REVIEW PROBLEMS (PINK SHEET LIST) - SOLUTIONS MIDTERM 3

READ THIS NOTE: I will be using parenthesis "(", ")" and brackets "[", "]" interchangeably (when there are too many parenthesis involved, I will put brackets to clear the situation a bit out, so you can see where one begins and where one ends an expression).

Also, I will be using exclusively the notation y', $f'(x)$, $h'(z)$ etc for the derivative. This doesn't, certainly, mean that notations such as $\frac{dy}{dx}$, $\frac{df}{dx}$ etc are not used, or invalid. If you prefer using the latter notation, kindly replace, without any penalty, accordingly: y' with $\frac{dy}{dx}$, $f'(x)$ with $\frac{df}{dx}$, etc.

Any comments or corrections regarding these solutions should be immediatly directed to me:

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Good luck!

(1) 1/p.681 : factor out the denominator, and check if we can cancel some of the factors

$$y = \frac{3x^2}{(x-4)(x+4)}$$

No cancellations possible, hence vertical asymptotes are at $x = -4$ and $x = 4$.

For horizontal asymptotes we need check the limit to infinity:

$$\lim_{x\to\infty} \frac{3x^2}{x^2 - 16}$$

(notice that I'm using the original form for y - it's much easier for the limit to use the "not-factored-out" form)

Dropping the lowest powers, respectively in the numerator and denominator, the limit becomes:

$$\lim_{x\to\infty} \frac{3x^2}{x^2} = \lim_{x\to\infty} 3 = 3$$

Same for $-\infty$. Hence we have a horizontal asymptote at level 3, namely $y = 3$.

(2) 3/p.681 : y is already factored out! only check where denominator is 0: $3x + 2 = 0 \Rightarrow x = -2/3$; check if $-2/3$ makes the numerator 0 - it doesn't. Hence vertical asymptote at $x = -2/3$

For the limit we could make use of the expanded form for the denominator: $9x^2 + 12x + 4$ (check it! multiply $(3x+2)(3x+2)$); the limit to both ∞ and $-\infty$ is $5/9$; that's the horizontal asymptote.

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(3) 5/p.681 :

$$f'(x) = \frac{7 \cdot 2x \cdot (2-x^2) - (-2x) \cdot 7x^2}{(2-x^2)^2} =$$
$$= \frac{28 - 14x^2 + 14x^2}{(2-x^2)^2} = \frac{28}{(2-x^2)^2}$$

Critical values (numbers) happen where $f'(x) = 0$ or $f'(x) = DNE$. A fraction is zero only if the NUMERATOR is zero, which in our case is impossible (28 is never zero!). A fraction Does Not Exist when denominator is zero, which in our case happens when $x = \sqrt{2}$ - but this value is NOT IN THE DOMAIN, as you can NOT plug it into the original f (by the way, this is what will ALWAYS happen when the function is a fraction).

Conclusion: no critical values.

(4) 7/p.681 :

$$f'(x) = \frac{\frac{1}{3}(x+1)^{-\frac{2}{3}} \cdot (3-4x) - (-4)(x+1)^{\frac{1}{3}}}{(3-4x)^2}$$

Takes too much time to simplify the numerator, so let's get straight to business: $f'(x) = 0$ first - that means numerator 0

$$\frac{1}{3}(x+1)^{-\frac{2}{3}} \cdot (3-4x) - (-4)(x+1)^{\frac{1}{3}} = 0$$
$$\frac{1}{3}(x+1)^{-\frac{2}{3}} \cdot (3-4x) = (-4)(x+1)^{\frac{1}{3}}$$

multiply both sides by $(x+1)^{\frac{2}{3}}$ - it will cancel the $(x+1)^{-\frac{2}{3}}$ in the left hand side.

$$\frac{1}{3}(3-4x) = (-4)(x+1)$$

(the power in the right hand side becomes 1! $1/3 + 2/3 = 1$); multiply now by 3

$$3 - 4x = -12x - 12$$
$$12x - 4x = -12 - 3$$
$$8x = -15$$

Critical value: $x = -15/8$

For the $f'(x) = DNE$ only 3/4 comes as candidate, but it cannot be used, since it's not in the domain (again, illustration of the point in the problem above)

(5) 9/p.681 :

$$f'(x) = -3x^2 + 6 \cdot 2x - 9 = -3(x^2 - 4x + 3) = -3(x-1)(x-3)$$

The derivative is 0 in 1 and 3. Draw the table (plug in 0, 2 and 4 to get the signs for $f'(x)$ to the left of 1, between 1 and 3 and to the right of 3, respectively:

x		1		3	
$f'(x)$	-	0	+	0	-
$f(x)$	↘		↗		↘

(6) 11/p.681 :

$$f'(x) = \frac{6 \cdot 4x^3 \cdot (x^2-3) - (2x) \cdot 6x^4}{(x^2-4)^2} =$$
$$= \frac{24x^5 - 72x^3 - 12x^5}{(x^2-4)^2} =$$
$$= \frac{12x^5 - 72x^3}{(x^2-4)^2} = \frac{12x^3(x^2-6)}{(x^2-4)^2} =$$
$$= \frac{12x^3(x-\sqrt{6})(x+\sqrt{6})}{(x^2-4)^2}$$

The sign of the denominator is obviously positive, so we ignore it. The numerator is 0 when x is either: 0 (the x^3 factor) or $\pm\sqrt{6}$. Draw the table ($\sqrt{6}$ is approximately 2.45; use, for example, -3, -1, 1 and 3 for the signs):

x		-2.45		0		2.45	
$f'(x)$	-	0	+	0	-	0	+
$f(x)$	↘		↗		↘		↗

(7) 13/p.681 :

$$f'(x) = 4x^3 - 3x^2$$

$$f''(x) = 4 \cdot 3x^2 - 3 \cdot 2x = 12x^2 - 6x = 6x(2x-1)$$

The second derivative is 0 in 0 and 1/2. Draw the signs' table:

x		0		.5	
$f''(x)$	+	0	-	0	+
$f(x)$	$\smile$		$\frown$		$\smile$

(8) 15/p.681 I'll leave you the pleasure to compute the second derivative - there is a fast way, if you use the fact that $f(x) = (2x-1)^{-1}$. But quotient rule has its advantages, I guess.

$$f''(x) = \frac{8}{(2x-1)^3}$$

The denominator has an ODD power, so its sign WILL change. The denominator is 0 in 1/2 (and that's where the second derivative will NOT exist). Draw the table:

x		.5	
$f''(x)$	-	DNE	+
$f(x)$	$\frown$		$\smile$

(9) 17/p.681 : Product rule at its best!

$$f'(x) = 3 \cdot (4x+1)^2 \cdot 4 \cdot (4x+9) + (4x+1)^3 \cdot 4$$

Factor out $4(4x+1)^2$:

$$f'(x) = 4(4x+1)^2 \cdot (12x+27+4x+1) = 4(4x+1)^2 \cdot (16x+28) = (4x+1)^2(64x+112)$$

$$f''(x) = 2(4x+1) \cdot 4 \cdot (64x+112) + (4x+1)^2 \cdot 64$$

Factor out $8(4x+1)$:

$$f''(x) = 8(4x+1)[64x+112+8(4x+1)] = 8(4x+1)(96x+120) = 8 \cdot 24 \cdot (4x+1)(4x+5)$$

The second derivative is 0 in $-1/4$ and $-5/4$; draw the table:

x		-1.25		$-.25$	
$f''(x)$	+	0	-	0	+
$f(x)$	$\smile$		$\frown$		$\smile$

(10) 19/p.681 : first derivative test

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

Critical numbers: $x = 1$, $x = 2$. Draw table:

x		1		2	
$f'(x)$	+	0	-	0	+
$f(x)$	$\nearrow$	rel max	$\searrow$	rel min	$\nearrow$

alternate method : you can certainly also use second derivative test We already have the critical numbers, 1 and 2. We don't draw the table this time, rather compute the second derivative (use the "not-factored-out" form:

$$f''(x) = 12x - 18$$

- $x = 1$: $f''(1) = 12 \cdot 1 - 18 = -6 \Rightarrow$ "$\frown$" $\Rightarrow$ rel. max
- $x = 2$: $f''(2) = 12 \cdot 2 - 18 = 6 \Rightarrow$ "$\smile$" $\Rightarrow$ rel. min

(11) 21/p.681 : first derivative test

$$f'(x) = x^5 + x^2 = x^2(x^3+1)$$

Find critical numbers: $f'(x) = 0 \Rightarrow x^2(x^3+1) = 0$ and so either $x = 0$ or $x^3 = -1 \Rightarrow x = -1$. Draw table:

x		-1		0	
$f'(x)$	-	0	+	0	+
$f(x)$	$\searrow$	rel min	$\nearrow$	neither(!)	$\nearrow$

note: second derivative test wouldn't have worked here! try it out, you'll get $f''(0) = 0$.

(12) 23/p.681 : first derivative test; for easier computation of the derivative, distribute the $x^{\frac{2}{3}}$ inside the paranthesis: $f(x) = x^{\frac{5}{3}} + x^{\frac{2}{3}}$.

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{2}{3}x^{-\frac{1}{3}}$$

Find critical numbers: $f'(x) = 0$

$$\frac{5}{3}x^{\frac{2}{3}} + \frac{2}{3}x^{-\frac{1}{3}} = 0$$

$$\frac{5}{3}x^{\frac{2}{3}} = -\frac{2}{3}x^{-\frac{1}{3}}$$

multiply both sides by $x^{1/3}$, in order to cancel the $x^{-1/3}$ in the right hand side:

$$\frac{5}{3}x = -\frac{2}{3} \Rightarrow x = -\frac{2}{5}$$

Draw table:

x		$-2/5$	
$f'(x)$	-	0	+
$f(x)$	$\searrow$	rel min	$\nearrow$

Note: second derivative test works here

(13) 25/p.681 : it's basically a table like for the first derivative test, yet we do it for the SECOND DERIVATIVE, and we look for sign change, AT POINTS ON THE GRAPH (that means that we make sure before stating "it is point of inflexion" that the original function is defined at that value - basically plug it in, make sure it works)

$$y'' = 20x^3 - 60x^2 = 20x^2(x - 3)$$

$y'' = 0$ when $x = 0$ or when $x = 3$. Draw the sign table:

x		0		3	
y''	-	0	-	0	+
y	$\frown$	nothing	$\frown$	point of inflexion	$\smile$

Check: $y(3) = 3^5 - 5 \cdot 3^4 + 3 \cdot 3$ works (you don't need to compute it, just make sure plugging in doesn't give you division by zero, or square root of negative, or something similar). So, $x = 3$ is point of inflexion.

(14) 27/p.681 : rewrite y, multiply it out.

$$y = 12x^5 - 20x^4 + 24x - 40$$

$$y'' = 240x^3 - 240x^2 = 240x^2(x-1)$$

$y'' = 0$ when $x = 0$ or $x = 1$ ... draw table (btw, by now you can probably tell that 0 is NOT going to be an inflexion point, since x is squared; yet 1 IS going to be one, as $x - 1$ is NOT squared ...

(15) 29/p.681 : you're probably better off rewriting y as $y = \frac{1}{5}x^2 \cdot e^{-x}$

$$y' = \frac{1}{5}(2xe^{-x} + x^2e^{-x} \cdot (-1)) = \frac{1}{5}(2xe^{-x} - x^2e^{-x})$$

notice I don't factor out anything at this point!

$$y'' = \frac{1}{5}[(2 \cdot e^{-x} + 2xe^{-x} \cdot (-1)) - (2x \cdot e^{-x} + x^2 \cdot e^{-x} \cdot (-1))]$$

$$y'' = \frac{1}{5}(2e^{-x} - 2xe^{-x} - 2xe^{-x} - (-1)x^2e^{-x})$$

$$y'' = \frac{e^{-x}}{5}(2 + x^2)$$

$y'' = 0$ has no solutions, as e^{-x} is never zero (no power will make e become zero ...) while $2 + x^2$ is positive anywhere.

Hence, no points of inflexion (the function is concave up everywhere, no change).

(16) 31/p.681 : find critical numbers first

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

$f'(x) = 0$ if either $x = 0$ or $x = 1$ - these are the critical numbers. Check to see if they are between 0 and 2 - yes, they are. Attach to these the endpoints: 0 and 2 (well, 0 is already considered, that's OK, makes our job easier).

Plug these numbers we found BACK into the original f:

$$f(0) = 0$$

$$f(1) = 3 - 4 = -1$$

$$f(2) = 48 - 32 = 16$$

Absolute max is 16, for $x = 2$; absolute min is -1, for $x = 1$.

(17) 33/p.681 : find critical numbers first

$$f'(x) = \frac{1 \cdot (5x-6)^2 - x \cdot 2(5x-6) \cdot 5}{(5x-6)^4}$$
$$f'(x) = \frac{(5x-6)(5x-6-10x)}{(5x-6)^4}$$

$f'(x) = 0$ if $5x - 6 = 0$ - but we can't have that, denominator of f is zero then; or, $-5x - 6 = 0 \Rightarrow x = -6/5$. We can also consider $f'(x) = DNE$, since we have a fraction, but we get the same $5x - 6 = 0$, so we discard this candidate for a critical number.

Is -1.2 between -2 and 0? yes. Attach to it these endpoints: -2 and 0. Compute values of f in these three points.

$$f(-2) = -2/256 = -0.0078125$$
$$f(-1.2) = -1.2/144 = -0.008279079$$
$$f(0) = 0$$

Absolute max is 0, in 0; absolute min is -0.0078125 in -2.

(I'm skipping the graphing problems, as it's taking to much time - the methods are the same as above, though; we'll do these problems at the review session, or during Monday's lecture, if time allows it)

(18) 49/p.682 : I doubt you'll need to know these, but they're food for thought ...

- a) no, not all critical numbers are relative extrema (max or min)
- b) $1/x$ is decreasing all the time, but it also has a VERTICAL ASYMPTOTE; looking at the graph we notice that the left side is below the x-axis, while the right side is above it, and so we CAN find (in fact ALL) negative numbers that play the role of x_1 and (ALL) positive numbers that play the role of x_2
- c) it does, but because x^4 is a "nice" function; the interval that is used is not CLOSED however, so we cannot apply our method here (must look at the graph, in fact, or use some other method)
- d) same as for a), not all point where the derivative is 0 are points where sign changes, so NO
- e) no; take $f = x^3 - \frac{1}{3}x$; it has a relative max at $-1/3$ - only one! - but as we try to get closer and closer to 2, we never get to it (2 is not part of the interval - see why it's so important to have endpoints?), and so we never obtain the value $8 - .66 = 7.33$ ... no absolute max (the value in the rel max is $-1/27 + 1/9 = 2/27 = 0.074$

(19) 51/p.682 : since c is total cost, MARGINAL cost is $c' = 3q^2 - 12q + 12$. Where is c' increasing? use the first derivative test's table.

$(c')' = 6q - 12 = 6(q-2)$; $(c')' = 0$ for $q = 2$. Draw the table:

x		2	
$(c')'$	-	0	+
$f(x)$	$\searrow$		$\nearrow$

Increase for $q \geq 2$.

Note: the major problem here is ... well, reading the problem properly! it's asking for the MARGINAL cost's increase, not TOTAL cost's ... so you need take a derivative first for DEFINING the function you will analyze. Then continue with the method you know ...