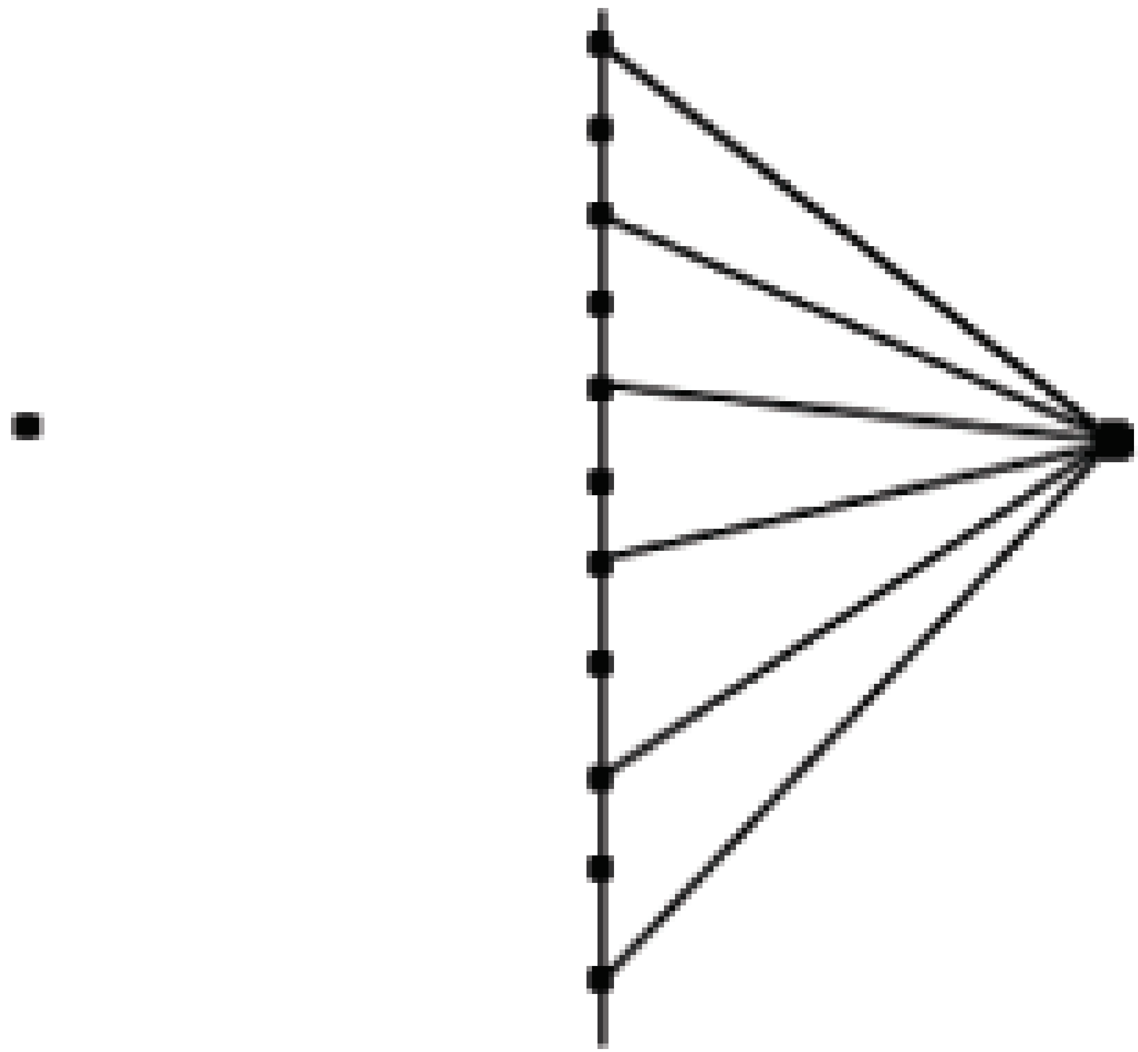
Theorem. *Suppose G is a type F_∞ group. G is isocohomological if and only if all higher Dehn functions are polynomially bounded.*

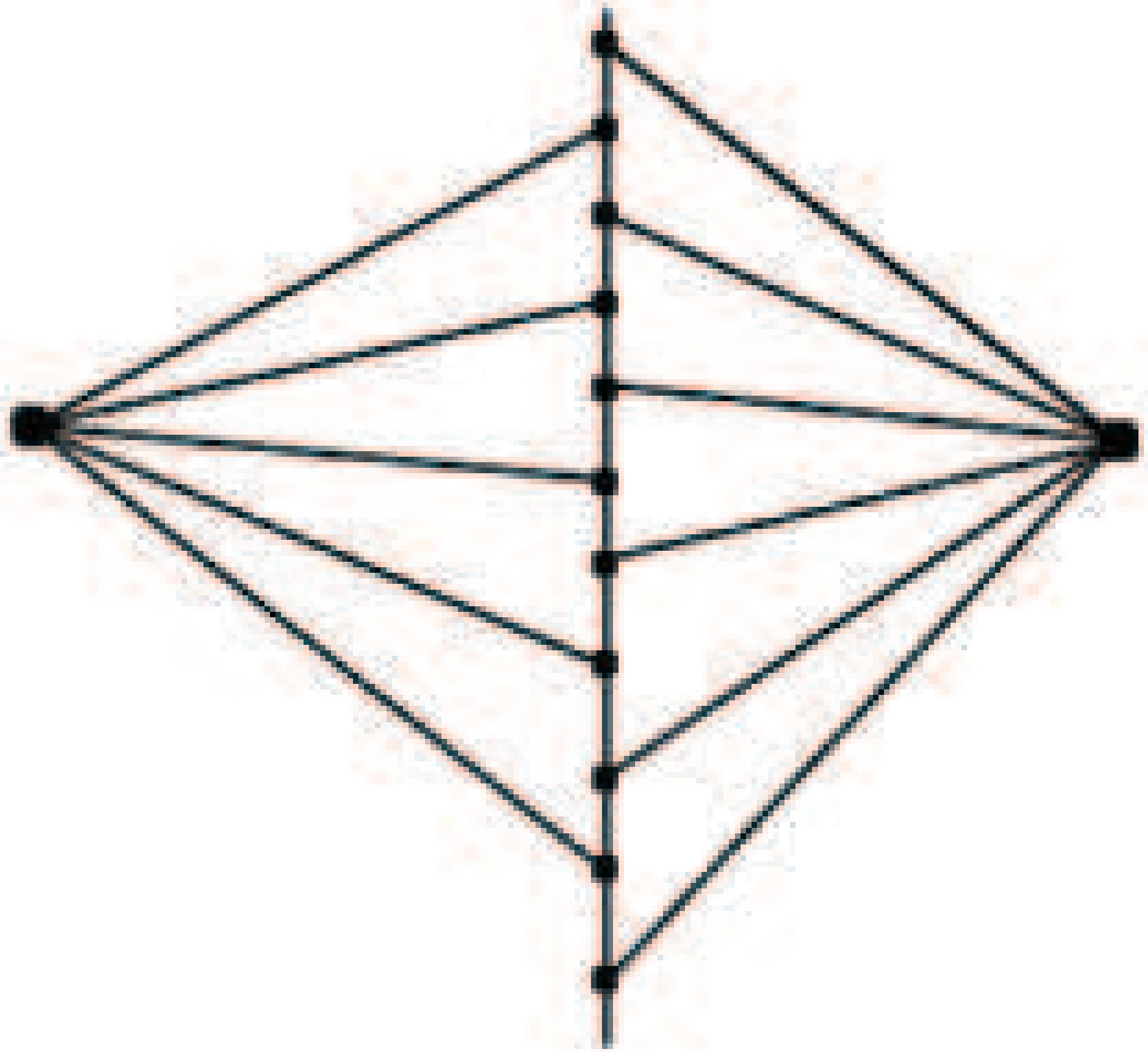
- Nilpotent groups have all Dehn functions polynomially bounded.
- $\mathbb{Z}^2 \rtimes \mathbb{Z}$ is not isocohomological.

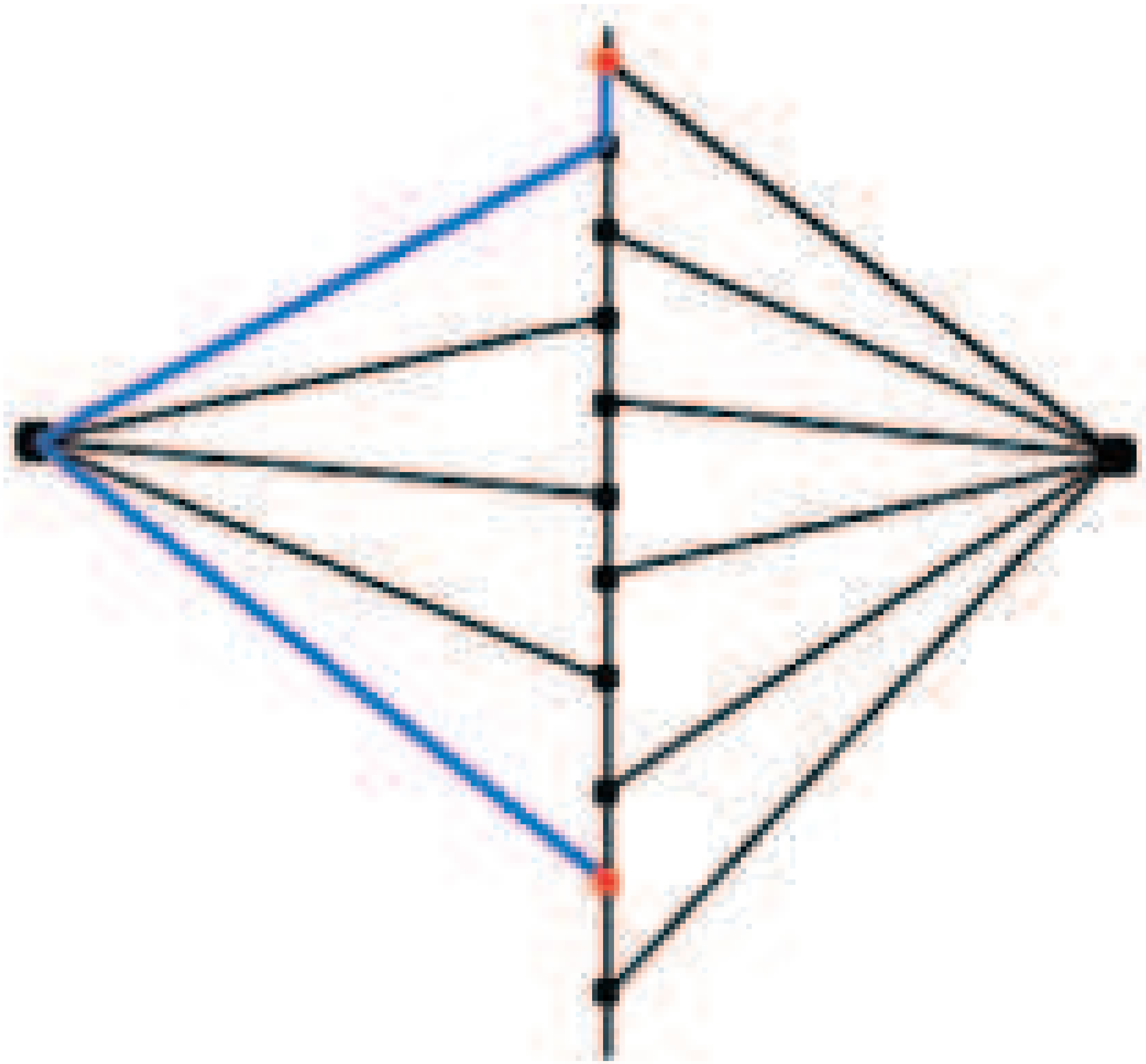
Currently, a lot of work proving theorems of the form:
Let G be hyperbolic relative to a subgroup H. Suppose that H has some property. Then G has the property, too.
This is true for uniform embeddability into Hilbert space, having finite asymptotic dimension, and the rapid decay property. In the same manner, one would like to be able to extend the isocohomological property for H, up to the isocohomological property for G.

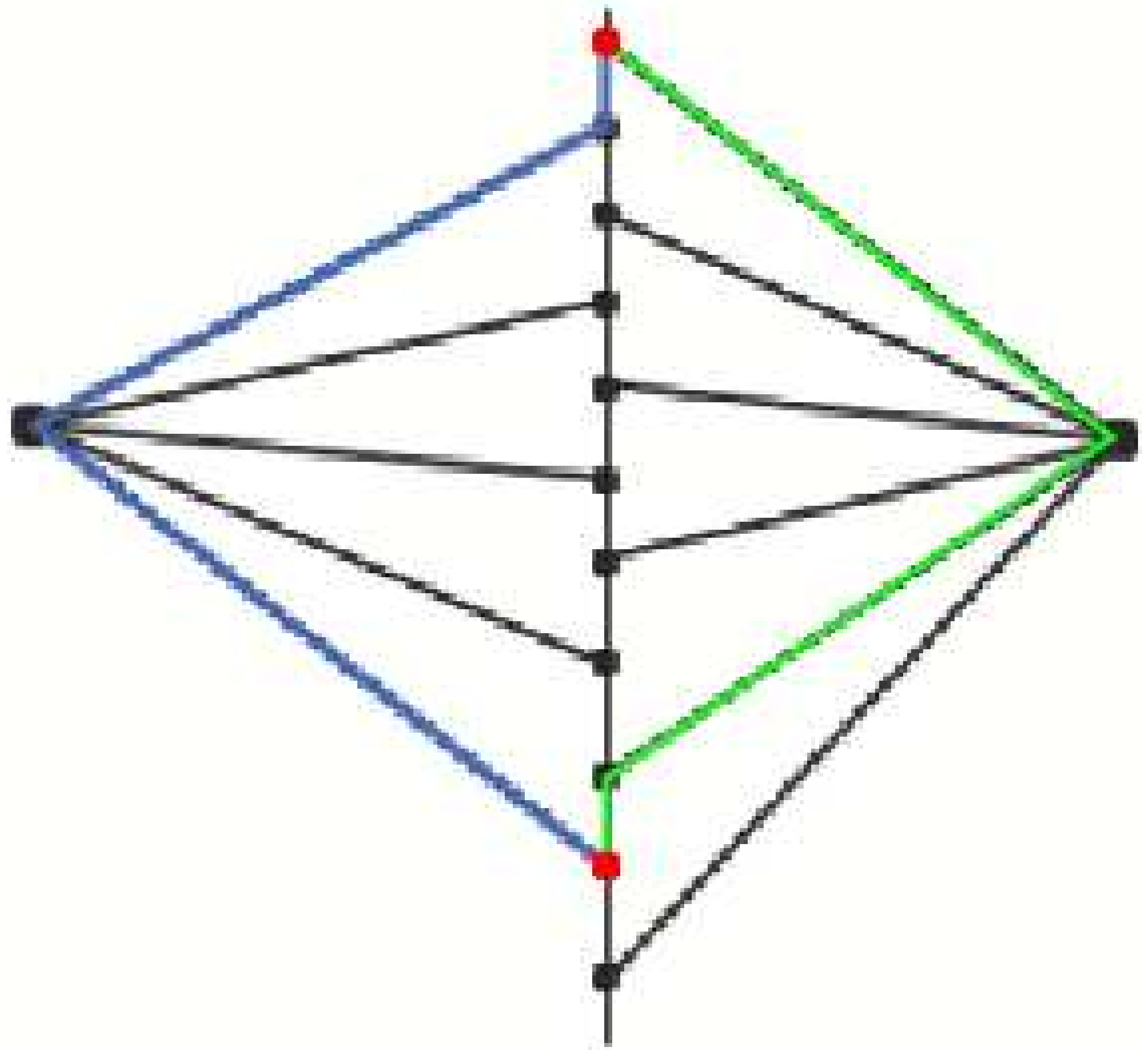
- Let M be a noncompact, complete, finite-volume Riemannian manifold with (pinched) negative sectional curvature $-b^2 \leq K(M) \leq -a^2 < 0$. The fundamental group of M hyperbolic relative to the collection of cusp subgroups.
- Free products
- Not $\mathbb{Z}^2$ with respect to $\mathbb{Z}$











G and H satisfy the bounded coset penetration property if for every λ there is a constant $c(\lambda)$ such that if p and q are two $(\lambda, 0)$-quasi-geodesics without backtracking, then:

- If p and q both penetrate a coset gH, the points at which p and q enter (respectively exit) gH are at a distance no more than $c(\lambda)$ from one another.
- If p penetrates a coset gH not penetrated by q, then the points where p enters the coset and where p exits the coset are within $c(\lambda)$ from one another.

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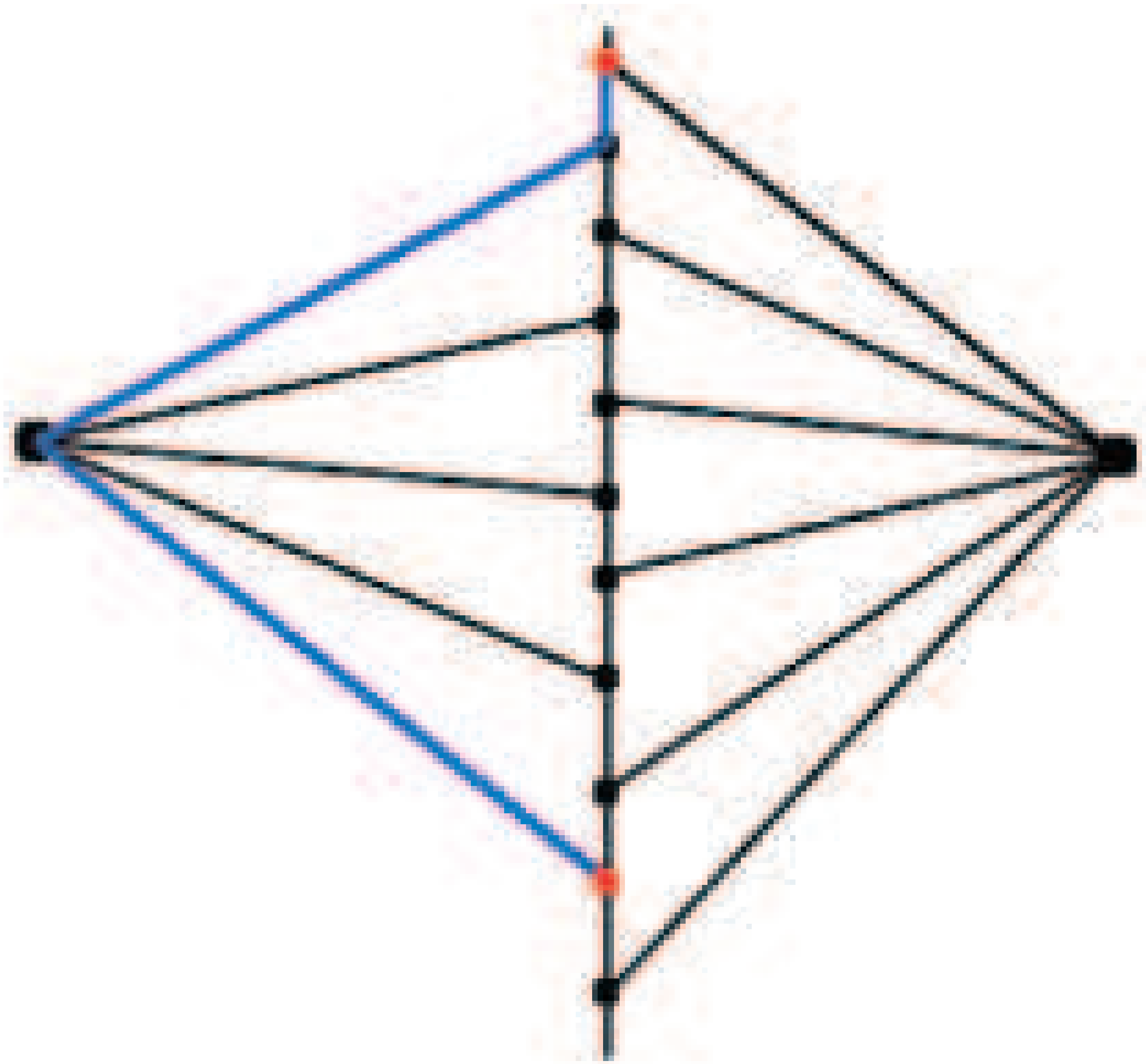
G is relatively hyperbolic with respect to H if the relative graph is hyperbolic and satisfies the bounded coset penetration property. Similar for finitely many subgroups $\{H_1, H_2, \dots H_n\}$.

Suppose H is polynomially combable. The relative graph is hyperbolic, so picking any geodesic from e to g as p_g yields a combing. We pick a tree of geodesics. That is, if p_g and p_h intersect, then they agree up to that point.



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A similar argument gives the following:

Theorem. *Suppose that H is F_∞. Then G is, too.*

Farb shows that the first Dehn function for G is equivalent to that of H.
By examining our construction of BG from BH, the higher Dehn functions of G are bounded by a polynomial of those of H.

Our characterization of the isocohomological property, and the Drutu-Sapir result mentioned above together gives that the groups for which the Connes-Moscovici approach to the Novikov conjecture holds, is closed under relatively hyperbolic extensions.