REVIEW PROBLEMS, CHAPTER 15 - MIDTERM 3

READ THIS NOTE: I will be using parenthesis "(", ")" and brackets "[", "]" interchangeably (when there are too many parenthesis involved, I will put brackets to clear the situation a bit out, so you can see where one begins and where one ends an expression).

Also, I will be using exclusively the notation y', $f'(x)$, $h'(z)$ etc for the derivative. This doesn't, certainly, mean that notations such as $\frac{dy}{dx}$, $\frac{df}{dx}$ etc are not used, or invalid. If you prefer using the latter notation, kindly replace, without any penalty, accordingly: y' with $\frac{dy}{dx}$, $f'(x)$ with $\frac{df}{dx}$, etc.

Any comments or corrections regarding these solutions should be immediatly directed to me:

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Good luck!

(1) **Exercise 2 :** we have two unknowns, x and y; we have the following relation between them:

$$x + y = 20$$

the relation that has to be maximized is

$$R = 2x \cdot y^2$$

Since we can only find max and min for functions containing a single variable, let's see if we can represent one of the x or y in terms of the other, using the given data. Yes! from $x + y = 20 \Rightarrow x = 20 - y$ (note: I chose x in terms of y since in R y gets squared, and squaring a sum is a tad more complex, isn't it?)

$$R = 2(20 - y)y^2 = -2y^3 + 40y^2$$

We have the function!!

Endpoints now: x and y must be nonnegative, and so one endpoint for y is 0. No other info is given, though, and so we **assume the right endpoint for y is a huge number**, state: "Assume right endpoint for y is 10000".

We have the elements, go for the solution: critical numbers are given by

$$R' = 0 \Rightarrow -6y^2 + 80y = 0 \Rightarrow$$
$$\Rightarrow 2y(-3y + 40) = 0$$

and so they are $y = 0$ and $y = 40/3 = 13.33$; attach to these 0 (already there) and 10000. Plug them into R:

$$R(0) = 0$$
$$R(13.33) = -4740.71 + 7111.07 = 2370.36$$
$$r(10000) = -2000000000000 + 4000000000 = -1996000000000$$

As you can see, the max happens in 13.33, and is equal to (approximately) 2370.6.

Date: March 9, 2003.

Note: another way to do it would be, of course, using the second derivative, and noticing it is negative, which implies the function is concave down, implying there is only one absolute max, and it must be the critical number ... same idea can be applied to all the problems below, in fact, since none of them will have conditions for right endpoint.

(2) **Exercise 4** : we have two unknowns, the length and the width; call them x and y, respectively. The relation between them is that the area equals 1000:

$$xy = 1000$$

The quantity to be minimized ("least number of ...") is the total fence length, including the inner fences, which account for 3 extra widths:

$$F = 2x + 2y + 3y = 2x + 5y$$

As usual, we need to rewrite it using a single variable; we can express y in terms of x: $y = 1000/x$, and so F becomes:

$$F = 2x + 5\frac{1000}{x} = 2x + \frac{5000}{x}$$

This the function to be minimized.

Endpoints: of course lengths must be positive, and so one endpoint is, well, 0 (there is a slight problem there). There is, again, no information concerning the right endpoint, and we assume, again, a value. State: "Assume right endpoint is 10000".

Problem with 0: since we cannot plug-in 0 into F, we resort to a little trick, and instead of 0 take a number close to 0, say 0.001 (in fact, looks like the assumption from above). These are things you can do, and in fact DO in practice (mathematically you can take any value, but practically there are bounds - mathematically you can produce millions of cars daily, but practically you cannot make more than, say, hundreds! mathematically you can use this really sub-atomic quantity, in practice you cannot measure quantities so small)

OK! Function, done! endpoints, ready! let's go for the solution.

Critical numbers: $F' = 0 \Rightarrow 2 - \frac{5000}{x^2} = 0 \Rightarrow x^2 = 5000/2 = 2500 \Rightarrow x = \pm 50$. Discard the negative value, and so we have $x = 50$. Attach 0.001 and 10000, and now plug them into F:

$$F(0.001) = 0.002 + 5000000 = 5000000.002$$
$$F(50) = 100 + 100 = 200$$
$$F(10000) = 20000 + .5 = 20000.5$$

Smallest amount: 200.

(3) **Exercise 22**: unknowns: x and y. Relation between them: area of box, no top - one square, of side x, four rectangles of sides x and y.

$$x^2 + 4xy = 192$$

Quantity to be maximized? volume:

$$V = x \cdot x \cdot y = x^2 y$$

We need to rewrite it, since it has two variables. We can express y in terms of x from the relation above (why y? because the relation above is **linear** in y, and quadratic in x ... which is easier to solve?)

$$x^2 + 4xy = 192 \Rightarrow 4xy = 192 - x^2 \Rightarrow y = \frac{192 - x^2}{4x}$$

Rewrite V:

$$V = x^2 \cdot y = x^2 \cdot \frac{192 - x^2}{4x} = \frac{1}{4}x(192 - x^2) = .25 \cdot (192x - x^3)$$
$$V = 48x - .25x^3$$

This is the function to be maximized.

Endpoints: lengths are always positive, so one endpoint is 0. Again, no information is given for the right endpoint, so go for 10000.

Critical numbers: $V' = 0 \Rightarrow 48 - .75x^2 = 0 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$; ignore the negative. Attach to 8 the 0 and 10000 (endpoints) and plug them into V:

$$V(0) = 0$$
$$V(8) = 384 - 128 = 256$$
$$V(10000) = 480000 - 250000000000 = -249999520000$$

Biggest value is 256, for $x = 8$ and $y = (192 - 64)/32 = 4$.

(4) **Exercise 8**: revenue is

$$R = pq$$

$q = 10000e^{-0.02p}$ and, well, $p = p$ give us revenue in terms of p:

$$R = p \cdot 10000e^{-0.02p}$$

This is the function to maximized.

Endpoints: smallest (practical) price is 0, of course. No word on the biggest? assume right endpoint to be 10000.

Critical numbers (product rule!):

$$R' = 0 \Rightarrow 1 \cdot 10000e^{-0.02p} + p \cdot 10000e^{-0.02p} \cdot (-0.02) = 0$$
$$10000e^{-0.02p}(1 - 0.02p) = 0 \Rightarrow p = 50$$

(the exponential is never zero)

Attach to 50 0 and 10000 and plug them into R:

$$R(0) = 0$$

$$R(50) = 50 \cdot 10000e^{-1} = 183939.72$$
$$R(10000) = 10000 \cdot 10000e^{-200} = 100000000/\text{HUGE NUMBER} = 0.000\ldots$$

Maximum is, certainly, 183939.72, obtained for $p = 50$.

(5) **exercise 12**: profit is revenue minus costs; revenue is $p \cdot q$ and cost is cost-per-unit times quantity (q); profit is, hence:

$$Pr = pq - 3 \cdot q$$

$p = 10/\sqrt{q}$ and we use this in the relation above, to express profit in terms of q alone:

$$Pr = \frac{10}{\sqrt{q}} \cdot q - 3q = 10\sqrt{q} - 3q$$

This is the function to be maximized ("greatest profit").
Endpoints: again, the classical pair 0 and 10000 (no info on right endpoint).
Critical numbers:

$$Pr' = 0 \Rightarrow 10 \cdot \frac{1}{2}\frac{1}{\sqrt{q}} - 3 = 0$$
$$\frac{5}{\sqrt{q}} - 3 = 0 \Rightarrow \sqrt{q} = 5/3 = 1.66 \Rightarrow q = 1.66^2 = 2.77$$

Attach 0 and 10000; plug all into Pr:

$$Pr(0) = 0$$
$$Pr(2.77) = 16.66 - 8.31 = 8.35$$
$$Pr(10000) = 10 \cdot 100 - 30000 = 1000 - 30000 = -29000$$

Maximum, by far, is 8.35, obtained for $q = 2.77$. Unfortunately the problem asks for the **price** that gives this max; yet we still have the relation $p = 10/\sqrt{q} = 10/1.33 = 6$.

(6) 14: Profit is, again, revenue minus costs; revenue is pq, cost is average cost (that is, cost per unit) times quantity (q). Put them together:

$$Pr = pq - \overline{c} \cdot q = \frac{50}{\sqrt{q}} \cdot q - (.5 + \frac{1000}{q}) \cdot q$$
$$Pr = 50\sqrt{q} - .5q - 1000$$

Endpoints ... you guessed it, 0 and 10000.
Critical numbers: $Pr' = 25/\sqrt{q} - .5$; $Pr' = 0 \Rightarrow 25/\sqrt{q} = .5 \Rightarrow q = 50^2 = 2500$
Plug all these values: 0, 2500, 10000 into Pr:

$$Pr(0) = -1000$$

$$Pr(2500) = 50 \cdot 50 - .5 \cdot 2500 - 1000 = 2500 - 1250 - 1000 = 250$$
$$Pr(10000) = 50 \cdot 100 - .5 \cdot 10000 - 1000 = 5000 - 5000 - 1000 = -1000$$

Maximum is 250. We get it for $q = 2500$ ("output") and price $p = 50/\sqrt{q} = 50/50 = 1$.

(7) **Exercise 41**: Total cost is $c = 3q^2 + 50q - 18q\ln(q) + 120$, but the function to be minimized is average cost, total cost - per - item:

$$\overline{c} = c/q = 3q + 50 - 18\ln(q) + 120/q$$

This is the function to be minimized.

Endpoints ... hmm, we should patent this: 0 and 10000. Problem with 0, though - cannot be plugged into ln, nor in the fraction. Assume left endpoint to be 0.001.

Critical numbers: $\overline{c}' = 3 - 18/q - 120/q^2$;set it equal to 0:

$$3 - 18/q - 120/q^2 = 0$$

Multiply both sides by q^2 to get rid of denominators:

$$3q^2 - 18q - 120 = 0 \Rightarrow 3(q^2 - 6q - 40) = 0 \Rightarrow 3(q+4)(q-10) = 0$$

We get $q = -4$ or $q = 10$; since the endpoints are positive, discard the negative value.

Attach to 10 the endpoints: 0.001 and 10000, plug them into $\overline{c}$:

$$\overline{c}(0.001) = .003 + 50 - (-2.6) + 120000 = 120052.603$$
$$\overline{c}(10) = 30 + 50 - 7.81 + 12 = 84.19$$
$$\overline{c}(10000) = 30000 + 50 - 1.95 + 0.012 = 30048.062$$

Minimum is 84.19, for $q = 10$.