INTRODUCTION TO SIGMA NOTATION

1. The notation itself

Sigma notation is a way of writing a sum of many terms, in a concise form. A sum in sigma notation looks something like this:

$$\sum_{k=1}^{5} 3k$$

The Σ (sigma) indicates that a sum is being taken. The variable k is called the index of the sum. The numbers at the top and bottom of the Σ are called the upper and lower limits of the summation. In this case, the upper limit is 5, and the lower limit is 1. The notation means that we will take every integer value of k between 1 and 5 (so 1, 2, 3, 4, and 5) and plug them each into the summand formula (here that formula is $3k$). Then those are all added together.

$$\sum_{k=1}^{5} 3k = 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5 = 45$$

Example 1. Write out what is meant by the following:

$$\sum_{k=0}^{3} \frac{1}{k+1}$$

Here, the index k takes the values 0, 1, 2, and 3. We'll plug those each into $\frac{1}{k+1}$ and add them together.

$$\sum_{k=0}^{3} \frac{1}{k+1} = \frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1}$$

Example 2. Write out what is meant by the following:

$$\sum_{i=1}^{8} (-1)^i$$

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The index variable here is written as i instead of k. That's ok. The most common variables to use for indexes include i, j, k, m, and n.

$$\begin{aligned}\sum_{i=1}^{8}(-1)^i &= (-1)^1+(-1)^2+(-1)^3+(-1)^4+(-1)^5+(-1)^6+(-1)^7+(-1)^8\\ &= -1+1+-1+1+-1+1+-1+1\\ &= 0\end{aligned}$$

Try one on your own.

Example 3. Write out what is meant by the following (no need to simplify):

$$\sum_{n=-1}^{4}\sqrt{n+1}$$

Let's try going the other way around.

Example 4. Write the following sum in sigma notation.

$$2+4+6+8+\ldots+22+24$$

Notice that we can factor a 2 out of each term to rewrite this sum as

$$2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4+\ldots+2\cdot 11+2\cdot 12$$

That means that we are adding together 2 times every number between 1 and 12. The sigma notation could be

$$\sum_{k=1}^{12}2k$$

There is no need to use k as our index variable. We could have just as easily used m or j instead.

$$\sum_{k=1}^{12}2k=\sum_{m=1}^{12}2m=\sum_{j=1}^{12}2j$$

Notice, that these are NOT the same as

$$\sum_{k=1}^{12}2m$$

Example 5. Write the following sum in sigma notation.

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots-\frac{1}{64}+\frac{1}{128}$$

This one is a little more complicated. We'll worry about the signs later, first we'll deal with the numbers themselves. Do you notice a pattern in the terms? Sure, we get to the next term by dividing by 2. That is:

$$
\begin{aligned}
1 &= \frac{1}{2^0} = \left(\frac{1}{2}\right)^0 \\
\frac{1}{2} &= \frac{1}{2^1} = \left(\frac{1}{2}\right)^1 \\
\frac{1}{4} &= \frac{1}{2^2} = \left(\frac{1}{2}\right)^2 \\
&\dots \\
\frac{1}{128} &= \frac{1}{2^7} = \left(\frac{1}{2}\right)^7
\end{aligned}
$$

If we call our index variable k, then k should go from 0 to 7, and the numbers themselves are just $\left(\frac{1}{2}\right)^k$. Now we need to deal with the signs. We say above that $(-1)^k$ will alternate between $+1$ and -1. That is, if we multiply our terms from above by $(-1)^k$, they will alternate between $+$ and $-$. We are starting with $k = 0$, so $(-1)^0 = +1$ will give us the alternation starting at the sign we want.

$$\sum_{k=0}^{7} (-1)^k \left(\frac{1}{2}\right)^k = \sum_{k=0}^{7} \left(\frac{-1}{2}\right)^k$$

Try one on your own.

Example 6. Write the following sum in sigma notation.

$$\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{4}} - \frac{4}{\sqrt{5}} + \ldots + \frac{51}{\sqrt{52}} - \frac{52}{\sqrt{53}}$$

2. Calculating with sigma notation

We want to use sigma notation to simplify our calculations. To do that, we will need to know some basic sums. First, let's talk about the sum of a constant. (Notice here, that our upper limit of summation is n. n is not the index variable, here, but the highest value that the index variable will take.)

$$\sum_{k=1}^{n} C$$

This is a sum of n terms, each of them having a value C. That is, we are adding n copies of C. This sum is just nC. The other basic sums that we need are much more complicated to derive. Rather than explaining where they come from, we'll just give you a list of the final formulas, that you should remember.

Formula 1. $\displaystyle\sum_{k=1}^{n} C = nC.$

Formula 2. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$

Formula 3. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$

Formula 4. $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$

Now that we have this list, let's use them to compute.

Example 7. Find the value of the sum $\displaystyle\sum_{k=1}^{5} 9.$

This is just the sum of a constant, with $C = 9$ and $n = 5$. The value is $nC = 5 \cdot 9 = 45$.

Example 8. Find the value of the sum $\displaystyle\sum_{k=1}^{100} k.$

This is the sum $1+2+3+\ldots+100$. According to Formula 2 above (with $n = 100$), this is $\frac{100(101)}{2} = 5050$.

Because sigma notation is just a new way of writing addition, the usual properties of addition still apply, but a couple of the important ones look a little different.

Property 1 (Commutativity)**.** $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$

Property 2 (Distribution)**.** $\displaystyle\sum_{k=1}^{n} c \cdot a_k = c\sum_{k=1}^{n} a_k.$

Example 9. Find the value of the sum $\displaystyle\sum_{k=1}^{10} \left(2k^2 + 5\right).$

First, we'll use the properties above to split this into two sums, then factor the 2 out of the first sum.

$$\begin{aligned}\sum_{k=1}^{10}\left(2k^2+5\right) &= \sum_{k=1}^{10}2k^2+\sum_{k=1}^{10}5\\ &= 2\sum_{k=1}^{10}k^2+\sum_{k=1}^{10}5\end{aligned}$$

The two sums we have left, can be found using formulas 1 and 3 above! We see that $\sum_{k=1}^{10}k^2=\frac{10(11)(21)}{6}=385$. Similarly, $\sum_{k=1}^{10}5=10\cdot 5=50$.

Putting all that together, $\sum_{k=1}^{10}\left(2k^2+5\right)=2\cdot 385+50=820$.

Example 10. Find the value of the sum $\sum_{k=1}^{200}\left(2k^3-6k^2+3\right)$.

Let's use the same approach as in the previous example. First, we'll use the properties to split this into individual sums, then factor out the coefficients. After that, we'll use the formulas above to evaluate it.

$$\begin{aligned}\sum_{k=1}^{200}\left(2k^3-6k^2+3\right) &= \sum_{k=1}^{200}2k^3-\sum_{k=1}^{200}6k^2+\sum_{k=1}^{200}3\\ &= 2\sum_{k=1}^{200}k^3-6\sum_{k=1}^{200}k^2+\sum_{k=1}^{200}3\\ &= 2\left(\frac{200(201)}{2}\right)^2-6\left(\frac{200(201)(401)}{6}\right)+200\cdot 3\\ &= 2\cdot 404010000-6\cdot 2686700+200\cdot 3\\ &= 791900400\end{aligned}$$

The numbers in this example were horribly ugly, but we were able to evaluate the sum without having to actually calculate all 200 terms, then add them all up. In 5 small lines, we were able to add 200 numbers.

Try one on your own.

Example 11. Find the value of the sum $\sum_{k=1}^{50}\left(4k^2-18k+2(-1)^k\right)$