

# **The Isocohomological Property, Higher Dehn Functions, and Relatively Hyperbolic Groups**

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**Theorem.** *Connes-Moscovici* Suppose a finitely generated group  $G$  has the Rapid Decay property of Jolissaint and has cohomology of polynomial growth. Then  $G$  satisfies the strong Novikov conjecture.

- Rapid Decay:  $\phi : G \rightarrow \mathbb{C}$  such that for all  $k$

$$\sum_{g \in G} |\phi(g)|^2 (1 + \ell(g))^{2k} < \infty$$

lie in  $C_r^*G$ .

- Cohomology of Polynomial Growth: For  $[c] \in H^*(G; \mathbb{C})$  there is a representative  $\phi$  and a polynomial  $P$  such that  $|\phi(g_0, \dots, g_n)| \leq P(\ell(g_0) + \dots + \ell(g_n))$ .

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$d : C^n(G) \rightarrow C^{n+1}(G)$  given by

$$(d\phi)(g_0, g_1, \dots, g_n) = \sum_{i=0}^n (-1)^i \phi(g_0, g_1, \dots, \widehat{g}_i, \dots, g_n)$$

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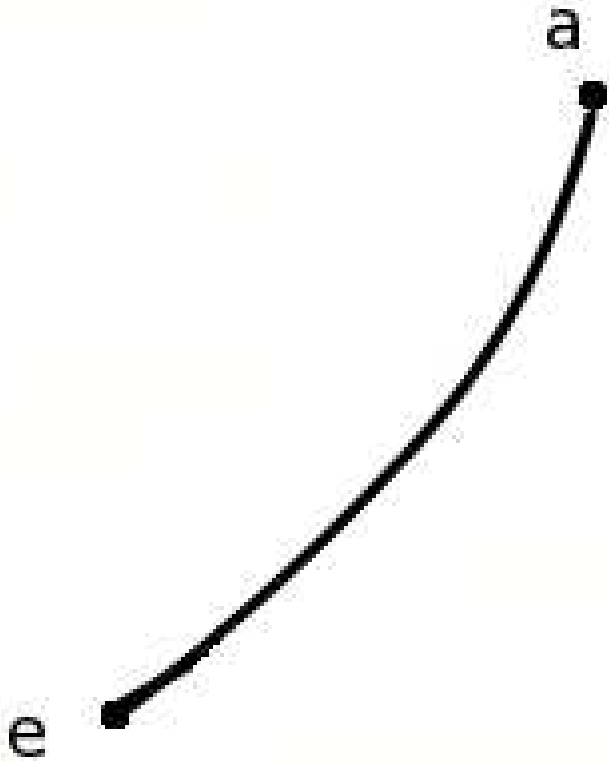


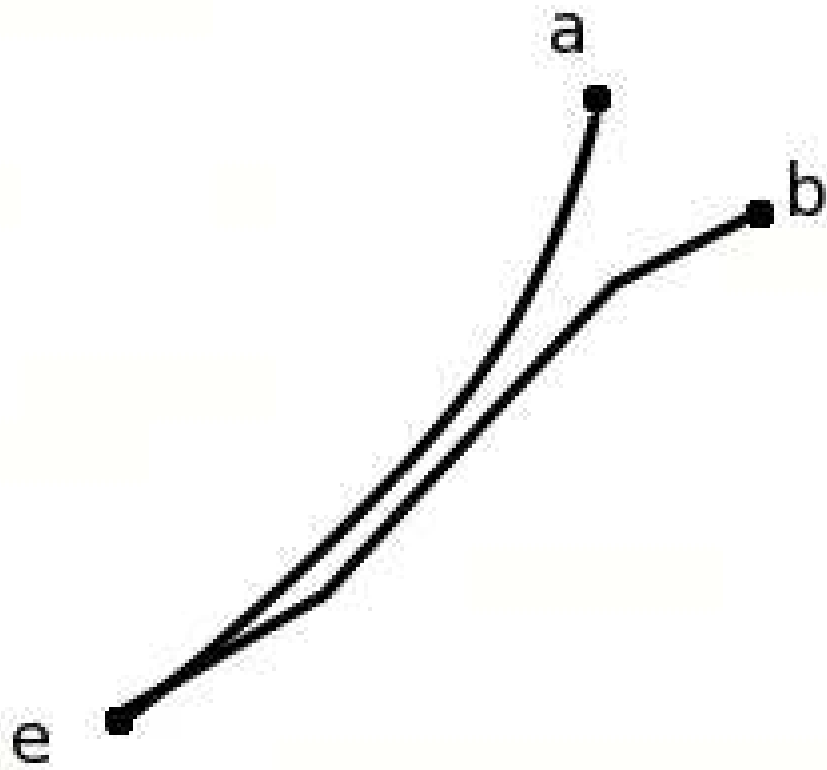
$$0 \leftarrow \mathbb{C} \leftarrow \mathcal{S}G \leftarrow \mathcal{S}G \hat{\otimes}_{\pi} \mathcal{S}G \leftarrow \mathcal{S}G \hat{\otimes}_{\pi}^3 \leftarrow \dots$$

**Definition 1.**  $G$  has the ‘weak isocohomological property’ if for all  $V$ ,  $HP^*(G; V) \rightarrow H^*(G; V)$  is an isomorphism.

The term ‘isocohomological’ comes from Meyer’s work on bornological analysis.  $f : A \rightarrow B$  is isocohomological if for any projective  $A$ -bimodule resolution  $P_*$  of  $A$ ,  $B \otimes_A P_* \otimes_A B$  is projective bimodule resolution of  $B$ .

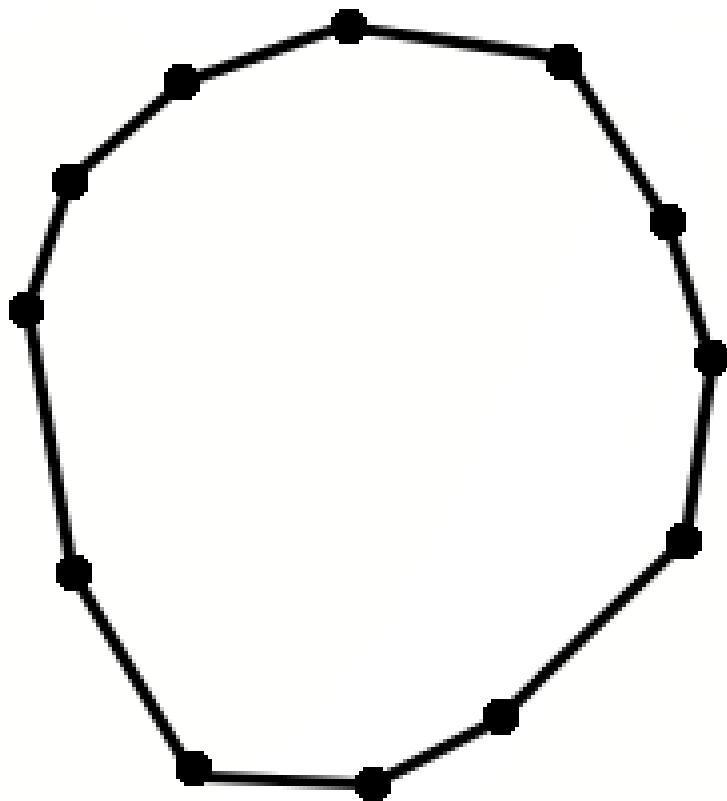
- Finitely generated nilpotent.
- Synchronous combing of polynomial length.

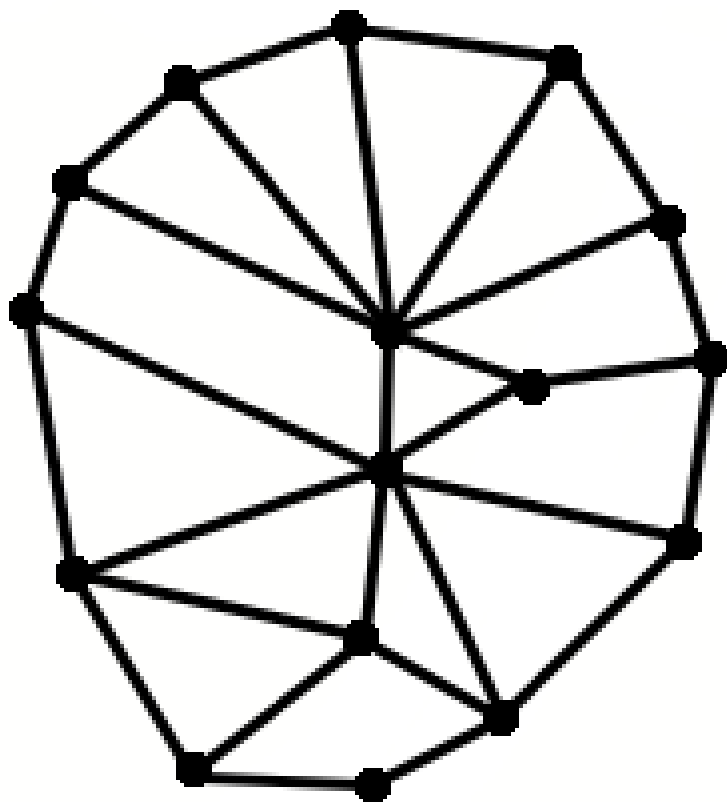




- $F_n$  if there is a  $BG$  with finite  $n$ -skeleton
- $F_\infty$  if there is a  $BG$  with all  $n$ -skeletons finite
- Ex: Torsion-free hyperbolic groups have Rips polyhedron
- Dehn functions in all degrees.





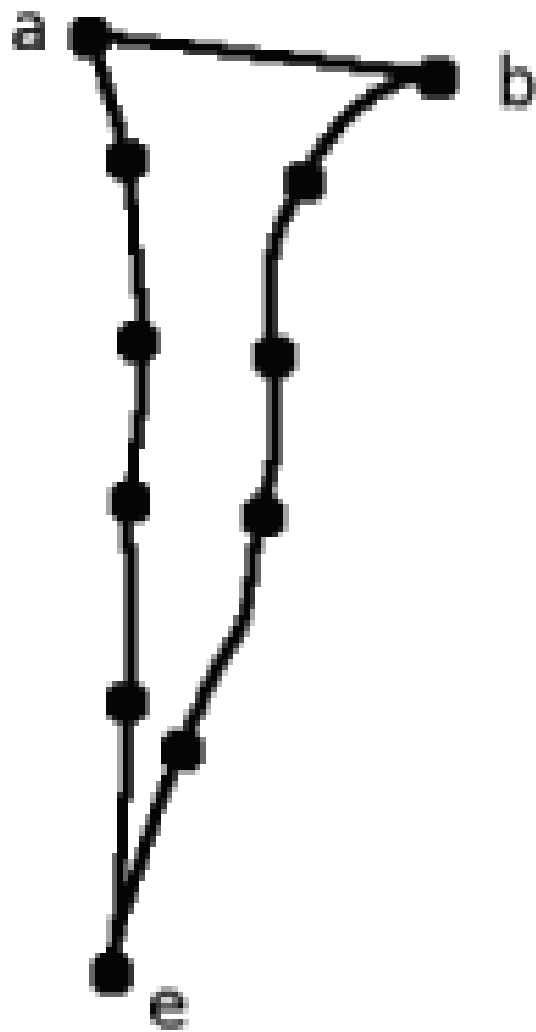


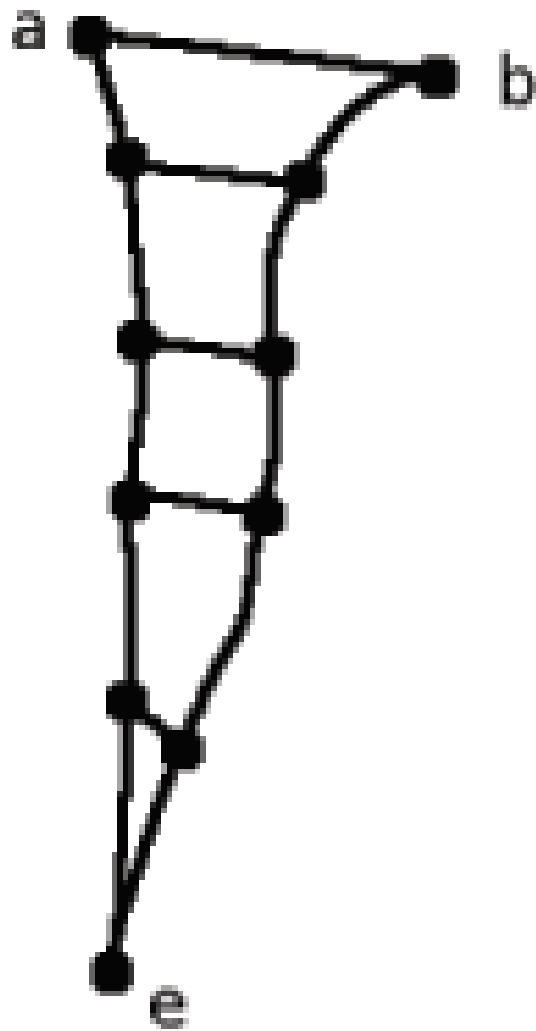
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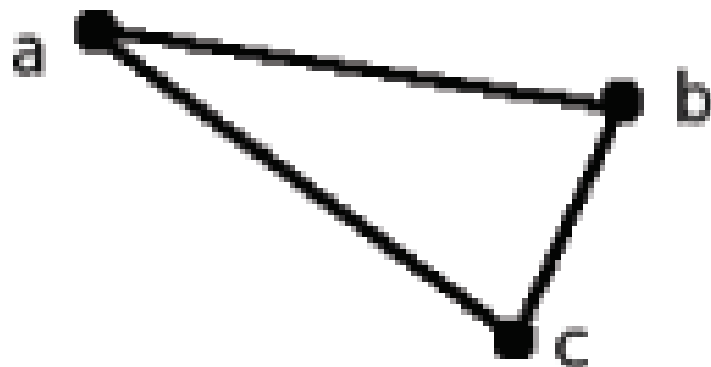
- Hyperbolicity is equivalent to linear first Dehn function. All Dehn functions are linearly bounded.
- Abelian groups have quadratic first Dehn function. All Dehn functions are polynomially bounded.
- Nilpotent groups have first Dehn function polynomially bounded.

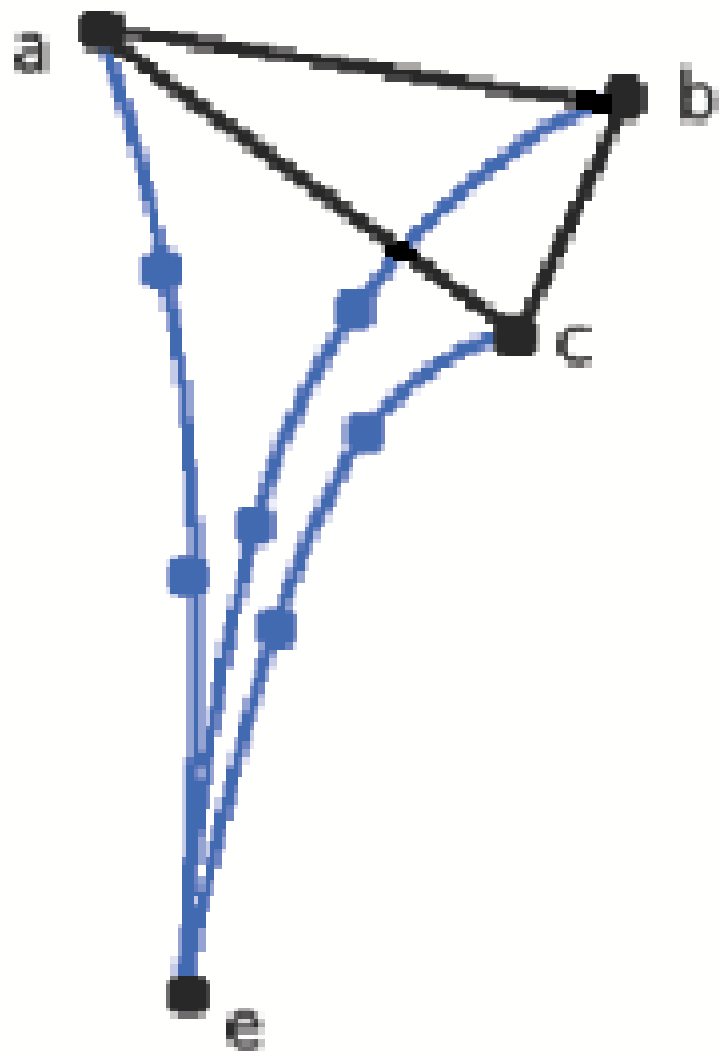


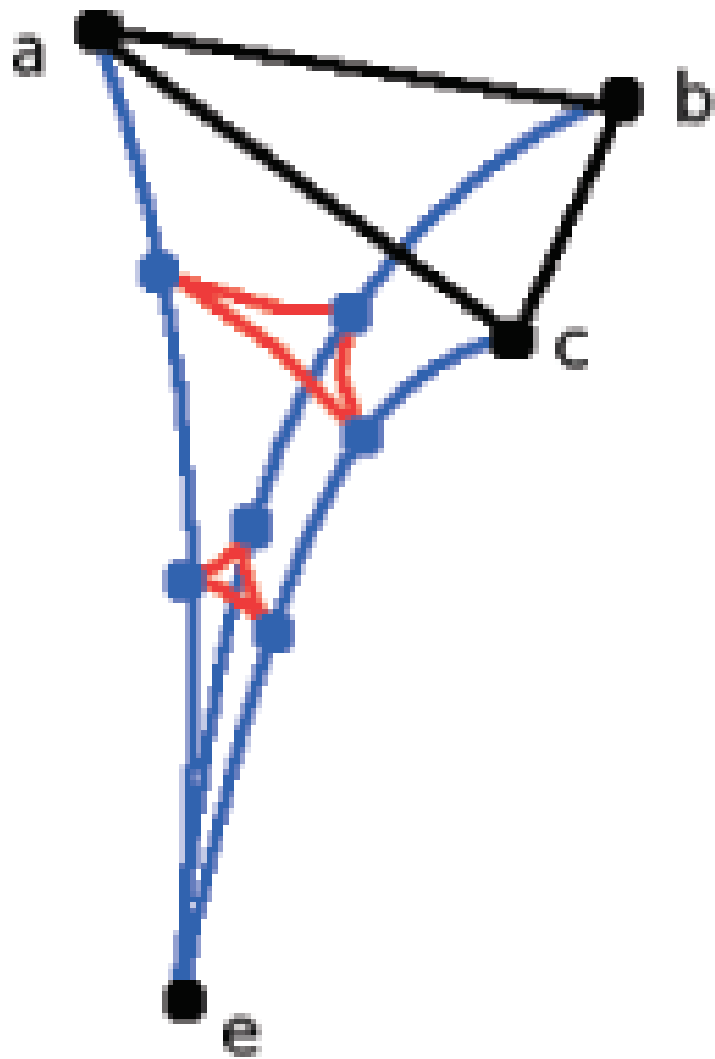












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If  $G$  is  $F_\infty$  with all Dehn functions polynomially bounded, we can use  $BG$  to compare  $HP^*(G; A)$  and  $H^*(G; A)$ .

**Theorem.** *Suppose  $G$  is a type  $F_\infty$  group. The following are equivalent.*

- (1)  *$G$  has all Dehn functions polynomially bounded.*
- (2)  *$G$  is isocohomological.*
- (3) *For all  $V$ ,  $HP^*(G; V) \rightarrow H^*(G; V)$  is surjective.*

# (1) implies (2)

- Denote by  $X$  is the universal cover of the  $F_\infty$   $BG$ .
- $C_*(X)$  is a projective resolution of  $\mathbb{C}$  over  $\mathbb{C}G$ .
- Length function on the vertices of  $X$ :  $\ell_X(x) = d_X(x, *)$
- Length function on  $X^{(n)}$ :  $\ell_X(\sigma) = \sum_{v \in \sigma} \ell_X(v)$
- $S_n(X)$  the completion of  $C_n(X)$  under the family of norms given by

$$\|\phi\|_k = \sum_{\sigma \in X^{(n)}} |\phi(\sigma)| (1 + \ell_X(\sigma))^k$$

# (1) implies (2)

- $S_*(X)$  a projective resolution of  $\mathbb{C}$  over  $\mathcal{S}G$ .
- For each  $n$  there is finite dimensional  $W_n$  with  $S_n(X) \cong \mathcal{S}G \otimes W_n$
- $C_n(X) \cong \mathbb{C}G \otimes W_n$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}G}(S_n(X), V) &\cong \mathrm{Hom}_{\mathcal{S}G}(\mathcal{S}G \otimes W_n, V) \\ &\cong \mathrm{Hom}(W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(W_n, V) \end{aligned}$$



# (2) implies (3)

Obvious

# (3) implies (1)

- $V$  be the collection of polynomially bounded  $(n - 1)$ -boundaries with complex coefficients
- $V$  is Frechet space with the family of filling norms
- $Y$  is the bar complex of  $G$
- $C_n(X), C_n(Y)$  the complex valued algebraic  $n$ -chains in  $X, Y$
- $\phi_* : C_*(Y) \rightarrow C_*(X)$  and  $\psi_* : C_*(X) \rightarrow C_*(Y)$

# (3) implies (1)

- $C^*(X, V) = \text{Hom}_{\mathbb{C}G}(C_*(X), V)$  and  
 $C^*(Y, V) = \text{Hom}_{\mathbb{C}G}(C_*(Y), V)$
- $\psi^* \circ \phi^*$  induces the identity map on cohomology  
 $H^*(G, V)$
- $u : C_n(X) \rightarrow V$  given by  $C_n(X) \xrightarrow{\partial} B_{n-1}(X) \hookrightarrow V$

# (3) implies (1)

- $u$  is an  $n$ -cocycle in  $C^n(X, V)$ .
- $u = (\psi^n \circ \phi^n)(u) + \delta v$ .
- $\phi^n(u)$  is a  $n$ -cocycle in  $C^n(Y, V)$
- There is a polynomially bounded  $n$ -cocycle  $u'$  such that  $\phi^n(u) = u' + \delta v'$ .
- For every  $k$  there exists a polynomial  $P_k$  such that for each  $\xi \in C_n(Y)$ ,  $\|u'(\xi)\|_{f,k} \leq P_k(\|\xi\|_k)$ .

# (3) implies (1)

For a simplex  $\sigma = [g_0, \dots, g_{n-1}]$  in  $Y$ , let  $[e, \sigma] = [e, g_0, \dots, g_{n-1}]$ , and extend by linearity.

- If  $b$  is a boundary,  $\partial[e, b] = b$  so  $[e, b]$  fills  $b$ , with  $\|[e, b]\|_k = \|b\|_k$  for all  $k$ .
- If  $\alpha$  is a cocycle in  $C^n(X, V)$ ,  $\langle \alpha | c \rangle = \langle \alpha | [e, \partial c] \rangle$ .

# (3) implies (1)

Let  $b$  be any  $(n - 1)$ -boundary in  $B_{n-1}(X)$ , and let  $a$  be any filling.

$$\begin{aligned} b &= \partial a \\ &= \langle u \mid a \rangle \\ &= \langle (\psi^n \circ \phi^n)(u) + \partial v \mid a \rangle \\ &= \langle (\psi^n \circ \phi^n)(u) \mid a \rangle + \langle v \mid b \rangle \\ &= \langle \phi^n(u) \mid \psi_n(a) \rangle + \langle v \mid b \rangle \\ &= \langle \phi^n(u) \mid [e, \partial\psi_n(a)] \rangle + \langle v \mid b \rangle \\ &= \langle \phi^n(u) \mid [e, \psi_{n-1}(b)] \rangle + \langle v \mid b \rangle \end{aligned}$$

# (3) implies (1)

( continuing )

$$\begin{aligned} b &= \langle u' + \delta v' \mid [e, \psi_{n-1}(b)] \rangle + \langle v \mid b \rangle \\ &= \langle u' \mid [e, \psi_{n-1}(b)] \rangle + \langle v' \mid \partial[e, \psi_{n-1}(b)] \rangle + \langle v \mid b \rangle \\ &= \langle u' \mid [e, \psi_{n-1}(b)] \rangle + \langle v' \mid \psi_{n-1}(b) \rangle + \langle v \mid b \rangle \\ &= \langle u' \mid [e, \psi_{n-1}(b)] \rangle + \langle \psi^{n-1}(v') + v \mid b \rangle \end{aligned}$$

# (3) implies (1)

For each  $k$ ,

$$\|b\|_{f,k} \leq P_k(\|\psi_{n-1}(b)\|_k) + \| \langle \psi^{n-1}(v') + v \mid b \rangle \|_{f,k}$$

Analysis of the length functions  $\ell$  and  $\ell_X$ , the  $F_\infty$  property and equivariance of  $\psi_*$  yields

$$\|\psi_{n-1}(b)\|_k \leq A_k \|\psi_{n-1}\|_{\infty,k} \|b\|_k.$$

Similarly equivariance and the  $F$  property show

$$\|(\psi^{n-1}(v') + v)(b)\|_{f,k} \leq B_k \|b\|_k.$$

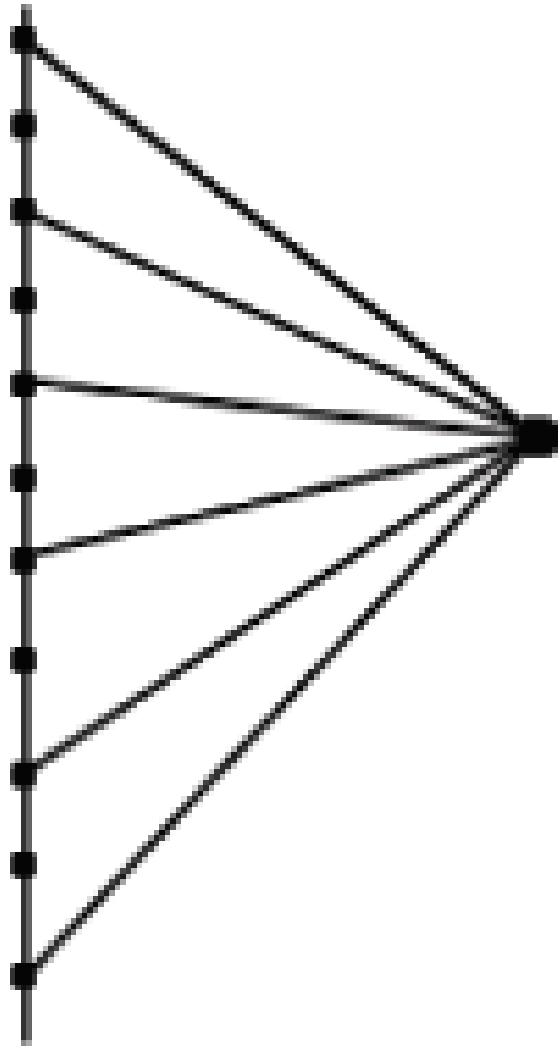
Together these verify the polynomial Dehn function in dimension  $n$ .

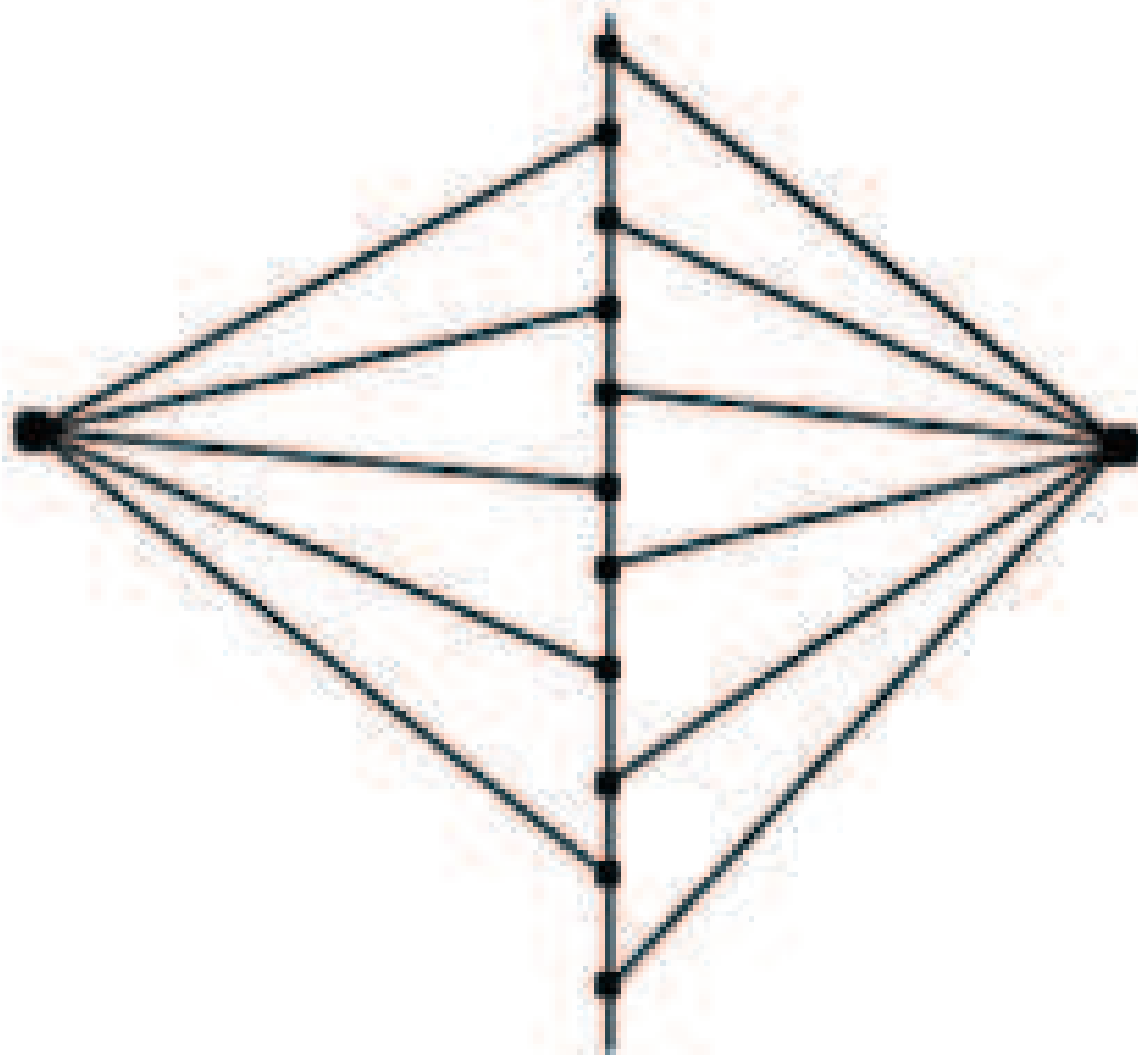


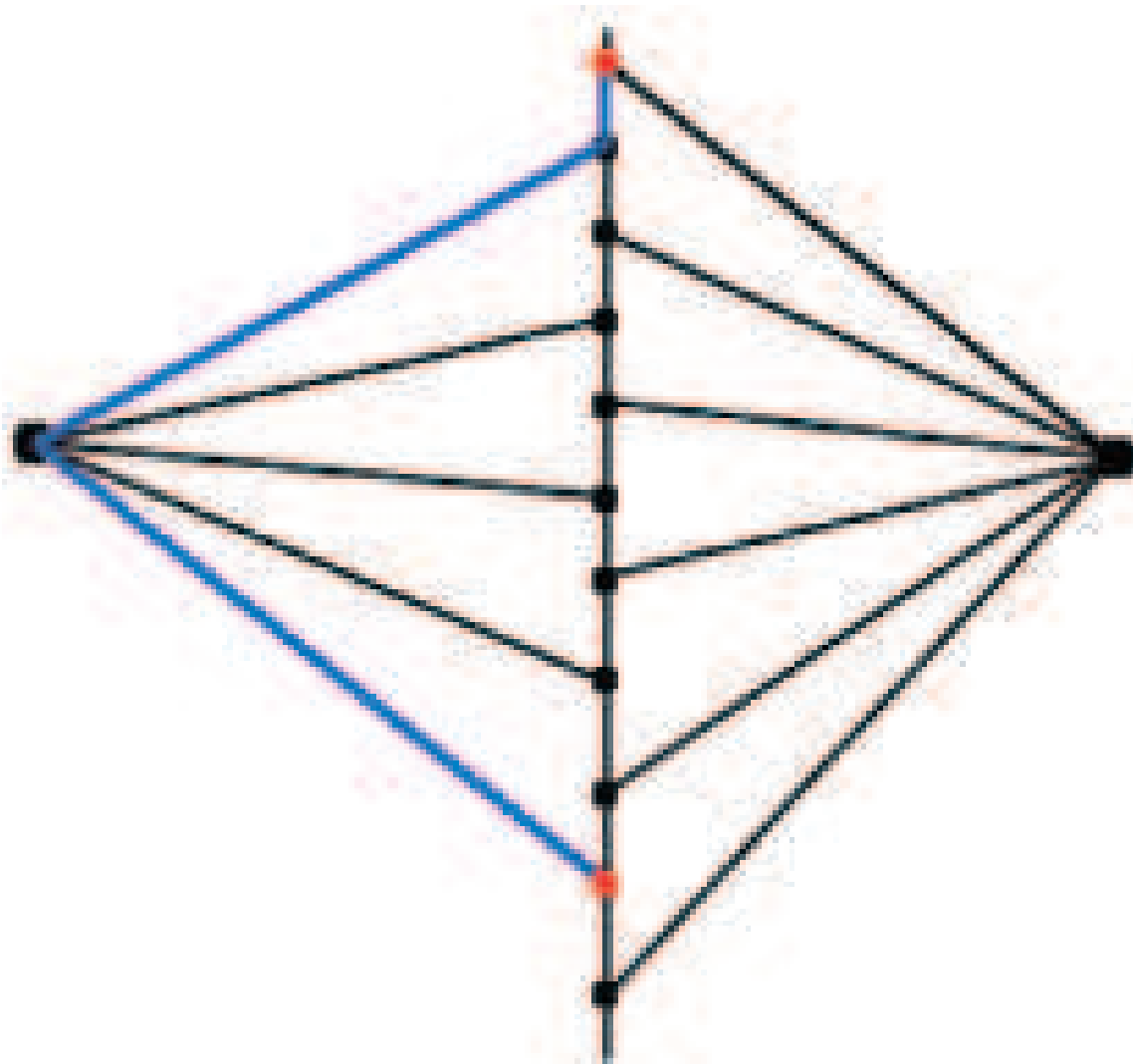
- Nilpotent groups have all Dehn functions polynomially bounded.
- $\mathbb{Z}^2 \rtimes \mathbb{Z}$  is not isocohomological.

Currently, a lot of work proving theorems of the form:  
Let  $G$  be hyperbolic relative to a subgroup  $H$ . Suppose that  $H$  has some property. Then  $G$  has the property, too.  
This is true for uniform embeddability into Hilbert space, having finite asymptotic dimension, and the rapid decay property. In the same manner, one would like to be able to extend the isocohomological property for  $H$ , up to the isocohomological property for  $G$ .











$G$  and  $H$  satisfy the bounded coset penetration property if for every  $\lambda$  there is a constant  $c(\lambda)$  such that if  $p$  and  $q$  are two  $(\lambda, 0)$ -quasi-geodesics without backtracking, then:

- If  $p$  and  $q$  both penetrate a coset  $gH$ , the points at which  $p$  and  $q$  enter ( respectively exit )  $gH$  are at a distance no more than  $c(\lambda)$  from one another.
- If  $p$  penetrates a coset  $gH$  not penetrated by  $q$ , then the points where  $p$  enters the coset and where  $p$  exits the coset are within  $c(\lambda)$  from one another.

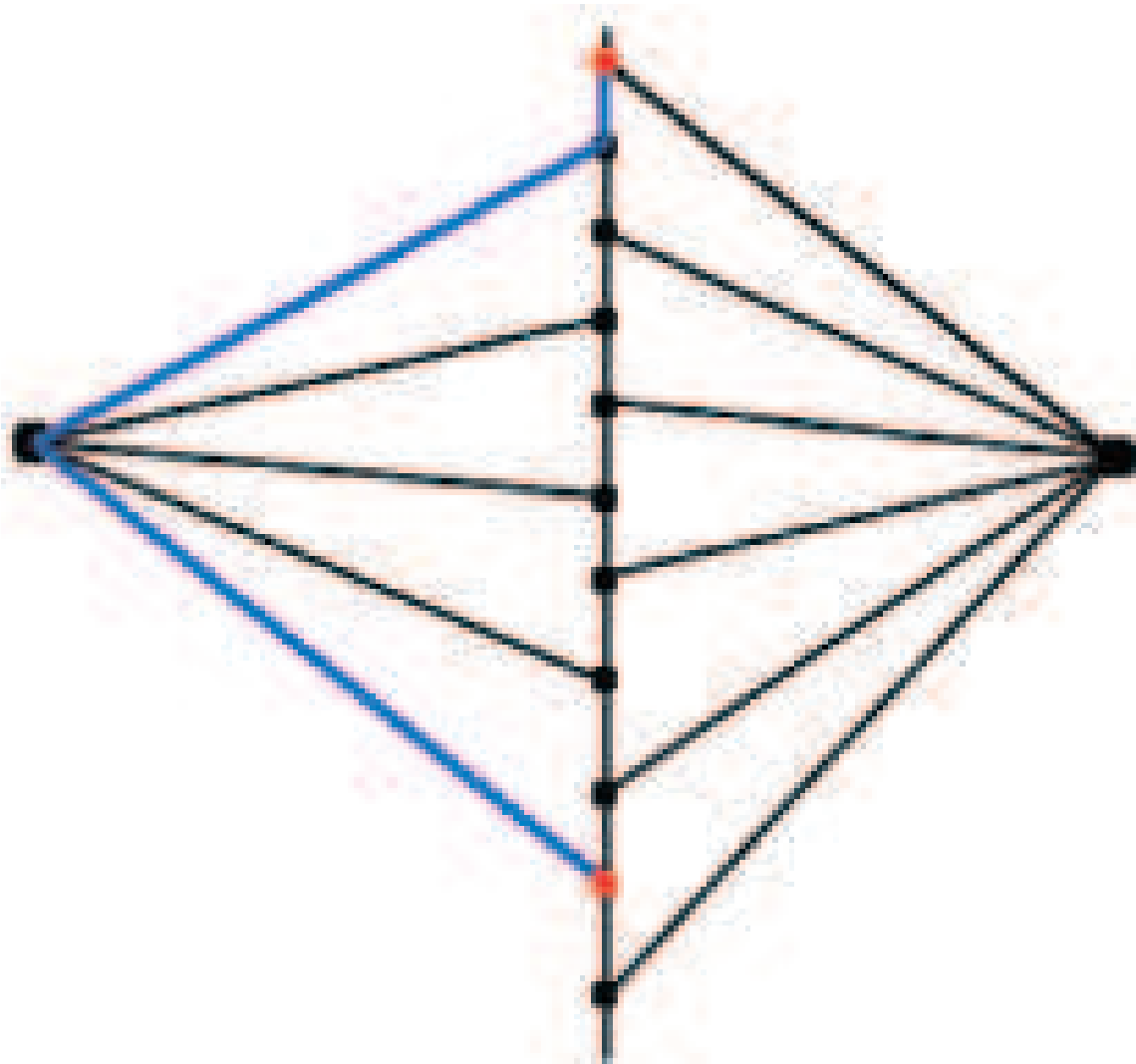


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$G$  is relatively hyperbolic with respect to  $H$  if the relative graph is hyperbolic and satisfies the bounded coset penetration property. Similar for finitely many subgroups  $\{H_1, H_2, \dots, H_n\}$ .

Suppose  $H$  is polynomially combable. The relative graph is hyperbolic, so picking any geodesic from  $e$  to  $g$  as  $p_g$  yields a combing. We pick a tree of geodesics. That is, if  $p_g$  and  $p_h$  intersect, then they agree up to that point.



**Theorem.** *Suppose that  $H$  is polynomially combable. Then  $G$  is polynomially combable.*

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A similar argument gives the following:

**Theorem.** *Suppose that  $H$  is  $F_\infty$ . Then  $G$  is, too.*

Farb shows that the first Dehn function for  $G$  is equivalent to that of  $H$ .

By examining our construction of  $BG$  from  $BH$ , the higher Dehn functions of  $G$  are bounded by a polynomial of those of  $H$ .

Our characterization of the isocohomological property, and the Drutu-Sapir result mentioned above together gives that the groups for which the Connes-Moscovici approach to the Novikov conjecture holds, is closed under relatively hyperbolic extensions.