Exact families of maps and embedding relative property A groups

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All metric spaces in this talk are uniformly discrete with bounded geometry.
All groups are countable and discrete.

Definition

(X, d) has *bounded geometry* if for every $r > 0$ there is an $N = N(r) > 0$ such that for all $x \in X$, $|B_r(x)| < N$.

Coarse Embeddability

Definition

A *coarse embedding* of (X, d) into a Hilbert space $\mathcal{H}$ is a map $\phi : X \to \mathcal{H}$ for which there exist nondecreasing $\rho_-, \rho_+ : [0, \infty) \to (0, \infty)$, with $\lim_{t\to\infty} \rho_\pm(t) = \infty$, and such that for all $x, y \in X$

$$\rho_-(d(x, y)) \leq \|\phi(x) - \phi(y)\| \leq \rho_+(d(x, y)).$$

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If X is coarsely embeddable, then the coarse Baum-Connes conjecture holds for X.

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$$\lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*(X))$$

Coarse Embeddability

Coarse embeddability has many consequences

- Strong Novikov conjecture
- Gromov-Lawson-Rosenberg conjecture
- Zero-in-the-spectrum conjecture

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Question

How to determine if a space is coarsely embeddable?

Property A

Definition

(X, d) has *property A* if for all $R > 0$ and $\epsilon > 0$, there exists nonempty finite subsets $\{A_x \subset X \times \mathbb{N}\}_{x \in X}$ and $S > 0$ such that

- If $(y, n) \in A_x$, then $d(y, x) < S$.
- If $d(x, y) < R$ then $\frac{|A_x \Delta A_y|}{|A_x|} < \epsilon$.

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Theorem (G. Yu, 2000)

If X has property A, then X is coarsely embeddable.

Equivalent conditions

Lemma (Dadarlat-Guentner, 2003)

X is coarsely embeddable if and only if for every $R > 0$ and $\epsilon > 0$ there is a map $\xi : X \to \mathcal{H}$, $x \mapsto \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that

- $\sup\{\|\xi_x - \xi_y\| : d(x,y) < R\} < \epsilon$
- $\lim_{S\to\infty} \sup\{|\langle \xi_x, \xi_y\rangle| : d(x,y) > S\} = 0$

Lemma (Tu, 2001)

X has property A if and only if for every $R > 0$ and $\epsilon > 0$ there is a map $\xi : X \to \mathcal{H}$, $x \mapsto \xi_x$ such that $\|\xi_x\| = 1$ for all $x \in X$ and such that

- *$d(x,y) < R$ implies $\|\xi_x - \xi_y\| < \epsilon$*
- *$d(x,y) > S$ implies $\langle \xi_x, \xi_y\rangle = 0$*

Equivalent conditions

- X has property A
- The uniform Roe algebra $C_u^*(X)$ is nuclear

If $X = G$ is a group, these are equivalent to the following.

- G acts amenably on its Stone-Cech compactification, βG. (The reduced crossed product is $C_u^*(G)$)
- $C_r^*(G)$ is exact
- $\mathcal{L}(G)$ is weakly exact

Spaces with property A

- Amenable groups
- Metric spaces with finite asymptotic dimension (Higson, Roe)
- Gromov-hyperbolic spaces (Roe)
- Relatively hyperbolic groups (Ozawa)
- One-relator groups (Guentner)
- Mapping class groups (Kida)
- Linear groups (Guentner, Higson, Weinberger)
- Discrete subgroup of a connected Lie group (Anantharaman-Delarcohe, Renault)
- Finite dimensional CAT(0) cube complexes (Brodzki, Campbell, Guentner, Niblo, Wright)

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Coarsely embeddable but without property A

Recently Arzhantseva, Guentner, and Spakula have constructed examples of coarsely embeddable bounded geometry metric spaces which do not have property A.

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Coarsely embeddable but without property A

The only groups known to not have property A (Gromov's Monster groups) do not coarsely embed.

Dadarlat-Guentner

Theorem (DG, 2003)

Let $0 \to N \to G \to Q \to 0$ *be an extension of countable discrete groups. If* N *is coarsely embeddable and* Q *has property* A*, then* G *is coarsely embeddable.*

Dadarlat-Guentner

Theorem (DG, 2003)

Let $0 \to N \to G \to Q \to 0$ be an extension of countable discrete groups. If N is coarsely embeddable and Q has property A, then G is coarsely embeddable.

How to generalize this?

Weaken Q to be only coarsely embeddable? The equivariant version of this is not true. ($\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is not a-T-menable, even though $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ are.)
Remove the normality condition!

Relative property A

Definition

A finitely generated group G has *relative property A* with respect to the family of subgroups $\mathfrak{H} = \{H_1, H_2, \ldots, H_k\}$ if for all $R > 0$ and $\epsilon > 0$, there exists nonempty finite subsets $\{A_g \subset G/\mathfrak{H} \times \mathbb{N}\}_{g \in G}$ and $S > 0$ such that

- If $(g'H_j, n) \in A_g$, then $d(g, g'H_j) < S$.
- If $d(g, g') < R$ then $\frac{|A_g \Delta A_{g'}|}{|A_g|} < \epsilon$.

$$(G/\mathfrak{H} = \sqcup_{i=1}^{k} G/H_i)$$

Equivalent characterizations

Theorem (JOR, 2012)

$(G, \mathfrak{H})$ *has relative property A if and only if exists a sequence of weak***-continuous* $\xi_n : \beta G \to \mathrm{Prob}(G/\mathfrak{H})$ *such that for all* $g \in G$*,*

$$\lim_{n\to\infty} \sup_{x\in\beta G} \|g\xi_n(x) - \xi_n(gx)\|_{\ell^1} = 0.$$

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Corollary

Suppose $N \triangleleft G$. If G/N has property A, then G has relative property A with respect to N.

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Corollary

Suppose $N \triangleleft G$. If G/N has property A, then G has relative property A with respect to N.

Corollary

Suppose G acts cocompactly on X. If X has property A, then G has relative property A with respect to the stabilizer of any point.

Equivalent characterizations

Brodzki, Niblo, Nowak, and Wright have recently characterized property A through bounded cohomology analogous to Johnson's characterization of amenability.

For each G they identify a collection of Banach G-modules, $\mathcal{N}(G)$, with the following property.

Theorem (BNNW, 2012)

G has property A if and only if $H_b^p(G; V^*) = 0$ for all $p \geq 1$ and every $V \in \mathcal{N}(G)$.

Equivalent characterizations

Theorem (JOR, 2011)

For every Banach G-module M, there are relative bounded cohomology groups $H_b^*(G,\mathfrak{H};M)$ fitting into the long-exact sequence:

$$\cdots \to H_b^k(G;M) \to H_b^k(\mathfrak{H};M) \to H_b^{k+1}(G,\mathfrak{H};M) \to H_b^{k+1}(G;M) \to \ldots$$

Here, $H_b^0(\mathfrak{H};V^*) = \prod_{H\in\mathfrak{H}} H_b^0(H;V^*)$.

Equivalent characterizations

Theorem (JOR, 2011)

For every Banach G-module M, there are relative bounded cohomology groups $H_b^*(G, \mathfrak{H}; M)$ fitting into the long-exact sequence:

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Here, $H_b^0(\mathfrak{H}; V^*) = \prod_{H \in \mathfrak{H}} H_b^0(H; V^*)$.

Theorem (JOR, 2012)

$(G, \mathfrak{H})$ has relative property A if and only if $H_b^0(\mathfrak{H}; V^*) \to H_b^1(G, \mathfrak{H}; V^*)$ is surjective for every $V \in \mathcal{N}(G)$.

Equivalent characterizations

Corollary

If $(G, \mathfrak{H})$ *has relative property A and if each* $H \in \mathfrak{H}$ *has property A, then* G *has property A.*

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If $(G, \mathfrak{H})$ has relative property A and if each $H \in \mathfrak{H}$ has property A, then G has property A.

Corollary

If $0 \to N \to G \to Q \to 0$ with Q and N with property A, then G has property A.

Corollary

If G acts cocompactly on a metric space X with property A, and there is an $x_0 \in X$ whose stabilizer has property A, the G has property A.

- Groups acting on finite dimensional CAT(0)-cube complexes with property A stabilizer.
- Fundamental groups of finite graphs of groups with property A vertex groups.

On to metric spaces

For $H < G$, we recast relative property A of (G, H) as a property of the map $\pi : G \to G/H$.

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(G, H) has relative property A if and only if for every $R > 0$ and $\epsilon > 0$ there exists an $S > 0$ and a map $\xi : X \to \ell^2(G/H)$, with $\|\xi_x\| = 1$ for all $x \in X$, satisfying the following.

1. For all $g, g' \in G$ if $d(g, g') < R$, then $\|\xi_g - \xi_{g'}\| < \epsilon$.
2. For all $g \in G$, $\operatorname{supp} \xi_g \subset \pi\,(B_S(g))$, where $B_S(g)$ denotes the ball of radius S in G centered at g.

Exact families of maps

Suppose (X, d) is a metric space. For a finite family of sets, $\{Y_i\}_{i=1}^n$, let $\mathcal{Y} = \sqcup_{i=1}^n Y_i$ denote the disjoint union.

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Definition

A family of set maps $\{\phi_i : X \to Y_i\}_{i=1}^n$ is an *exact family of maps* if for every $R, \epsilon > 0$ there exists an $S > 0$ and a map $\xi : X \to \ell^2(\mathcal{Y})$, with $\|\xi_x\| = 1$ for all $x \in X$, and satisfying the following.

1. For all $x, y \in X$ if $d(x, y) \leq R$, then $\|\xi_x - \xi_y\| \leq \epsilon$.
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Lemma

$(G, \mathfrak{H})$ has relative property A if and only if $\{\pi_i : G \to G/H_i | H_i \in \mathfrak{H}\}$ is an exact family of maps.

Revisiting property A and coarse embeddability

Definition (Dadarlat-Guenter, 2007)

A family $\{(X_j, d_j)\}_{j\in\mathcal{J}}$ of metric spaces is *equi-coarsely embeddable* if for every $R > 0$ and $\epsilon > 0$ there is a family of Hilbert space valued maps $\xi_j : X_j \to \mathcal{H}$ with $\|\xi_j(x)\| = 1$ for all $x \in X_j$ and satisfying:

1. For all $j \in \mathcal{J}$ and all $x, y \in X_j$, if $d_j(x, y) < R$, then $\|\xi_j(x) - \xi_j(y)\| < \epsilon$.
2. $\lim_{S\to\infty} \sup_{j\in\mathcal{J}} \sup \{|\langle\xi_j(x), \xi_j(y)\rangle| : d(x, y) < S\ ,\ x, y \in X_j\} = 0.$

Revisiting property A and coarse embeddability

Definition

A family $\{(X_j, d_j)\}_{j\in\mathcal{J}}$ of metric spaces have *uniform property A* if for every $R > 0$ and $\epsilon > 0$ there is a family of Hilbert space valued maps $\xi_j : X_j \to \mathcal{H}$, with $\|\xi_j(x)\| = 1$ for all $x \in X_j$, and an $S > 0$ and satisfying:

1. For all $j \in \mathcal{J}$ and all $x, y \in X_j$, if $d_j(x, y) < R$, then $\|\xi_j(x) - \xi_j(y)\| < \epsilon$.
2. For all $j \in \mathcal{J}$ and all $x, y \in X_j$, $\langle \xi_j(x), \xi_j(y)\rangle = 0$ if $d(x, y) > S$.

Theorem

Suppose (X, d) *is a metric space and* $\{\phi_i : X \to Y_i\}_{i=1}^{n}$ *is an exact family of maps. If* $\left\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\right\}$ *has uniform property A, then* X *has property A.*

Theorem

Suppose (X, d) is a metric space and $\{\phi_i : X \to Y_i\}_{i=1}^n$ is an exact family of maps. If $\left\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\right\}$ has uniform property A, then X has property A.

Outline of proof for one map $\phi : X \to Y$

For $x \in X$ and $w \in Y$, let $\eta(x, w)$ be a point in $\phi^{-1}(w)$ closest to x. For all $x, y \in X$ and $w \in Y$,

$$d(x, y) \leq d(\eta(x, w), \eta(y, w)) + d(x, \phi^{-1}(w)) + d(y, \phi^{-1}(w))$$
$$d(\eta(x, w), \eta(y, w)) \leq d(x, y) + d(x, \phi^{-1}(w)) + d(y, \phi^{-1}(w)).$$

Fix $R, \epsilon > 0$.

There is a map $\alpha : X \to \ell^2(Y)$, with each $\|\alpha_x\| = 1$, and an $S_X > 0$ such that

- For each $x, y \in X$, if $d(x, y) \leq R$, then $|1 - \langle \alpha_x, \alpha_y \rangle| < \frac{\epsilon}{2}$.
- For each $x \in X$, if $\alpha_x(w) \neq 0$ then $w \in \phi(B_{S_X}(x))$.

There is an $S_Y > 0$, a Hilbert space $\mathcal{H}$, and for each $w \in Y$, a $\beta_w : \phi^{-1}(w) \to \mathcal{H}$ with $\|\beta_w(s)\| = 1$ for all $s \in \phi^{-1}(w)$, and such that for all $s, t \in \phi^{-1}(w)$

- If $d(s, t) < 2S_X + R$, then $|1 - \langle \beta_w(s), \beta_w(t) \rangle| \leq \frac{\epsilon}{2}$.
- If $d(s, t) > S_Y$, then $\langle \beta_w(s), \beta_w(t) \rangle = 0$.

Define $\xi : X \to \ell^2(Y, \mathcal{H})$ by

$$\xi_x(w) = \alpha_x(w)\beta_w(\eta(x, w)), \quad \text{for all } x \in X, w \in Y.$$

If $x, y \in X$ with $d(x, y) \leq R$, then

$$|1 - \langle \xi_x, \xi_y \rangle| \leq \left| \sum_{w \in \mathcal{Y}} \left(1 - \langle \beta_{i_w}(\eta(x, w)), \beta_{i_w}(\eta(y, w)) \rangle\right) \alpha_x(w)\alpha_y(w) \right| + |1 - \langle \alpha_x, \alpha_y \rangle|.$$

The first sum is over $w \in \phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y))$ and is bounded by

$$\sup \left\{ |1 - \langle \beta_{i_w}(\eta(x, w)), \beta_{i_w}(\eta(y, w)) \rangle| : w \in \phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y)) \right\}.$$

Each such w satisfies $d(\eta(x, w), \eta(y, w)) \leq R + 2S_X$, this sum is bounded by $\frac{\epsilon}{2}$.
The second term is bounded by $\frac{\epsilon}{2}$.

For $d(x, y) > 2S_X + S_Y$,

$$\langle \xi_x, \xi_y \rangle = \sum_{w \in \mathcal{Y}} \alpha_x(w) \alpha_y(w) \langle \beta_{i_w}(\eta(x, w)), \beta_{i_w}(\eta(y, w)) \rangle.$$

The sum is over $w \in \cup_i (\phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y)))$.
For $w \in \phi_i(B_{S_X}(x)) \cap \phi_i(B_{S_X}(y))$,

$$d(\eta(x, w), \eta(y, w)) \geq d(x, y) - d(x, \phi_i^{-1}(w)) - d(y, \phi_i^{-1}(w)) > S_Y$$

Thus $\langle \xi_x, \xi_y \rangle = 0$. $\square$

Coarse embeddability

Theorem

Let $\{\phi_i : X \to Y_i\}_{i=1}^{n}$ be an exact family of maps. If $\left\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\right\}$ is an equi-coarsely embeddable family of metric spaces, then X is coarsely embeddable.

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Corollary

Let $(G, \mathfrak{H})$ have relative property A. If each $H \in \mathfrak{H}$ is coarsely embeddable, then G is coarsely embeddable.

Relative coarse embeddability?

In the last theorem, an exact family of maps is more than is actually needed.

Relative coarse embeddability?

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Definition

A family of set maps $\{\phi_i : X \to Y_i\}_{i=1}^n$ is a *weakly exact family of maps* if for every $R, \epsilon > 0$ there exists a map $\xi : X \to \ell^2(\mathcal{Y})$, with $\|\xi_x\| = 1$ for all $x \in X$, satisfying the following:

1. For all $x, y \in X$ if $d(x,y) < R$, then $\|\xi_x - \xi_y\| < \epsilon$.
2. $$\lim_{S\to\infty} \sup \left\{ \sum_{w \notin \cup_i(\phi_i(B_S(x)) \cap \phi_i(B_S(y)))} |\xi_x(w)\xi_y(w)| \right\} = 0.$$

Relative coarse embeddability?

Theorem

Let $\{\phi_i : X \to Y_i\}_{i=1}^n$ be a weakly exact family of maps. X is coarsely embeddable if and only if $\left\{\phi_i(w)^{-1} : w \in Y_i, i = 1, \ldots, n\right\}$ is an equi-coarsely embeddable family of metric spaces.