The Asymptotics of Finite Index Subgroups of F_2

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Abstract

Lagrange's Theorem says that a subgroup which has index k in a finite group takes up exactly $\frac{1}{k}$ of the entire group. Using coset dynamics, we can explore a similar concept for free groups; namely, if H is a subgroup of F_2 and if H contains an element of odd length, then we have

$$\lim_{n\to\infty} \frac{|\, H\cap S_n \,|}{|\, S_n \,|} = \frac{1}{[F_2 : H]}$$

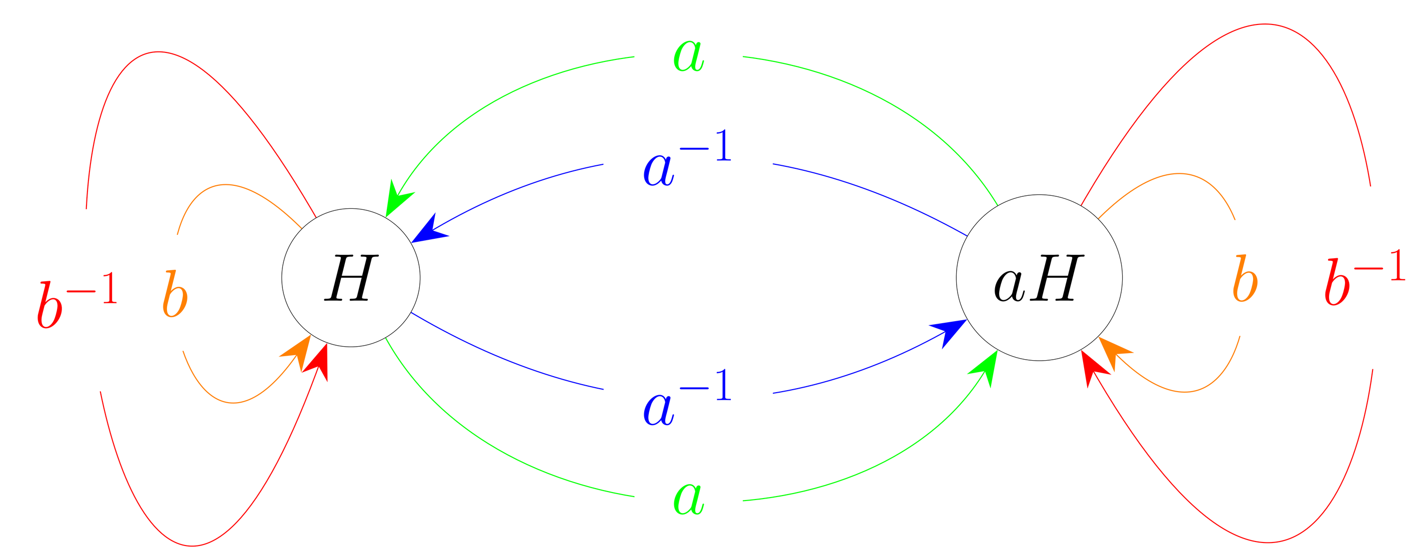
where $S_n \subseteq F_2$ that denotes the set of words which have length n.

Definitions

- **A Free group (F_2)**
  $F_2 := \langle a, b\rangle$ is the set of all finite words in the letters a, b, a^{-1}, b^{-1}. Words can be formally multiplied by writing them next to each other, such that adjacent a and a^{-1} or b and b^{-1} will cancel.
- **Subgroup (H)**
  A **subgroup** of F_2 is a subset which is closed under multiplication, taking inverses, and which contains the empty word.
- **Coset (gH)**
  For $g \in F_2$, the set $gH := \{gh : h \in H\}$ is called a **coset** of H. The distinct cosets of H partition F_2.
- **Index ($[G : H]$)**
  The **index** of subgroup H in a group G, denoted $[G : H]$, is the number of distinct cosets of H in G. Lagrange's Theorem states that in a finite group G, $[G : H]$ is exactly $|G|/|H|$.

Example

Let $H = \langle a^2, aba, b\rangle$, which is an index 2 subgroup of F_2. For $x \in \{a, b, a^{-1}, b^{-1}\}$, let $p_x^{(n)}$ denote the elements of $H \cap S_n$ which begin with x and let $q_x^{(n)}$ denote the elements of $aH \cap S_n$ which begin with x.



We have the following recurrence relations for $p_x^{(n)}$, and there are similar ones for $q_x^{(n)}$:

$$\begin{aligned}
p_a^{(n)} &= q_a^{(n-1)} + q_b^{(n-1)} + q_{b^{-1}}^{(n-1)}\\
p_b^{(n)} &= p_a^{(n-1)} + p_b^{(n-1)} + p_{a^{-1}}^{(n-1)}\\
p_{a^{-1}}^{(n)} &= q_b^{(n-1)} + q_{a^{-1}}^{(n-1)} + q_{b^{-1}}^{(n-1)}\\
p_{b^{-1}}^{(n)} &= p_a^{(n-1)} + p_{a^{-1}}^{(n-1)} + p_{b^{-1}}^{(n-1)}
\end{aligned}$$

Since $|H \cap S_n| = p_a^{(n)} + p_b^{(n)} + p_{a^{-1}}^{(n)} + p_{b^{-1}}^{(n)}$, and similarly with q's for aH, we can apply these recurrence relations several times to arrive at

$$|H \cap S_n| = |aH \cap S_{n-1}| + |H \cap S_{n-1}| + 3|aH \cap S_{n-2}|$$

and

$$|aH \cap S_n| = |H \cap S_{n-1}| + |aH \cap S_{n-1}| + 3|H \cap S_{n-2}|$$

from which we can deduce that

$$\lim_{n\to\infty} \frac{|H \cap S_n|}{|S_n|} = \lim_{n\to\infty} \frac{|aH \cap S_n|}{|S_n|} = \frac{1}{2}$$

Results

This method can give recurrence relations for other subgroups.

- When $H = \langle a, b^3, bab, ba^{-1}b, ba^2b\rangle$, which has index 3,
  $$|H \cap S_n| = |H \cap S_{n-1}| + 3|bH \cap S_{n-2}| + 3|b^{-1}H \cap S_{n-2}|$$
  and
  $$|H \cap S_n| = |S_{n-1}| + |H \cap S_{n-1}| - 3|H \cap S_{n-2}|$$
- When $H = \langle a^2, b^3, ab, b^{-1}a, bab^{-1}\rangle$, which has index 3,
  $$|H \cap S_n| = 2|aH \cap S_{n-1}| + 2|b^{-1}H \cap S_{n-1}| + 3|H \cap S_{n-3}|$$
- When $H = \varphi^{-1}(\text{Fix}(4))$ for $\varphi : F_2 \to S_4$ with $\varphi(a) = (12)$ and $\varphi(b) = (1234)$, which has index 4,
  $$\begin{aligned}|H \cap S_n| = |\, H \cap S_{n-1} \,| &+ 2|b^{-1}H \cap S_{n-2}|\\ &+ 3|b^{-2}H \cap S_{n-2}| + 3|b^{-2}H \cap S_{n-3}|\end{aligned}$$

Conjectures and Future Work

Conjecture: This method will always give a recurrence relation only in terms of the subgroup and its cosets.

Conjecture: There is always a recurrence relation just in terms of the subgroup, that is, not its cosets.

Question: Is there a way to describe the coefficients that appear in these recurrence relations? Can they be described in terms of the coset dynamics?

Question: Can we use function composition of relations in $\mathbb{Z}^2$ to be sure there is a recurrence relation?