Exact families of maps and embedding relative property
A groups

Bobby Ramsey (joint with Ronghui Ji and Crichton Ogle)

Sept. 29, 2013
All metric spaces in this talk are uniformly discrete with bounded geometry. All groups are countable and discrete.

**Definition**

$(X, d)$ has *bounded geometry* if for every $r > 0$ there is an $N = N(r) > 0$ such that for all $x \in X$, $|B_r(x)| < N$. 
Coarse Embeddability

Definition

A coarse embedding of \((X, d)\) into a Hilbert space \(\mathcal{H}\) is a map \(\phi : X \to \mathcal{H}\) for which there exist nondecreasing \(\rho_-, \rho_+ : [0, \infty) \to (0, \infty)\), with \(\lim_{t \to \infty} \rho_\pm(t) = \infty\), and such that for all \(x, y \in X\)

\[ \rho_-(d(x, y)) \leq \|\phi(x) - \phi(y)\| \leq \rho_+(d(x, y)). \]

\((X, d)\) is coarsely embeddable if such a coarse embedding exists.
Coarse Embeddability

Definition
A coarse embedding of \((X, d)\) into a Hilbert space \(\mathcal{H}\) is a map \(\phi : X \to \mathcal{H}\) for which there exist nondecreasing \(\rho_-, \rho_+ : [0, \infty) \to (0, \infty)\), with \(\lim_{t \to \infty} \rho_\pm(t) = \infty\), and such that for all \(x, y \in X\)

\[\rho_-(d(x, y)) \leq \|\phi(x) - \phi(y)\| \leq \rho_+(d(x, y)).\]

\((X, d)\) is coarsely embeddable if such a coarse embedding exists.

Theorem (G. Yu, 2000)
If \(X\) is coarsely embeddable, then the coarse Baum-Connes conjecture holds for \(X\).
Coarse Embeddability

Definition

A coarse embedding of \((X, d)\) into a Hilbert space \(\mathcal{H}\) is a map \(\phi : X \to \mathcal{H}\) for which there exist nondecreasing \(\rho_-, \rho_+ : [0, \infty) \to (0, \infty)\), with \(\lim_{t \to \infty} \rho_{\pm}(t) = \infty\), and such that for all \(x, y \in X\)

\[\rho_-(d(x, y)) \leq \|\phi(x) - \phi(y)\| \leq \rho_+(d(x, y)).\]

\((X, d)\) is coarsely embeddable if such a coarse embedding exists.

Theorem (G. Yu, 2000)

If \(X\) is coarsely embeddable, then the coarse Baum-Connes conjecture holds for \(X\).

\[\lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X)).\]
Coarse Embeddability

Coarse embeddability has many consequences

- Strong Novikov conjecture
- Gromov-Lawson-Rosenberg conjecture
- Zero-in-the-spectrum conjecture
Coarse Embeddability

Coarse embeddability has many consequences

- Strong Novikov conjecture
- Gromov-Lawson-Rosenberg conjecture
- Zero-in-the-spectrum conjecture

Question

How to determine if a space is coarsely embeddable?
Property A

Definition

\((X, d)\) has property A if for all \(R > 0\) and \(\epsilon > 0\), there exists nonempty finite subsets \(\{A_x \subset X \times \mathbb{N}\}_{x \in X}\) and \(S > 0\) such that

- If \((y, n) \in A_x\), then \(d(y, x) < S\).
- If \(d(x, y) < R\) then \(\frac{|A_x \Delta A_y|}{|A_x|} < \epsilon\).
Property A

Definition

$(X, d)$ has property A if for all $R > 0$ and $\epsilon > 0$, there exists nonempty finite subsets $\{A_x \subset X \times \mathbb{N}\}_{x \in X}$ and $S > 0$ such that

- If $(y, n) \in A_x$, then $d(y, x) < S$.
- If $d(x, y) < R$ then $\frac{|A_x \Delta A_y|}{|A_x|} < \epsilon$.

Theorem (G. Yu, 2000)

*If $X$ has property A, then $X$ is coarsely embeddable.*
Equivalent conditions

Lemma (Dadarlat-Guentner, 2003)

$X$ is coarsely embeddable if and only if for every $R > 0$ and $\epsilon > 0$ there is a map $\xi : X \to H$, $x \mapsto \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that

- $\sup \{\|\xi_x - \xi_y\| : d(x, y) < R\} < \epsilon$
- $\lim_{S \to \infty} \sup \{|\langle \xi_x, \xi_y \rangle| : d(x, y) > S\} = 0$

Lemma (Tu, 2001)

$X$ has property $A$ if and only if for every $R > 0$ and $\epsilon > 0$ there is a map $\xi : X \to H$, $x \mapsto \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that

- $d(x, y) < R$ implies $\|\xi_x - \xi_y\| < \epsilon$
- $d(x, y) > S$ implies $\langle \xi_x, \xi_y \rangle = 0$
Equivalent conditions

- $X$ has property A
- The uniform Roe algebra $C_u^*(X)$ is nuclear

If $X = G$ is a group, these are equivalent to the following.
- $G$ acts amenably on its Stone-Cech compactification, $\beta G$. (The reduced crossed product is $C_u^*(G)$)
- $C_r^*(G)$ is exact
- $\mathcal{L}(G)$ is weakly exact
Spaces with property A

- Amenable groups
- Metric spaces with finite asymptotic dimension (Higson, Roe)
- Gromov-hyperbolic spaces (Roe)
- Relatively hyperbolic groups (Ozawa)
- One-relator groups (Guentner)
- Mapping class groups (Kida)
- Linear groups (Guentner, Higson, Weinberger)
- Discrete subgroup of a connected Lie group (Anantharaman-Delaroche, Renault)
- Finite dimensional CAT(0) cube complexes (Brodzki, Campbell, Guentner, Niblo, Wright)
Spaces with property A

- Amenable groups
- Metric spaces with finite asymptotic dimension (Higson, Roe)
- Gromov-hyperbolic spaces (Roe)
- Relatively hyperbolic groups (Ozawa)
- One-relator groups (Guentner)
- Mapping class groups (Kida)
- Linear groups (Guentner, Higson, Weinberger)
- Discrete subgroup of a connected Lie group (Anantharaman-Delarcohe, Renault)
- Finite dimensional CAT(0) cube complexes (Brodzki, Campbell, Guentner, Niblo, Wright)

Coarsely embeddable but without property A

Recently Arzhantseva, Guentner, and Spakula have constructed examples of coarsely embeddable bounded geometry metric spaces which do not have property A.
Spaces with property A

- Amenable groups
- Metric spaces with finite asymptotic dimension (Higson, Roe)
- Gromov-hyperbolic spaces (Roe)
- Relatively hyperbolic groups (Ozawa)
- One-relator groups (Guentner)
- Mapping class groups (Kida)
- Linear groups (Guentner, Higson, Weinberger)
- Discrete subgroup of a connected Lie group (Anantharaman-Delaroche, Renault)
- Finite dimensional CAT(0) cube complexes (Brodzki, Campbell, Guentner, Niblo, Wright)

Coarsely embeddable but without property A

The only groups known to not have property A (Gromov’s Monster groups) do not coarsely embed.
Theorem (DG, 2003)

Let $0 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 0$ be an extension of countable discrete groups. If $N$ is coarsely embeddable and $Q$ has property A, then $G$ is coarsely embeddable.
Dadarlat-Guentner

Theorem (DG, 2003)

Let $0 \to N \to G \to Q \to 0$ be an extension of countable discrete groups. If $N$ is coarsely embeddable and $Q$ has property A, then $G$ is coarsely embeddable.

How to generalize this?

Weaken $Q$ to be only coarsely embeddable? The equivariant version of this is not true. ($\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is not a-T-menable, even though $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ are.) Remove the normality condition!
Relative property A

Definition

A finitely generated group $G$ has relative property A with respect to the family of subgroups $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ if for all $R > 0$ and $\epsilon > 0$, there exists nonempty finite subsets $\{A_g \subset G/\mathcal{H} \times \mathbb{N}\}_{g \in G}$ and $S > 0$ such that

- If $(g'H_j, n) \in A_g$, then $d(g, g'H_j) < S$.
- If $d(g, g') < R$ then $\frac{|A_g \Delta A_{g'}|}{|A_g|} < \epsilon$.

\[
(G/\mathcal{H} = \bigsqcup_{i=1}^{k} G/H_i)
\]
Equivalent characterizations

Theorem (JOR, 2012)

\((G, \mathcal{H})\) has relative property A if and only if exists a sequence of weak*-continuous \(\xi_n : \beta G \rightarrow \text{Prob}(G/\mathcal{H})\) such that for all \(g \in G\),

\[
\lim_{n \to \infty} \sup_{x \in \beta G} \| g\xi_n(x) - \xi_n(gx) \|_{\ell^1} = 0.
\]
Equivalent characterizations

Theorem (JOR, 2012)

\((G, \mathcal{H})\) has relative property A if and only if exists a sequence of weak*-continuous \(\xi_n : \beta G \rightarrow \text{Prob}(G/\mathcal{H})\) such that for all \(g \in G\),

\[
\lim_{n \to \infty} \sup_{x \in \beta G} \| g \xi_n(x) - \xi_n(gx) \|_{\ell^1} = 0.
\]

Corollary

Suppose \(N \triangleleft G\). If \(G/N\) has property A, then \(G\) has relative property A with respect to \(N\).
Equivalent characterizations

Theorem (JOR, 2012)

\((G, \mathcal{H})\) has relative property A if and only if exists a sequence of \(\text{weak}^*-\text{continuous}\) \(\xi_n : \beta G \to \text{Prob}(G/\mathcal{H})\) such that for all \(g \in G\),

\[
\lim_{n \to \infty} \sup_{x \in \beta G} \| g\xi_n(x) - \xi_n(gx) \|_{\ell^1} = 0.
\]

Corollary

Suppose \(N \triangleleft G\). If \(G/N\) has property A, then \(G\) has relative property A with respect to \(N\).

Corollary

Suppose \(G\) acts cocompactly on \(X\). If \(X\) has property A, then \(G\) has relative property A with respect to the stabilizer of any point.
Equivalent characterizations

Brodzki, Niblo, Nowak, and Wright have recently characterized property A through bounded cohomology analogous to Johnson’s characterization of amenability.

For each $G$ they identify a collection of Banach $G$-modules, $\mathcal{N}(G)$, with the following property.

Theorem (BNNW, 2012)

$G$ has property A if and only if $H^p_b(G; V^*) = 0$ for all $p \geq 1$ and every $V \in \mathcal{N}(G)$. 

Equivalent characterizations

Theorem (JOR, 2011)

For every Banach $G$-module $M$, there are relative bounded cohomology groups $H^*_b(G, \mathfrak{H}; M)$ fitting into the long-exact sequence:

$$\cdots \rightarrow H^k_b(G; M) \rightarrow H^k_b(\mathfrak{H}; M) \rightarrow H^{k+1}_b(G, \mathfrak{H}; M) \rightarrow H^{k+1}_b(G; M) \rightarrow \cdots$$

Here, $H^0_b(\mathfrak{H}; V^*) = \prod_{H \in \mathfrak{H}} H^0_b(H; V^*)$. 
Theorem (JOR, 2011)

For every Banach $G$-module $M$, there are relative bounded cohomology groups $H^*_b(G, \mathfrak{H}; M)$ fitting into the long-exact sequence:

$$
\cdots \to H^k_b(G; M) \to H^k_b(\mathfrak{H}; M) \to H^{k+1}_b(G, \mathfrak{H}; M) \to H^{k+1}_b(G; M) \to \cdots
$$

Here, $H^0_b(\mathfrak{H}; V^*) = \prod_{H \in \mathfrak{H}} H^0_b(H; V^*)$.

Theorem (JOR, 2012)

$(G, \mathfrak{H})$ has relative property A if and only if $H^0_b(\mathfrak{H}; V^*) \to H^1_b(G, \mathfrak{H}; V^*)$ is surjective for every $V \in \mathcal{N}(G)$. 
Equivalent characterizations

Corollary

If \((G, \mathcal{H})\) has relative property A and if each \(H \in \mathcal{H}\) has property A, then \(G\) has property A.
Corollary

If \((G, \mathcal{H})\) has relative property A and if each \(H \in \mathcal{H}\) has property A, then \(G\) has property A.

Corollary

If \(0 \to N \to G \to Q \to 0\) with \(Q\) and \(N\) with property A, then \(G\) has property A.
Corollary

If $G$ acts cocompactly on a metric space $X$ with property A, and there is an $x_0 \in X$ whose stabilizer has property A, the $G$ has property A.

- Groups acting on finite dimensional CAT(0)-cube complexes with property A stabilizer.
- Fundamental groups of finite graphs of groups with property A vertex groups.
For $H < G$, we recast relative property A of $(G, H)$ as a property of the map $\pi : G \rightarrow G/H$. 
For $H < G$, we recast relative property A of $(G, H)$ as a property of the map $\pi : G \to G/H$.

$(G, H)$ has relative property A if and only if for every $R > 0$ and $\epsilon > 0$ there exists an $S > 0$ and a map $\xi : X \to \ell^2(G/H)$, with $\|\xi_x\| = 1$ for all $x \in X$, satisfying the following.

1. For all $g, g' \in G$ if $d(g, g') < R$, then $\|\xi_g - \xi_{g'}\| < \epsilon$.
2. For all $g \in G$, supp $\xi_g \subset \pi(B_S(g))$, where $B_S(g)$ denotes the ball of radius $S$ in $G$ centered at $g$. 
Suppose \((X, d)\) is a metric space. For a finite family of sets, \(\{Y_i\}_{i=1}^n\), let \(Y = \sqcup_{i=1}^n Y_i\) denote the disjoint union.

Definition

A family of set maps \(\{\phi_i: X \to Y_i\}_{i=1}^n\) is an exact family of maps if for every \(R, \epsilon > 0\) there exists an \(S > 0\) and a map \(\xi: X \to \ell^2(Y)\), with \(\|\xi_x\| = 1\) for all \(x \in X\), and satisfying the following.

1. For all \(x, y \in X\) if \(d(x, y) \leq R\), then \(\|\xi_x - \xi_y\| \leq \epsilon\).
2. For all \(x \in X\), \(\text{supp} \, \xi_x \subset \bigcup_{i} \phi_i(B_{S}(x))\), where \(B_S(x)\) denotes the ball of radius \(S\) in \(X\) centered at \(x\).

Lemma

\((G, H)\) has relative property A if and only if \(\{\pi_i: G \to G/H_i\mid H_i \in H\}\) is an exact family of maps.
Exact families of maps

Suppose \((X, d)\) is a metric space. For a finite family of sets, \(\{Y_i\}_{i=1}^n\), let \(Y = \bigcup_{i=1}^n Y_i\) denote the disjoint union.

**Definition**

A family of set maps \(\{\phi_i : X \to Y_i\}_{i=1}^n\) is an **exact family of maps** if for every \(R, \epsilon > 0\) there exists an \(S > 0\) and a map \(\xi : X \to \ell^2(Y)\), with \(\|\xi_x\| = 1\) for all \(x \in X\), and satisfying the following.

1. For all \(x, y \in X\) if \(d(x, y) \leq R\), then \(\|\xi_x - \xi_y\| \leq \epsilon\).
2. For all \(x \in X\), supp \(\xi_x \subset \bigcup_i \phi_i(B_S(x))\), where \(B_S(x)\) denotes the ball of radius \(S\) in \(X\) centered at \(x\).
Exact families of maps

Suppose \((X, d)\) is a metric space. For a finite family of sets, \(\{Y_i\}_{i=1}^n\), let \(Y = \bigsqcup_{i=1}^n Y_i\) denote the disjoint union.

**Definition**

A family of set maps \(\{\phi_i : X \to Y_i\}_{i=1}^n\) is an **exact family of maps** if for every \(R, \epsilon > 0\) there exists an \(S > 0\) and a map \(\xi : X \to \ell^2(Y)\), with \(\|\xi_x\| = 1\) for all \(x \in X\), and satisfying the following.

1. For all \(x, y \in X\) if \(d(x, y) \leq R\), then \(\|\xi_x - \xi_y\| \leq \epsilon\).
2. For all \(x \in X\), \(\text{supp} \xi_x \subseteq \bigcup_i \phi_i(B_S(x))\), where \(B_S(x)\) denotes the ball of radius \(S\) in \(X\) centered at \(x\).

**Lemma**

\((G, \mathcal{F})\) has relative property A if and only if \(\{\pi_i : G \to G/H_i | H_i \in \mathcal{F}\}\) is an exact family of maps.
Revisiting property A and coarse embeddability

Definition (Dadarlat-Güenter, 2007)

A family \( \{(X_j, d_j)\}_{j \in J} \) of metric spaces is \textit{equi-coarsely embeddable} if for every \( R > 0 \) and \( \epsilon > 0 \) there is a family of Hilbert space valued maps \( \xi_j : X_j \to \mathcal{H} \) with \( \|\xi_j(x)\| = 1 \) for all \( x \in X_j \) and satisfying:

1. For all \( j \in J \) and all \( x, y \in X_j \), if \( d_j(x, y) < R \), then
   \[ \|\xi_j(x) - \xi_j(y)\| < \epsilon. \]

2. \( \lim_{S \to \infty} \sup_{j \in J} \sup \{|\langle \xi_j(x), \xi_j(y) \rangle| : d(x, y) < S, \ x, y \in X_j\} = 0. \)
Exact families of maps

Revisiting property A and coarse embeddability

Definition

A family \( \{(X_j, d_j)\}_{j \in J} \) of metric spaces have \textit{uniform property A} if for every \( R > 0 \) and \( \epsilon > 0 \) there is a family of Hilbert space valued maps \( \xi_j : X_j \to \mathcal{H} \), with \( \|\xi_j(x)\| = 1 \) for all \( x \in X_j \), and an \( S > 0 \) and satisfying:

1. For all \( j \in J \) and all \( x, y \in X_j \), if \( d_j(x, y) < R \), then \( \|\xi_j(x) - \xi_j(y)\| \leq \epsilon \).

2. For all \( j \in J \) and all \( x, y \in X_j \), \( \langle \xi_j(x), \xi_j(y) \rangle = 0 \) if \( d(x, y) > S \).
Theorem

Suppose $(X, d)$ is a metric space and $\{\phi_i : X \to Y_i\}_{i=1}^n$ is an exact family of maps. If $\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\}$ has uniform property $A$, then $X$ has property $A$. 
Theorem

Suppose $(X, d)$ is a metric space and $\{\phi_i : X \rightarrow Y_i\}_{i=1}^{n}$ is an exact family of maps. If $\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\}$ has uniform property A, then $X$ has property A.

Outline of proof for one map $\phi : X \rightarrow Y$

For $x \in X$ and $w \in Y$, let $\eta(x, w)$ be a point in $\phi^{-1}(w)$ closest to $x$. For all $x, y \in X$ and $w \in Y$,

\[
d(x, y) \leq d(\eta(x, w), \eta(y, w)) + d(x, \phi^{-1}(w)) + d(y, \phi^{-1}(w))
\]

\[
d(\eta(x, w), \eta(y, w)) \leq d(x, y) + d(x, \phi^{-1}(w)) + d(y, \phi^{-1}(w)).
\]
Fix $R, \epsilon > 0$.

There is a map $\alpha : X \to \ell^2(Y)$, with each $\|\alpha_x\| = 1$, and an $S_X > 0$ such that

- For each $x, y \in X$, if $d(x, y) \leq R$, then $|1 - \langle \alpha_x, \alpha_y \rangle| < \frac{\epsilon}{2}$.
- For each $x \in X$, if $\alpha_x(w) \neq 0$ then $w \in \phi(B_{S_X}(x))$.

There is an $S_Y > 0$, a Hilbert space $H$, and for each $w \in Y$, a $\beta_w : \phi^{-1}(w) \to H$ with $\|\beta_w(s)\| = 1$ for all $s \in \phi^{-1}(w)$, and such that for all $s, t \in \phi^{-1}(w)$

- If $d(s, t) < 2S_X + R$, then $|1 - \langle \beta_w(s), \beta_w(t) \rangle| \leq \frac{\epsilon}{2}$.
- If $d(s, t) > S_Y$, then $\langle \beta_w(s), \beta_w(t) \rangle = 0$. 
Define $\xi : X \to \ell^2(Y, \mathcal{H})$ by

$$\xi_x(w) = \alpha_x(w)\beta_w(\eta(x, w)), \quad \text{for all } x \in X, w \in Y.$$ 

If $x, y \in X$ with $d(x, y) \leq R$, then

$$|1 - \langle \xi_x, \xi_y \rangle| \leq \left| \sum_{w \in Y} (1 - \langle \beta_{iw}(\eta(x, w)), \beta_{iw}(\eta(y, w)) \rangle) \alpha_x(w)\alpha_y(w) \right| + |1 - \langle \alpha_x, \alpha_y \rangle|.$$ 

The first sum is over $w \in \phi_i(B_{Sx}(x)) \cap \phi_i(B_{Sx}(y))$ and is bounded by

$$\sup \{ |1 - \langle \beta_{iw}(\eta(x, w)), \beta_{iw}(\eta(y, w)) \rangle| : w \in \phi_i(B_{Sx}(x)) \cap \phi_i(B_{Sx}(y)) \}.$$ 

Each such $w$ satisfies $d(\eta(x, w), \eta(y, w)) \leq R + 2S_X$, this sum is bounded by $\frac{\epsilon}{2}$.

The second term is bounded by $\frac{\epsilon}{2}$. 

---

B. Ramsey (OSU)  
Sept. 29, 2013 22 / 26
For \( d(x, y) > 2S_X + S_Y \),

\[
\langle \xi_x, \xi_y \rangle = \sum_{w \in Y} \alpha_x(w)\alpha_y(w)\langle \beta_{i_w}(\eta(x, w)), \beta_{i_w}(\eta(y, w)) \rangle.
\]

The sum is over \( w \in \bigcup_i (\phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y))) \).

For \( w \in \phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y)) \),

\[
d(\eta(x, w), \eta(y, w)) \geq d(x,y) - d(x, \phi_i^{-1}(w)) - d(y, \phi_i^{-1}(w)) > S_Y.
\]

Thus \( \langle \xi_x, \xi_y \rangle = 0. \)
Coarse embeddability

Theorem

Let \( \{ \phi_i : X \to Y_i \}_{i=1}^n \) be an exact family of maps. If \( \{ \phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n \} \) is an equi-coarsely embeddable family of metric spaces, then \( X \) is coarsely embeddable.
Coarse embeddability

Theorem

Let \( \{\phi_i : X \to Y_i\}_{i=1}^n \) be an exact family of maps. If \( \{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\} \) is an equi-coarsely embeddable family of metric spaces, then \( X \) is coarsely embeddable.

Corollary

Let \((G, \mathcal{H})\) have relative property A. If each \( H \in \mathcal{H} \) is coarsely embeddable, then \( G \) is coarsely embeddable.
Relative coarse embeddability?

In the last theorem, an exact family of maps is more than is actually needed.
In the last theorem, an exact family of maps is more than is actually needed.

**Definition**

A family of set maps \( \{ \phi_i : X \to Y_i \}_{i=1}^n \) is a weakly exact family of maps if for every \( R, \epsilon > 0 \) there exists a map \( \xi : X \to \ell^2(Y) \), with \( \|\xi_x\| = 1 \) for all \( x \in X \), satisfying the following:

1. For all \( x, y \in X \) if \( d(x, y) < R \), then \( \|\xi_x - \xi_y\| < \epsilon \).
2. \[
\lim_{S \to \infty} \sup \left\{ \sum_{w \notin \bigcup_i (\phi_i(B_S(x)) \cap \phi_i(B_S(y)))} |\xi_x(w)\xi_y(w)| \right\} = 0.
\]
Relative coarse embeddability?

Theorem

Let \( \{ \phi_i : X \to Y_i \}_{i=1}^n \) be a weakly exact family of maps. \( X \) is coarsely embeddable if and only if \( \{ \phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n \} \) is an equi-coarsely embeddable family of metric spaces.