

The Additive Combinatorics of Normal Sets



Abstract

An infinite sequence of 1's and 0's is normal if each word (any finite series of 1's and 0's) shows up with the 'correct frequency'. Given some normal sequence of 1's and 0's we can then craft a 'normal set' of natural numbers out of this sequence. Our main focus is on the probability that $A + A$ is cofinite for a randomly chosen normal set A .

Introduction

Additive Combinatorics is the theory of counting additive structures in sets. In our case we are looking at the structure of $A + A$ where A is a normal set, and in particular we would like to know when this set is cofinite.

Definitions

Asymptotic Density

$$d(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [1, n]|}{n}$$

A+A

Given a set $A \subseteq \mathbb{N}$, we can define $A + A = \{x + y : x, y \in A\}$.

Cofinite

A set is cofinite if its complement is finite.

Examples

- The set of even numbers $E = \{2n : n \in \mathbb{N}\}$ has asymptotic density $d(E) = \frac{1}{2}$, since half of all natural numbers are even.
- If $O = \{2n - 1 : n \in \mathbb{N}\}$ denotes the set of odd numbers then $O + O = E$.

What is normal?

First, we define a 'word' as any finite string of 1's and 0's, such as the word '101'. With 3 different positions, each with 2 possibilities (1 or 0) we have 8 words of length 3. In a normal sequence, each of these 8 words will appear exactly $1/8$ th of the time. This also means there are an equal number of 1's and 0's.

To check that a sequence $(a_n)_{n \in \mathbb{N}}$ is normal, for each finite word $w = w_0 w_1 \dots w_{\ell-1}$ we consider the set

$$S_w = \{n \in \mathbb{N} : a_n = w_0, a_{n+1} = w_1, \dots, a_{n+\ell-1} = w_{\ell-1}\}$$

and we verify that $d(S_w) = 1/2^\ell$.

We can move between sequences and sets by taking a sequence $(a_n)_{n \in \mathbb{N}}$ and constructing a set $A = \{n \in \mathbb{N} : a_n = 1\}$. This allows us to say that a set is normal if the sequence it comes from is normal.

Normality and Randomness

We also arrive at the conclusion that with probability 1 any random sequence of 1's and 0's will be normal. The best way to think about this is through coin flipping. If we were to flip a coin randomly assigning heads to 1's and tails to 0's, we would get a sequence that is normal. Using asymptotic density again we see that

$$d((A - k_1) \cap (A - k_2) \cap \dots \cap (A - k_n)) = \frac{1}{2^n}$$

where $x \in A - n$ if and only if $x + n \in A$. This is the same as being normal.

Borel-Cantelli Lemma

If E_1, E_2, \dots are events such that

$$\sum_{n=1}^{\infty} \Pr(E_n) < \infty$$

then the probability that infinitely many of these events happen is 0.

Main Result

It is not the case that $A + A$ is cofinite for any normal set A [1]. However if we randomly choose a subset A of \mathbb{N} by uniformly and independently including each number with probability $\frac{1}{2}$, then the resulting set is almost surely normal.

Now for such a random set A , $n \notin A + A$ only when at least one from each pair of numbers

$$(1, n-1), (2, n-2), \dots, \left(\frac{n}{2}, \frac{n}{2}\right) \text{ are not in } A$$

but these events are independent, giving that the probability of the event $n \notin A + A$ is $\frac{\sqrt{3}^n}{2}$.

We can use the Borel-Cantelli Lemma along with this observation to see that $A + A$ is almost surely cofinite.

References

- [1] Kabir Belgikar. personal communication, 1 2023.
- [2] D. G. Champernowne. The construction of decimals normal in the scale of ten. *Journal of the London Mathematical Society*, 8(4):254–260, 1933.