THE ALGEBRA STRUCTURE OF $K(l)_*SO(2^{l+1}-1)$

Vidhyānāth K. Rao

The Ohio State University

Abstract. We determine the algebra structure of $K(l)_*SO(2^{l+1}-1)$.

0. Introduction

In two earlier papers, [R2] and [R3], we calculated the E^∞-term of the bar spectral sequence converging to the Morava K-theories of the special orthogonal groups. This gives the additive structure, and coalgbra structure modulo the algebra structure. We also obtained a system of algebra generators. One algebra relation was obtained in [R4] and was used to prove that $Spin(n)$ is not homotopy nilpotent if $n \geq 7$. However, a complete set of relations is yet to be determined.

In this paper, we make a start of that objective, by determining the algebra structure of $P(l)_*SO(2^{l+1}-1)$. The methods used have some applicability to $K(l)_*SO(n)$. In particular, Proposition 4.1 below will used in a later paper to prove that $SO(5)$ and $SO(6)$ are not homotopy nilpotent.

The results of this paper can also be used to give another derivation of the Hopf algebra structure of $H^*(Spin(n), \mathbb{Z}/2)$, by proving the main result of [MZ]. We will not use any of the results of [MZ], although a proof or two can be shortened thereby.

Throughout this paper BP will refer to the 2-local theory. For background information on BP and related topics, see [Wi].

1. Preliminaries on Morava K-theories

In this section, we recall some basic facts on the Morava K-theories, and related spectra.

It is well known that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ where the degree of v_i is $2(2^i-1)$ (see, for example, [Qu]). Let $l > 0$. Using the Sullivan-Baas technique ([Su], [Bs]), we can kill $\{v_i \mid i < l\}$ to get $P(l)$, a BP-module theory with coefficient ring $P(l)_* = \mathbb{Z}/2[v_l, v_{l+1}, \dots]$. (This and the next few statements are due to Jack Morava. See [JW] for a source in print.)

1991 *Mathematics Subject Classification.* Primary: ; Secondary:.

Key words and phrases. Orthogonal groups, Morava K-theory.

There are Bockstein maps $Q_i : P(l) \to P(l)$ of degree $2^{i+1} - 1$ for $0 \leq i < l$. These cover the Milnor Bocksteins on $H\mathbb{Z}/2$ under the canonical map $P(l) \to H\mathbb{Z}/2$.

Lemma 1.1. *Suppose that X is n-connected and that $l \leq m$. Then $\widetilde{P(l)}_i X \to \widetilde{P(m)}_i X$ is an isomorphism if $0 \leq i \leq n + 2^{l+1} - 2$.*

If $0 < l < \infty$, there are two products on $P(l)$ which make it into a BP-algebra theory [Wu]. The Bocksteins Q_i are derivations with respect to either of these two products (see, for example, [SY, Theorem 3.14 (iii)]). Neither of these is commutative. We make $P(l)$ into an associative ring spectrum by arbitrarily selecting one of these two products.

Proposition 1.2. *Let X and Y be spectra and $\tau : X \wedge Y \to Y \wedge X$ be the switch map. Then $P(l)_* \tau(x \wedge y) = y \wedge x + v_l(Q_{l-1}y) \wedge (Q_{l-1}x)$.*

This is an immediate consequence of the commutator formula of Würgler [Wu, 2.5].

Note that if X is an H-space with $P(l)_*X$ free as a $P(l)_*$-module, then $P(l)_*X$ is both an algebra and a coalgebra. But it need not be a Hopf algebra, as $P(l)$ is not commutative.

Proposition 1.3. *Suppose that X is an H-space and that $P(l)_*X$ is free as a$P(l)_*$-module. Let x and y be elements of $P(l)_*X$ and Δ be the diagonal of the latter. Then*

$$\Delta(xy) = \Delta(x)\Delta(y) + v_l((\mathrm{id} \otimes Q_{l-1})\Delta(x))((Q_{l-1} \otimes \mathrm{id})\Delta(y)) .$$

See [R4, Corollary 1.6].

Throughout this paper, H_*X will denote the mod-2 homology of X. We will consider Q_i as a differential on H_*X, with degree $2^{i+1} - 1$.

2. Preliminaries on $P(l)_*SO(2^{l+1} - 1)$

In this section, we collect some results from the earlier papers [R1], [R2] and [R3]. We will also set up some notation.

The results of the next paragraph follows from [R1].

Let $G_n = SO(n+2)/(SO(2) \times SO(n))$ be the generating variety for the homology of $\Omega_0 SO(n+2)$; *i.e.* there is a map $G_n \to \Omega_0 SO(n+2)$ such that $H_*(G_n, \mathbb{Z})$ maps monomorphicaly into $H_*(\Omega_0 SO(n+2), \mathbb{Z})$ and the image of the former generates the latter as an algebra [Bt]. G_n has no torsion in homology. Note that $G_\infty = \mathbb{C}P^\infty$. Let x be the image of the standard generator of $MU^*\mathbb{C}P^\infty$ in MU^*G_n. $MU\mathbb{Q}^*G_{2n-1} = MU^*[x]/(x^{2n} = 0)$. Let $\{\beta_0', \beta_1, \ldots \beta_{2n-1}\}$ be the basis of MU_*G_{2n-1} that is dual to $\{1, x, \ldots, x^{2n-1}\}$. Let $\beta_0 \in \widetilde{MU}_0 \Omega SO(2n+1)$ be the unique element such that $\beta_0^2 = 2\beta_0$. Define $\alpha_i = \sum_{j=0}^{i} c_{i-j}\beta_j$ and $\alpha_i' = \sum_{j=1}^{i} c_{i-j}\beta_j$ where c_j is the coefficient of t^{j+1} in the MU-[2]-series. If $1 \leq i \leq 2n-1$ and $1 \leq j \leq n-1$, then $\alpha_i \in \widetilde{MU}_*\Omega SO(2n+1)$ and $\alpha_i', \beta_j \in \widetilde{MU}_*\Omega Spin(2n+1) \subset \widetilde{MU}_*\Omega SO(2n+1)$.

If h is an MU-algebra theory, we will denote the images of the β's and α's in $h_*\Omega SO(2n+1)$ by the same symbols. These elements are independent of n in the sense that if $0 \leq i <$

$n < q$ and $j < 2n$, then

$$h_*\Omega SO(2n+1) \xrightarrow{\text{incl}_*} h_*\Omega SO(2q+1)$$

preserves β_i, α'_j and α_j. The image of u under the homology suspension $\widetilde{h}_*\Omega SO(2n+1) \to h_{*+1}SO(2n+1)$ will be denoted by $\overline{u}$.

For the rest of this paper we will fix an $l > 0$. For $0 \le i < 2^{l-1}$, define $k(i)$ by $2^l \le 2^{k(i)}(2i+1) < 2^{l+1}$. Let $k'(i) = 2^{k(i)}$.

Fix a ground ring of characteristic 2. Let $\Gamma_k(u)$ denote the divided power algebra of height k on u. This is the dual of the primitively generated truncated polynomial algebra $P(x)/(x^{2^k})$. The j^{th} divided power of u will be denoted by $\gamma_j(u)$. These are characterized by the relations $\gamma_1(u) = u$, $\gamma_s(u)\gamma_t(u) = (s,t)\gamma_{s+t}(u)$ where (s,t) is the binomial coefficient $(s+t)!/s!t!$; and $\Delta(\gamma_s(u)) = \sum_{j=0}^{s} \gamma_j(u) \otimes \gamma_{s-j}(u)$.

We need some facts concerning the Bar spectral sequence; See [R2, Section 3] for proofs. If G is a compact connected Lie group and h a BP-algebra theory, then the bar spectral sequence

$$E^2_{**}(G,h) = \operatorname{Tor}^{h_*\Omega G}_{**}(h_*, h_*) \Rightarrow h_*G$$

is a spectral sequence of commutative algebras. If $E^r_{**}(G,h)$ is free over h_* for all r, then it a spectral sequence of bicommutative, biassociative Hopf algebras.

The next result is proved in [R2]:

Proposition 2.1. (1) *Let $m \ge l$. Then*

$$E^\infty_{**}\left(SO(2^{l+1}-1), P(m)\right) = \bigotimes_{i=0}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)$$

$$E^\infty_{**}\left(Spin(2^{l+1}-1), P(m)\right) = E(\overline{\alpha}'_{2^l-1}) \otimes \bigotimes_{i=1}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)$$

(2) *If $i \ge 2^l - 1$, then $\overline{\alpha}_{2i+1} = 0$ in $P(l)_*SO(2^{l+1}-1)$.*
(3) *If $m > l$, then $P(m)_*SO(2^{l+1}-1)$ is a bicommutative, biassociative Hopf algebra isomorphic to $\bigotimes_{i=0}^{2^l-2} \Gamma_{k(i)+1}(\overline{\beta}_i)$.*

Corollary 2.2. *$P(l)_*Spin(2^{l+1}-1) \to P(l)_*SO(2^{l+1}-1)$ is injective and sends $\overline{\alpha}'_{2^l-1}$ to $v_l\overline{\beta}_0$.*

Proof. Note that $\alpha'_{2^l-1} = \alpha_{2^l-1} - c_{2^l-1}\beta_0$ in $BP_*\Omega SO(2^{l+1}-1)$, where c_{2^l-1} is the coefficient of t^{2^l} in the [2]-series. But c_{2^l-1} is 0 if $m > l$ and v_l if $m = l$. The rest of the proof is easy.

The elements $\gamma_j(\overline{\beta}_i)$ are independent of l and m in sense that if $l' \ge l$, $m' \ge m$ and $m' > l'$ then the map $P(m)_*SO(2^{l+1}-1) \to P(m')_*SO(2^{l'+1}-1)$ preserves $\gamma_j(\overline{\beta}_i)$.

Lemma 2.3. *$\gamma_j(\overline{\beta}_i) \in \operatorname{Im}\left(H_*Spin(2^{l+1}-1) \to H_*SO(2^{l+1}-1)\right)$ if $i \geq 1$ and $j < 2^{k(i)+1}$.*

Proof. Dualizing Proposition 2.1, we see that

$$H^*SO(2^{l+1}-1) = E_\infty^{**}(SO(2^{l+1}-1), H\mathbb{Z}/2) = \bigotimes_{i=0}^{2^{l-1}-1} P(\widetilde{w}_{2i+1})/(\widetilde{w}_{2i+1}^{2k'(i)})$$

where $\widetilde{w}_{2i+1}$ is a primitive element of degree $2i+1$ and

$$\langle \gamma_s(\overline{\beta}_i), \widetilde{w}_{2j+1}^{2^t} \rangle = \begin{cases} 1 & s = 2^t \text{ and } i = j \\ 0 & \text{otherwise} \end{cases}$$

($w_{2j+1}^{2^t}$ is, of course, the cohomology suspension of the $(2^t(2j+1)+1)$th Stiefel-Whitney class.)

Now $\widetilde{w}_1$ maps to 0 in $H^*Spin(2^{l+1}-1)$ for degree reasons. On the other hand, if $i > 0$ and $j < 2k'(i)$, then $\widetilde{w}_{2i+1}^j$ maps non-trivially to $H^*Spin(2^{l+1}-1)$: By Proposition 2.1,

$$E_\infty^{**}(Spin(2^{l+1}-1), H\mathbb{Z}/2) = E(\overline{\alpha}_{2^l-1}) \otimes \bigotimes_{i=1}^{2^{l-1}-1} P(\widetilde{w}_{2i+1})/(\widetilde{w}_{2i+1}^{2k'(i)})\,.$$

Thus the kernel of the map $H^*SO(2^{l+1}-1) \to H^*Spin(2^{l+1}-1)$ is the ideal generated by $\widetilde{w}_1$.

Let $i \geq 1$ and $w \in H^*SO(2^{l+1}-1)$. Then

$$\langle \gamma_s(\overline{\beta}_i), \widetilde{w}_1 w \rangle = \langle \Delta(\gamma_s(\overline{\beta}_i)), \widetilde{w}_1 \otimes w \rangle = \sum_{t=0}^{s} \langle \gamma_t(\overline{\beta}_i), \widetilde{w}_1 \rangle \langle \gamma_{s-t}(\overline{\beta}_i), w \rangle = 0\,.$$

Let $0 \leq i$ and $j \leq k'(i)$. By Lemma 1.1 and Proposition 2.1, there exist unique elements γ'_{ij} in $P(l)_*SO(2^{l+1}-1)$ that map to $\gamma_j(\overline{\beta}_i)$ in $H_*SO(2^{l+1}-1)$. From Lemma 2.3, Corollary 2.2 and Lemma 1.1 we see that if $i \geq 1$, then there is a unique element in $P(l)_*Spin(2^{l+1}-1)$ that maps to $\gamma_j(\overline{\beta}_i)$. This element will also denoted by γ'_{ij}. Clearly this notation is consistent with the map $P(l)_*Spin(2^{l+1}-1) \to P(l)_*SO(2^{l+1}-1)$. For typographical convenience, we will write $\widetilde{\gamma}_i$ for $\gamma_{i,k'(i)}$

Proposition 2.4. *The module of primitives of $P(l)_*SO(2^{l+1}-1)$ and $P(l)_*Spin(2^{l+1}-1)$ are generated by*

$$\{\overline{\beta}_i \mid 0 \leq i \leq 2^l-2\} \text{ and } \{\overline{\beta}_i \mid 1 \leq i \leq 2^l-2\} \cup \{\overline{\alpha}_{2^l-1}\}$$

respectively.

Proof. The case of $SO(2^{l+1}-1)$ was proved in [R4]. The case of *Spin* follows by Corollary 2.2.

3. The algebra structure of $P(l)_*SO(2^{l+1}-1)$ I

Let A be an algebra over a commutative ring R. A is said to be simply generated by $g_1, g_2, \ldots$ over R if A is a free R-module with basis

$$\{1\} \cup \{g_{i_1} g_{i_2} \ldots g_{i_s} \mid i_1 < i_2 < \cdots < i_s\}.$$

As we do not assume that A is commutative, the order of the elements is important.

We have the following corollary of Proposition 2.1:

Proposition 3.1. *As a $P(l)_*$ algebra, $P(l)_*SO(2^{l+1}-1)$ is simply generated by*

$$\left\{\overline{\beta}_{2^l-2-j} \mid 0 \le j \le 2^l-2\right\} \cup \left\{\gamma'_{i,2^s} \mid 0 \le i < 2^{l-2}, 1 \le s < k(i)\right\} \cup \left\{\widetilde{\gamma}_i \mid 0 \le i < 2^{l-1}\right\}.$$

It turns out that the elements of the second set above commute with each other. So how they are ordered is not material. Otherwise, the generators above will be ordered by the sequence in which they are given.

Thus to determine the algebra structure of $P(l)_*SO(2^{l+1}-1)$, we must determine the squares and the commutators of the generators listed above. The results can be summarized by saying that these squares and commutators are as non-trivial as possible (that is to say, consistent with Proposition 2.1).

We will repeatedly use the following observation: If x, y are elements of $P(l)_*SO(2^{l+1}-1)$, then $[x,y] = xy - xy$ is in $v_l\widetilde{P(l)}_*SO(2^{l+1}-1)$. It is so because both $P(l)_*\{1\}$ and $P(l+1)_*SO(2^{l+1}-1)$ are commutative. Also note that $xy + yx = [x,y]$ as we are working in characteristic 2.

Lemma 3.2. *If $i, j \le 2^l-2$, then $[\overline{\beta}_i, \overline{\beta}_j] = 0$ and $\overline{\beta}_i^2 = 0$ in $P(l)_*SO(2^{l+1}-1)$. If $i, j \ge 1$ then those equations hold in $P(l)_*Spin(2^{l+1}-1)$.*

Proof. As the Bar ss is commutative, $[\overline{\beta}_i, \overline{\beta}_j]$ and $\overline{\beta}_i^2$ must have Bar filtration < 2. But $E^\infty_{1,2s+1} = 0$ and $E^\infty_{0*} = P(l)_*\{pt\}$.

Lemma 3.3. *Suppose that $0 \le j \le 2^{l-1}-1$, $1 \le s \le k'(j)$ and $i < 2^l - 1 - s(2j+1)/2$. Then $[\overline{\beta}_i, \gamma'_{js}] = 0$.*

Proof. $[\overline{\beta}_i, \gamma'_{js}]$ is divisible by v_l, has degree $\le 2^{l+1}-2$, and its augmentation is trivial. Hence it must be zero.

Lemma 3.4. *Let $i, j \le 2^{l-1}-1$. Then the following equations hold in $P(l)_*SO(2^{l+1}-1)$:*

$$\Delta(\gamma'_{is}) = \sum_{t=0}^{s} \gamma'_{it} \otimes \gamma'_{i,s-t} \text{ if } s \le k'(i);$$
$$\gamma'_{is}\gamma'_{it} = (s,t)\gamma'_{i,s+t} \text{ if } s,\ t < k'(i);$$
$$[\gamma'_{is}, \gamma'_{jt}] = 0 \text{ if } s < k'(i) \text{ and } t < k'(j).$$

Proof. The first equation follows from the definition of γ_{is} and Lemma 1.1. Note that

$$\bigotimes_{i=0}^{2^{l-1}-2} \Gamma_{k(i)}(\overline{\beta}_i) = P(l)_* SO(2^l - 1) \to P(l)_* SO(2^{l+1} - 1)$$

takes $\gamma_s(\overline{\beta}_i)$ to γ'_{is}. Using Proposition 2.1, we get the second and the third equalities of the lemma for $i, j \leq 2^{l-1} - 2$.

Suppose that $i = 2^{l-1} - 1$. Then $k(i) = 1$. This the second equation follows from Lemma 3.2. The third equation is vacuous if $j = 2^{l-1} - 1$. Let $j < 2^{l-1} - 1$. If $t \leq 2^{k(j)-1}$, then Lemma 3.3 implies the result. Finally, if $2^{k(j)-1} < t < k'(j)$ because $\gamma'_{jt} = \gamma'_{j,t-2^{k(j)-1}} \gamma'_{j,2^{k(j)-1}}$ commutes with $\overline{\beta}_{2^{l-1}-1}$.

4. The main lemma

Let P be the projective plane of $SO(n)$. There is a cofiber sequence

$$\Sigma SO(n) \wedge SO(n) \xrightarrow{\widetilde{\mu}} \Sigma SO(n) \xrightarrow{f} P \xrightarrow{g} \Sigma SO(n) \wedge \Sigma SO(n)$$

where $\widetilde{\mu}$ is the Hopf construction on the multiplication of $SO(n)$. This section is devoted to proving the following result.

Proposition 4.1. *Suppose that* $2j+1 < n$, $i < 2^{l-1}$, $1 \leq t \leq k(i)$ *and* $2^t(2i+1)+2j+1 \geq \max(n-1, 2^{l+1})$. *Then the image of the composition*

$$\widetilde{k}(l)_{*+2} P \to \widetilde{H}_{*+2} P \xrightarrow{g_*} \widetilde{H}_{*+2}(\Sigma SO(n) \wedge \Sigma SO(n)) \cong \widetilde{H}_* SO(n) \otimes \widetilde{H}_* SO(n)$$

does not contain the element $\overline{\beta}_j \otimes \gamma'_{i,2^t} - \gamma'_{i,2^t} \otimes \overline{\beta}_j$.

We start by studying some identities of symmetric polynomials. Let R be the ring of symmetric polynomials in n variables x_i, $1 \leq i \leq n$, with coefficients in $\mathbb{Z}/2$. Denote the elementary symmetric polynomials by σ_i, $1 \leq i \leq n$. By convention $\sigma_0 = 1$ and $\sigma_i = 0$ if $i < 0$ or $i > n$. Let I be the ideal generated by the σ_i's. Define ϕ_{is} by

$$\phi_{is} = \sum \left\{ x_{j_0}^s x_{j_1} \dots x_{j_i} \mid j_1 < \dots < j_i;\ j_0 \neq j_t \text{ for } 1 \leq t \leq i \right\}.$$

Note that $\phi_{0s} = \sum x_j^s$, which we will denote by π_s.

Lemma 4.2. *In* R/I^3, $\phi_{2k+1,2s} = \sigma_{2k+2s+1} + \sum_{j=1}^{s-1} \sigma_{2k+2j}\sigma_{2s+1-2j}$.

Proof. We have the following well-known Newton relations:

$$\pi_i + \sum_{j=1}^{i-1} (-1)^j \sigma_j \pi_{i-j} + (-1)^i i \sigma_i = 0$$

It follows that $\pi_i \equiv i\sigma_i \mod I^2$. Using the Newton relations once again, we have, in R/I^3,

$$\begin{aligned}\pi_{2s+1} &= \sum_{i=1}^{2s} \sigma_i(2s+1-i)\sigma_{2s+1-i} + (2s+1)\sigma_{2s+1} \\ &= \sum_{t=1}^{s} \sigma_{2t}\sigma_{2s+1-2t} + \sigma_{2s+1}\end{aligned}$$

An easy calculation shows that $\phi_{ts} = \pi_s\sigma_t - \phi_{t-1,s+1}$. Iterating gives

$$\phi_{ts} = \sum_{i=0}^{t} (-1)^i \pi_{s+i}\sigma_{t-i}\,.$$

Thus, in R/I^3,

$$\begin{aligned}\phi_{2k+1,2s} &= \sum_{i=0}^{2k} \pi_{2s+i}\sigma_{2k+1-i} + \pi_{2k+2s+1} \\ &= \sum_{t=0}^{k-1} \sigma_{2s+2t+1}\sigma_{2k-2t} + \sum_{j=0}^{k+s-1} \sigma_{2k+2s-2j}\sigma_{2j+1} + \sigma_{2k+2s+1} \\ &= \sum_{t=0}^{k-1} \sigma_{2s+2t+1}\sigma_{2k-2t} + \sum_{j=0}^{s-1} \sigma_{2k+2s-2j}\sigma_{2j+1} + \sum_{j=s}^{s+k-1} \sigma_{2k+2s-2j}\sigma_{2j+1} + \sigma_{2k+2s+1}\end{aligned}$$

The first and third sums above cancel each other, and the second sum is clearly the sum in the statement of the lemma.

Next, we study the action of Q_l on H^*P.

Extend the notation of Lemma 2.3 by letting $\widetilde{w}_s$ be the unique primitive element in $H^sSO(n)$. We have the inclusions $P \to BSO(n) \to BO(n)$. Denote the image, in H^*P, of the s-th Stiefel-Whitney class by w'_s. Note that $f^* : H^*P \to H^*\Sigma SO(n) \cong H^{*-1}SO(n)$ takes w'_{s+1} to $\widetilde{w}_s$.

Lemma 4.3. *In H^*P,* $Q_l w'_{2k+2} = w'_{2k+1+2^{l+1}} + \sum_{j=1}^{2^l-1} w'_{2k+2j} w'_{2^{l+1}+1-2j}$

Proof. Identify $H^*BO(n)$ with its image in $H^*(\mathbb{R}P^\infty)^n = \mathbb{Z}/2[x_1,\ldots,x_n]$. The i-th Steifel-Whitney class is the i-th elementary symmetric polynomial in x_i's. In $H^*(\mathbb{R}P^\infty) = \mathbb{Z}/2[x]$, $Q_l x = x^{2^{l+1}}$. It follows that $Q_l\sigma_{2k+2} = \phi_{2k+1,2^{l+1}}$ in $H^*((\mathbb{R}P^\infty)^n)$. The homomorphism $H^*BO(n) \to H^*P$ annihilates I^3 and w'_1. Now apply Lemma 4.2.

Proof of Proposition 4.1. Denote $\overline{\beta}_j \otimes \gamma_{i,2^t} - \gamma_{i,2^t}\overline{\beta}_j$ by x'. It is of degree $m = 2^{t+1}(2i+1) + 2j + 1$.

The cokernal of $\mu_* : \widetilde{H}_*SO(n)\otimes\widetilde{H}_*SO(n) \to \widetilde{H}_*SO(n)$ is the module of indecomposables of the latter. Hence the kernel of f_* is trivial in degrees $> n$. On the other hand, x' is in the kernel of μ_* because $H_*SO(n)$ is commutative. So x' is in the image of g_*. Thus there is a unique element $x \in H_{m+2}P$ such that $g_*x = x'$. We will show that $xQ_l \neq 0$. For typographical convenience, we put $k = 2^{t-1}(2i+1) + j + 1 - 2^l$.

$$\langle xQ_l, w'_{m+3-2^{l+1}}\rangle = \langle x, Q_l w'_{m+3-2^{l+1}}\rangle = \langle x, w'_{m+2}\rangle + \sum_{s=1}^{2^l-1} \langle x, w'_{2k+2s} w'_{2^{l+1}+1-2s}\rangle$$

Now $w'_{m+2} = 0$, for $m+2 > n$. It is well known that the composition

$$\widetilde{H}^*SO(n) \otimes \widetilde{H}^*SO(n) \cong \widetilde{H}^{*+2}\left(\Sigma SO(n) \wedge \Sigma SO(n)\right) \xrightarrow{g_*} \widetilde{H}^{*+2}P$$

takes $\widetilde{w}_r \otimes \widetilde{w}_s$ to $w'_{r+1}w'_{s+1}$ [Th]. By definition, $g_*x = \overline{\beta}_j \otimes \gamma_{i,2^t} - \gamma_{i,2^t} \otimes \overline{\beta}_j$. So

$$\langle xQ_l, w'_{m+3-2^{l+1}}\rangle = \sum_{s=1}^{2^l-1} \langle \overline{\beta}_j, \widetilde{w}_{2k+2s-1}\rangle\langle \gamma_{i,2^t}, \widetilde{w}_{2^{l+1}-2s}\rangle - \sum_{s=1}^{2^l-1} \langle \gamma_{i,2^t}, \widetilde{w}_{2k+2s-1}\rangle\langle \overline{\beta}_j, \widetilde{w}_{2^{l+1}-2s}\rangle$$

Using the remarks in the proof of Lemma 2.3 we see that all the terms in the second sum vanish. So do all the terms in the first, except for the one with $s = 2^l - 2^{t-1}(2i+1)$; that term is 1.

The proposition follows by noting that Q_l annihilates the image of the reduction map $k(l)_*P \to H_*P$.

5. The algebra structure of $P(l)_*SO(2^{l+1}-1)$ II

In this section we determine a complete set of algebra relations for the generators introduced in Section 3.

The next result is a useful technical lemma which is fairly well-known for commutative Hopf-algebras. But we will use it in $P(l)_*SO(2^{l+1}-1)$ whose diagonal is not an algebra homomorphism. So we give a statement that uses minimal hypothesis.

Lemma 5.1. *Let B be both an algebra and coalgebra, and let $A \subset B$ be both a subalgebra and a subcoalgebra. Let $\delta : A \to B$ be both an algebra derivation and a coalgebra derivation. Suppose that $\gamma_i \in A$, $0 \le i \le 2^{r+1}$ have the following properties: $\gamma_0 = 1$, $\Delta(\gamma_i) = \sum_{j=0}^{i} \gamma_j \otimes \gamma_{i-j}$ if $i \le 2^{r+1}$; $[\gamma_i, \delta(\gamma_j)] = 0$ and $\gamma_i\gamma_j = (i,j)\gamma_{i+j}$ if $i, j \le 2^r$; and $\Delta(\delta(\gamma_{2^r})\gamma_{2^r}) = \Delta(\delta(\gamma_{2^r}))\Delta(\gamma_{2^r})$. Then $\delta(\gamma_{2^{r+1}}) - \delta(\gamma_{2^r})\gamma_{2^r}$ is primitive.*

Proof. Now $\gamma_i\gamma_j = (i,j)\gamma_{i+j}$ if $i, j < 2^{r+1}$: if $2^r < i < 2^{r+1}$, then $\gamma_i = \gamma_{2^r}\gamma_{i-2^r}$. In the same way $[\gamma_i, \delta(\gamma_j)] = 0$ if $i, j < 2^{r+1}$.

If $0 < k < 2^{r+1}$, then $\sum_{0 \le i < k-i \le 2^r} \binom{k}{i}\binom{2^{r+1}-k}{2^r-i} = 1 \pmod 2$: Note that $\binom{2^s+i}{j} \equiv \binom{i}{j}$ mod 2 if $0 \le i, j < 2^s$. Thus, if $i \le k < 2^r$, then

$$\binom{k}{i}\binom{2^{r+1}-k}{2^r-i} = \binom{k}{i}\binom{2^r-k}{2^r-i} \pmod 2 = \begin{cases} 1 & i = k \\ 0 & i < k \end{cases} \pmod 2,$$

while if $i \leq 2^r \leq k$,

$$\binom{k}{i}\binom{2^{r+1}-k}{2^r-i} = \binom{k-2^r}{i}\binom{2^{r+1}-k}{2^r-i} \pmod 2 = \begin{cases} 1 & i = k-2^r \\ 0 & i \neq k-2^r \end{cases} \pmod 2 .$$

Now the result follows from the next calculation:

$$\begin{aligned}
\Delta(\delta(\gamma_{2^r})\gamma_{2^r}) &= \left(\sum_{i=0}^{2^r} \delta(\gamma_i)\otimes\gamma_{2^r-i} + \gamma_i\otimes\delta(\gamma_{2^r-i})\right)\left(\sum_{j=0}^{2^r}\gamma_j\otimes\gamma_{2^r-j}\right) \\
&= \sum_{i=0}^{2^r}(\delta(\gamma_i)\gamma_i\otimes\gamma_{2^r-i}\gamma_{2^r-i} + \gamma_i\gamma_i\otimes\delta(\gamma_{2^r-i})\gamma_{2^r-i}) \\
&\quad + \sum_{i\neq j}(\delta(\gamma_i)\gamma_j\otimes\gamma_{2^r-i}\gamma_{2^r-j} + \gamma_i\gamma_j\otimes\delta(\gamma_{2^r-i})\gamma_{2^r-j}) \\
&= \delta(\gamma_{2^r})\gamma_{2^r}\otimes 1 + 1\otimes\delta(\gamma_{2^r})\gamma_{2^r} \\
&\quad + \sum_{0\leq i<j\leq 2^r}(\delta(\gamma_i\gamma_j)\otimes\gamma_{2^r-i}\gamma_{2^r-j} + \gamma_i\gamma_j\otimes\delta(\gamma_{2^r-i}\gamma_{2^r-j})) \\
&= \delta(\gamma_{2^r})\gamma_{2^r}\otimes 1 + 1\otimes\delta(\gamma_{2^r})\gamma_{2^r} \\
&\quad + \sum_{0\leq i<k-i\leq 2^r}\binom{k}{i}\binom{2^{r+1}-k}{2^r-i}(\delta(\gamma_k)\otimes\gamma_{2^{r+1}-k} + \gamma_k\otimes\delta(\gamma_{2^{r+1}-k})) \\
&= \delta(\gamma_{2^r})\gamma_{2^r}\otimes 1 + 1\otimes\delta(\gamma_{2^r})\gamma_{2^r} + \Delta(\delta(\gamma_{2^{r+1}})) - \delta(\gamma_{2^{r+1}})\otimes 1 - 1\otimes\delta(\gamma_{2^{r+1}})
\end{aligned}$$

Lemma 5.2. *If $0 \leq i \leq 2^l - 2$ and $0 \leq r < l$, then $Q_r\overline{\beta}_i = 0$ in $P(l)_*SO(2^{l+1}-1)$. If $0 \leq i \leq 2^{l-1}-1$ and $1 \leq s \leq k(i)$, then*

$$Q_r\gamma'_{i,2^s} = \sum_{t=1}^{s}\overline{\beta}_{2^{t-1}(2i+1)-2^r}\gamma'_{i,2^s-2^t} .$$

By convention, $\overline{\beta}_t = 0$ if $t < 0$.

Proof. The first sentence follows from the fact that

$$\overline{\beta}_i \in \operatorname{im}\left(BP_*\Sigma G_{2^{l+1}-3} \to P(l)_*SO(2^{l+1}-1)\right) .$$

We will prove the second by induction on s. Note that the claim is vacuously true if $s = 0$, and true by degree considerations if $2^s(2i+1) \leq 2^{r+1}-1$. In particular $Q_{l-1}\gamma_{i,2^s} = 0$ if $s < k(i)$.

Suppose that $2^{s-1}(2i+1) \geq 2^r$. Then $Q_r\gamma'_{i,2^s}$ is indecomposable in $H_*SO(2^{l+1}-1)$: Recall that $H_*\mathbb{R}P^{2^{l+1}-2}$ is isomorphic to the module of indecomposables of $H_*SO(2^{l+1}-1)$

as modules over the Steenrod algebra. Let x_t be the non-trivial element of the former of degree t. It is well known that if $t \geq 2^r$, then $Q_r x_{2t} \neq 0$.

Thus $x = Q_r\gamma'_{i,2^s} - \gamma'_{i,2^{s-1}}Q_r\gamma'_{i,2^{s-1}}$ is non-trivial. The last hypothesis of Lemma 5.1 holds by Proposition 1.3. So x is primitive. By Proposition 2.4, $x = \overline{\beta}_{2^{s-1}(2i+1)-2^r}$. The induction hypothesis, and an easy calculation complete the proof.

Corollary 5.3. *Let $i \leq 2^{l-1}-1$. If $s < k'(i)$, then $Q_{l-1}\gamma'_{is} = 0$. Also, $Q_{l-1}\widetilde{\gamma}_i = \overline{\beta}_{2^{k(i)-1}(2i+1)-2^{l-1}}$.*

Proposition 5.4. *Suppose that $0 \leq i,j \leq 2^{l-1}-1$ and $s < k'(j)$. Then $[\widetilde{\gamma}_i, \gamma'_{js}] = [\widetilde{\gamma}_i, \overline{\beta}_j]\gamma'_{j,s-1}$.*

Proof. The proof is by induction on s. Note that it is true by definition if $s = 1$. Let $s = 2^t + r$ with $0 < r < 2^t$. Then

$$[\widetilde{\gamma}_i, \gamma'_{js}] = [\widetilde{\gamma}_i, \gamma'_{j,2^t}\gamma'_{jr}] = \gamma'_{j,2^t}[\widetilde{\gamma}_i, \gamma'_{jr}] + [\widetilde{\gamma}_i, \overline{\beta}_j]\gamma_{j,2^t-1}\gamma_{jr}\,.$$

The second term is trivial. Thus we are left with the case $s = 2^{t+1}$.

The induction hypothesis, Corollary 5.3, Lemma 3.4 and Proposition 1.3 imply that Lemma 5.1 applies to γ'_{jr} and

$$[\widetilde{\gamma}_i, .]\colon P(l)_*SO(2^l-1) \to P(l)_*SO(2^{l+1}-1)\,.$$

Thus $[\widetilde{\gamma}_i, \gamma'_{js}] - [\widetilde{\gamma}_i, \gamma'_{j,2^t}]\gamma'_{j,2^t}$ is an even dimensional primitive. It must be trivial by Proposition 2.4.

Proposition 5.5. *Let $0 \leq i \leq 2^{l-1}-1$, $1 \leq t \leq k'(i)$ and $0 \leq j \leq 2^l-2$. If $s = 2^{t-1}(2i+1)+j-2^l+1 \geq 0$, then $[\gamma_{i,2^t}, \overline{\beta}_j] = [\gamma_{i,2^{t-1}}, \overline{\beta}_j]\gamma_{i,2^{t-1}} + v_l\overline{\beta}_s$.*

Proof. It follows from Lemma 5.1 that $x = [\gamma_{i,2^t}, \overline{\beta}_j] - [\gamma_{i,2^{t-1}}, \overline{\beta}_j]\gamma_{i,2^{t-1}}$ is primitive. It is divisible by v_l for $P(l+1)_*SO(2^{l+1}-1)$ is commutative. By Proposition 2.4, x is either 0 or $v_l\overline{\beta}_s$.

Let P be the projective plane of $SO(2^{l+1}-1)$. If x were trivial,

$$\overline{\beta}_j \wedge \gamma_{i,2^t} - \gamma_{i,2^t} \wedge \overline{\beta}_j - [\gamma_{i,2^{t-1}}, \overline{\beta}_j] \wedge \gamma_{i,2^{t-1}}$$

would be in the image of

$$\widetilde{P(l)}_*P \to \widetilde{P(l)}_*(\Sigma SO(2^{l+1}-1) \wedge \Sigma SO(2^{l+1}-1)) \cong \widetilde{P(l)}_{*-2}(SO(2^{l+1}-1) \wedge SO(2^{l+1}-1))\,.$$

This contradicts Proposition 4.1 because the reduction of $[\gamma_{i,2^{t-1}}, \overline{\beta}_j]$ to ordinary homology is trivial.

Corollary 5.6. *Let $0 \le i \le 2^{l-1}-1$, $0 \le j < 2^l - 1$. If $2^s(2i+1)+2j+1 = 2^{l+1}-1$, then $[\gamma'_{i,2^s}, \overline{\beta}_j] = \overline{\alpha}'_{2^l-1}$ in $P(l)_*Spin(2^{l+1}-1)$.*

Proof. Combine Corollary 2.2 with the previous proposition.

This corollary, combined with Proposition 2.1, implies the dual of the main result of [MZ].

Lemma 5.7. *Let $0 \le i, j \le 2^{l-1}-1$. Then*

$$[\widetilde{\gamma}_i, \widetilde{\gamma}_j] = [\widetilde{\gamma}_i, \overline{\beta}_j]\gamma_{j,k'(j)-1} + [\widetilde{\gamma}_j, \overline{\beta}_i]\gamma_{i,k'(i)-1} + (Q_{l-1}\widetilde{\gamma}_i)(Q_{l-1}\widetilde{\gamma}_j)\,.$$

Proof. Note that $(\mathrm{id} \otimes Q_{l-1})(\Delta(\widetilde{\gamma}_i)) = 1 \otimes (Q_{l-1}\widetilde{\gamma}_i)$. Thus, by Proposition 1.3, Proposition 5.4 and Lemma 3.4,

$$\begin{aligned}
\Delta([\widetilde{\gamma}_i, \widetilde{\gamma}_j]) &= [\Delta(\widetilde{\gamma}_i), \Delta(\widetilde{\gamma}_j)] + v_l(1 \otimes (Q_{l-1}\widetilde{\gamma}_i))((Q_{l-1}\widetilde{\gamma}_j) \otimes 1) \\
&= \sum_{s=0}^{k'(j)} ([\widetilde{\gamma}_i, \gamma'_{js}] \otimes \gamma'_{j,k'(j)-s} + \gamma'_{js} \otimes [\widetilde{\gamma}_i, \gamma'_{j,k'(j)-s}]) \\
&\qquad + \sum_{s=0}^{k'(i)} ([\widetilde{\gamma}_j, \gamma'_{is}] \otimes \gamma'_{i,k'(i)-s} + \gamma'_{is} \otimes [\widetilde{\gamma}_j, \gamma'_{i,k'(i)-s}]) + v_l Q_{l-1}\widetilde{\gamma}_j \otimes Q_{l-1}\widetilde{\gamma}_i \\
&= [\widetilde{\gamma}_i, \widetilde{\gamma}_j] \otimes 1 + 1 \otimes [\widetilde{\gamma}_i, \widetilde{\gamma}_j] + \Delta([\widetilde{\gamma}_i, \overline{\beta}_j])\Delta(\gamma'_{j,k'(j)-1}) \\
&\qquad + \Delta([\widetilde{\gamma}_j, \overline{\beta}_i])\Delta(\gamma'_{i,k'(i)-1}) + v_l Q_{l-1}\widetilde{\gamma}_j \otimes Q_{l-1}\widetilde{\gamma}_i
\end{aligned}$$

As $Q_{l-1}\widetilde{\gamma}_r$ are primitive, we see that

$$[\widetilde{\gamma}_i, \widetilde{\gamma}_j] - ([\widetilde{\gamma}_i, \overline{\beta}_j]\gamma_{j,k'(j)-1} + [\widetilde{\gamma}_j, \overline{\beta}_i]\gamma_{i,k'(i)-1} + (Q_{l-1}\widetilde{\gamma}_i)(Q_{l-1}\widetilde{\gamma}_j))$$

is an even dimensional primitive and hence trivial.

Proposition 5.8. *Let $0 \le i \le 2^{l-1}-1$ and $j = (2i+1)2^{k(i)-1} - 2^{l-1}$. Then*

$$\widetilde{\gamma}_i^2 = [\widetilde{\gamma}_i, \overline{\beta}_i]\gamma'_{i,k'(i)-1} + \gamma'_{j,2}\,.$$

Proof. We have a by-now-familiar type of calculation:

$$\begin{aligned}
\Delta(\widetilde{\gamma}_i^2) &= (\Delta(\widetilde{\gamma}_i))^2 + v_l\overline{\beta}_j \otimes \overline{\beta}_j \\
&= \widetilde{\gamma}_i^2 \otimes 1 + 1 \otimes \widetilde{\gamma}_i^2 + v_l\overline{\beta}_j \otimes \overline{\beta}_j \\
&\qquad + \sum_{s=1}^{k'(i)-1} ([\widetilde{\gamma}_i, \gamma'_{is}] \otimes \gamma'_{i,k'(i)-s} + \gamma'_{is} \otimes [\widetilde{\gamma}_i, \gamma'_{i,k'(i)-s}]) \\
&= \widetilde{\gamma}_i^2 \otimes 1 + 1 \otimes \widetilde{\gamma}_i^2 + v_l\overline{\beta}_j \otimes \overline{\beta}_j + \Delta([\widetilde{\gamma}_i, \overline{\beta}_i])\Delta(\gamma'_{i,k'(i)-1})
\end{aligned}$$

It follows that $\widetilde{\gamma}_i^2 - ([\widetilde{\gamma}_i, \overline{\beta}_i]\gamma_{i,k'(i)-1} + \gamma'_{j,2})$ is an even dimensional primitive and hence trivial.

References

[Bs] N. A. Baas, *On the bordism theory of manifolds with singularities*, Math. Scandinavia **33** (1973), 279–302.

[Bt] R. Bott, *The space of loops on a Lie group*, Michigan J. Math. **5** (1958), 35–61.

[Hp] M. J. Hopkins, *Nilpotence and finite H-spaces*, Israel J. Math. **66** (1989), 238–246.

[JW] D. C. Johnson and W. S. Wilson, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975), 55–75.

[MM] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. Math. **(2) 18** (1965), 211–264.

[Po] G. J. Porter, *Homotopy nilpotency of S^3.*, Proc. Amer. Math. Soc. **15** (1964), 681–682.

[Qu] D. Quillen, *Formal group laws and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[R1] V. K. Rao, *The Hopf algebra strucure of $MU_*(\Omega SO(n))$*, Indiana Univ. J. Math. **38** (1989), 277–291.

[R2] ———, *The bar spectral sequence converging to $h_*SO(2n+1)$*, Manuscripta Math. **65** (1989), 47–61.

[R3] ———, *On the Morava K-theories of $SO(2n+1)$*, Proc. Amer. Math. Soc. **108** (1990), 1031–1038.

[R4] ———, *$SO(n)$ is not homotopy nilpotent for $n \neq 2^m - 3$, $2^m - 2$*, submitted to *Topology*..

[SY] N. Shimada and N. Yagita, *Multiplication in the complex bordism theories with singularities*, Publ. research Inst. Math. Kyoto U. **12** (1976), 259–293.

[Sl] D. Sullivan, *Singularities in spaces*, Proc. Liverpool Singularities Symposium II, Lect. Notes in Math. vol. 209, Springer Verlag, Berlin, 1971, pp. 196–207.

[Th] E. Thomas, *Functional cup operations and the Transgression operator*, Arch. Math. **12** (1961), 435–444.

[Wl] W. S. Wilson, *Brown-Peterson homology – An introduction and sampler*, CBMS Regional Conference Series in Math. no. 48, Amer. Math. Soc., Providence RI, 1980.

[Wu] U. Würgler, *Commutative ring spectra of charactersitic 2*, Comm. Math. Helvetici **61** (1986), 33–45.

[Zb] A. Zabrodsky, *Hopf spaces*, North Holland Math. Studies 22, North Holland, Amsterdam, 1976.

Department of Mathematics, The Ohio State University at Newark, Newark, OH 43055.
E-mail address: nathrao+@osu.edu