SAMPLE MIDTERM 3 - SOLUTIONS

READ THIS NOTE: I will be using parenthesis "(", ")" and brackets "[", "]" interchangeably (when there are too many parenthesis involved, I will put brackets to clear the situation a bit out, so you can see where one begins and where one ends an expression).

Also, I will be using exclusively the notation y', $f'(x)$, $h'(z)$ etc for the derivative. This doesn't, certainly, mean that notations such as $\frac{dy}{dx}$, $\frac{df}{dx}$ etc are not used, or invalid. If you prefer using the latter notation, kindly replace, without any penalty, accordingly: y' with $\frac{dy}{dx}$, $f'(x)$ with $\frac{df}{dx}$, etc.

Any comments or corrections regarding these solutions should be immediatly directed to me:

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Good luck!

(1) For the function $f(x) = x + \frac{4}{x}$

(a) find all critical points for $f(x)$

$$f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 = \frac{4}{x^2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$f'(x) = DNE \Rightarrow x = 0$, but $x = 0$ is NOT in the domain - cannot plug into f.

(b) use derivatives to determine the interval(s) (if any) on which $f(x)$ is increasing and the interval(s) (if any) on which $f(x)$ is decreasing (if there are none, please say so)

draw the sign table (plug in -3, 1, 3 into f' and obtain $1-2/9 = 7/9 = +$, $1-2 = -1 = -$ and $1 - 2/9 = 7/9 = +$, respectively) (cannot plug in 0!):

x		-2		2	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

(c) use information obtained in part (a) to find the values of x for which $f(x)$ has relative max and relative min (if there are none please say so)

as can be seen from the table, -2 is a rel max and 2 is a relative min

Date: March 8, 2003.

(2) Let $f(x) = x^4 - 4x^2 + 9$.

(a) find its y-intercept

$f(0) = 9$

(b) use derivatives to find the interval(s) where $f(x)$ is increasing and where $f(x)$ is decreasing

$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$
critical numbers: $f'(x) = 0$; $x = 0$ or $x^2 = 2 \Rightarrow x = \pm\sqrt{2} = \pm 1.4142$
draw the sign table (plug in -2, -1, 1, 2, for example, into f' to find sign)

x		-1.4142		0		1.4142	
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

(c) use information obtained in part (b) to find its points of relative max and relative min

by checking the table above, both the 1.4142 and -1.4142 are relative minima, while 0 is a relative maximum

(d) use derivatives to determine the interval(s) where it is concave up and where it is concave down

$f''(x) = 12x^2 - 8$; $f''(x) = 0 \Rightarrow 12x^2 = 8 \Rightarrow x = \pm\sqrt{8/12} = \pm 0.81649$
draw the table, using for the intermediate signs -1, 0 and 1 plugged into f'' (they give you, respectively, $4 = +$, $-8 = -$ and $4 = +$)

x		$-.81649$		$.81649$	
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$		$\frown$		$\smile$

(e) where are its point(s) of inflection?

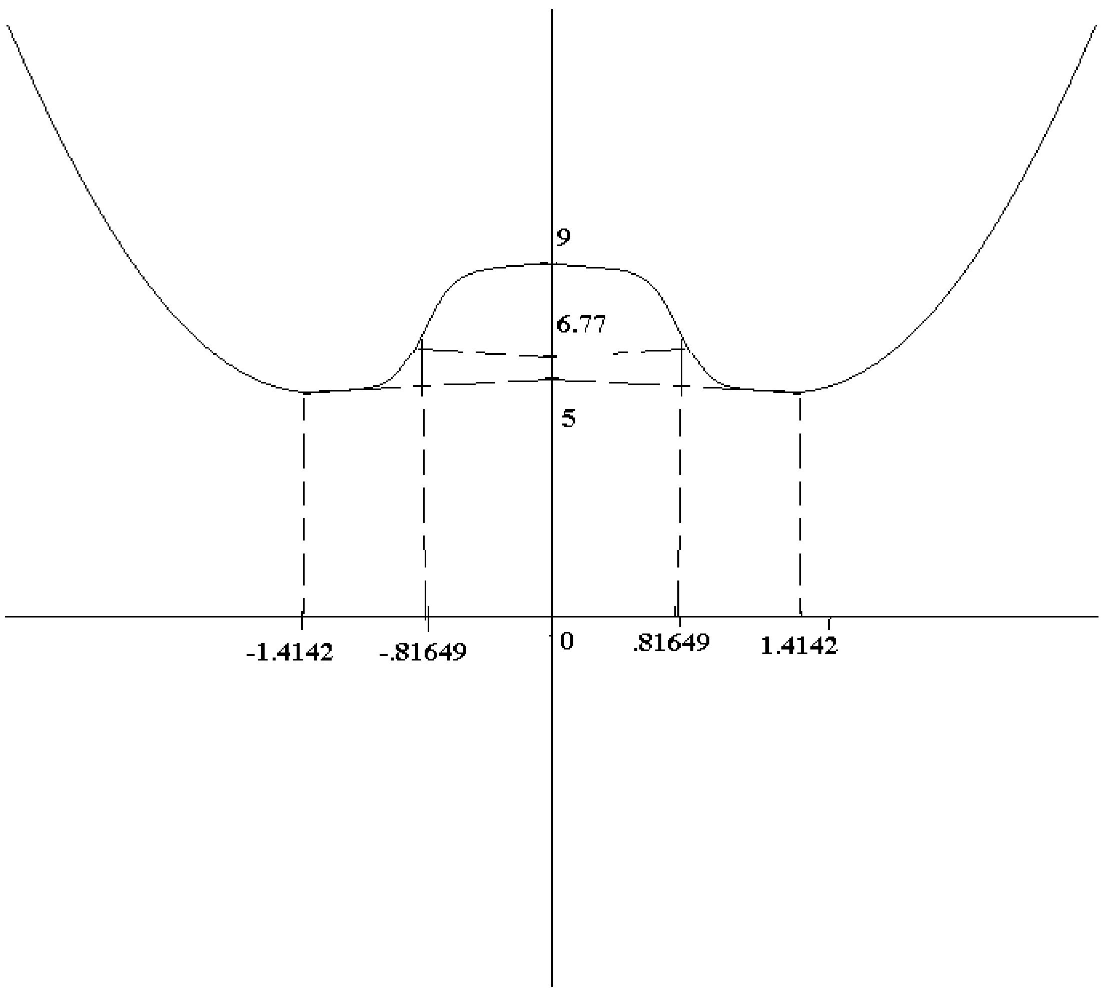
as seen from the table above, $-.81649$ and $.81649$ are both points of inflection

(f) sketch a graph of the function $f(x)$ showing all the information obtained in parts (a)-(e); labeling the points of relative extrema and also the inflection points

x		-1.4142		$-.81649$		0		$.81649$		1.4142	
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
$f(x)$	$\searrow\smile$		$\smile\nearrow$		$\nearrow\frown$		$\frown\searrow$		$\searrow\smile$		$\smile\nearrow$

$$f(-1.4142) = 4 - 8 + 9 = 5$$
$$f(-.81649) = .44 - 2.66 + 9 = 6.77$$
$$f(0) = 9$$
$$f(.81649) = .44 - 2.66 + 9 = 6.77$$
$$f(1.4142) = 4 - 8 + 9 = 5$$

(I'm including a very approximate graph here - I'm lacking the dexterity to draw these things with the mouse, bear with me ... make corrections as necessary; you can always check your work against the calculator ... it's just that you cannot simply use the calculator to sketch graphs)



(3) For the graph of the function

$$f(x) = \frac{2x^2 - 8}{x + 9}$$

(a) find the x-intercept(s) and the y-intercept if there are any. If there are none please say so.

x-intercepts are where $f(x) = 0$; a fraction is zero when the numerator is zero: $2x^2 - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
y-intercept: $f(0) = -8/9 = -.8888$

(b) find all its horizontal asymptote(s). If there are none please say so (Show all work)

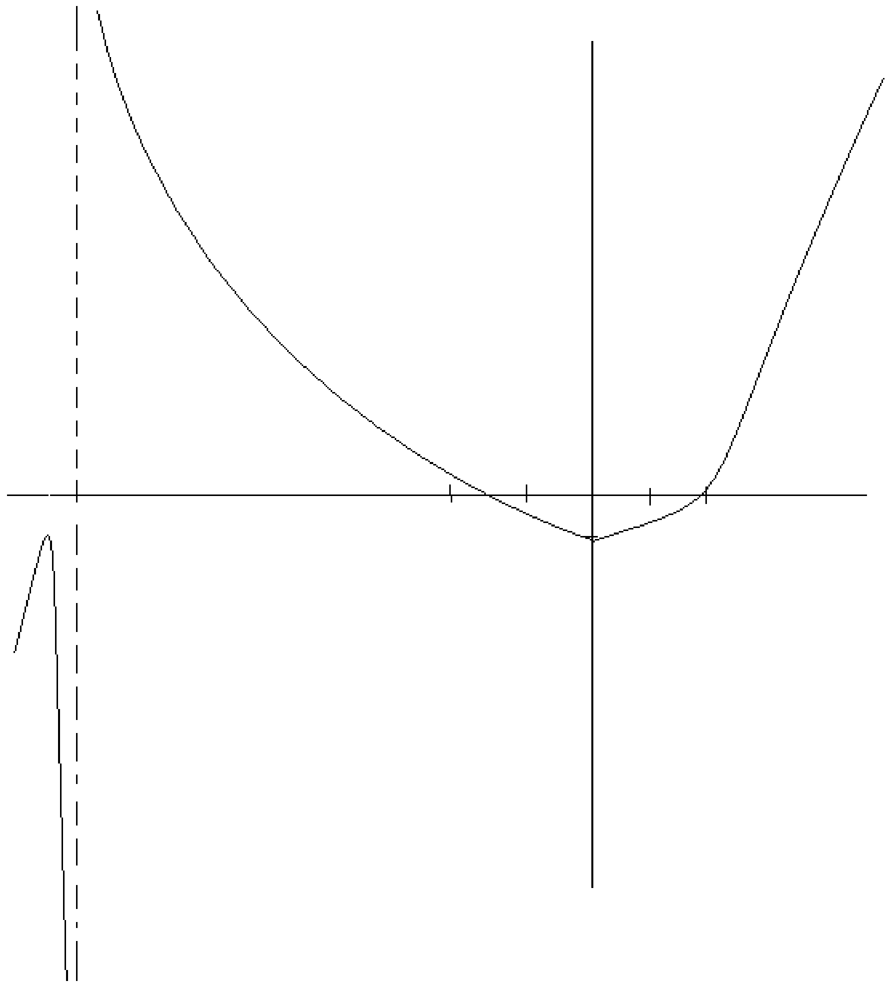
f is a fraction, so there is a chance; yet the power in the numerator (2) is bigger than the power in the denominator (1), so we DO NOT have a horizontal asymptote.

$$\lim_{x \to \infty} \frac{2x^2 - 8}{x + 9} = \lim_{x \to \infty} \frac{2x^2}{x} =$$
$$= \lim_{x \to \infty} 2x = \infty$$

(c) find all its vertical asymptote(s). If there are none please say so (show all work)

f is a fraction, let's look at its denominator: $x + 9 = 0 \Rightarrow x = -9$ - that's where we have a vertical asymptote, $x = -9$ (the numerator does not get canceled by -9, $2 \cdot 81 - 8 = 154 \neq 0$
look at values left - right of vertical asymptote: $f(-9.5) =$ (something positive)/$(-.5)$ =negative; $f(-8.5) =$ (something positive)/$(.5)$ = positive. So, the graph plunges down to the left of -9 and goes high to the right.

(d) sketch a graph of the function $f(x)$ using information obtained in parts (a) and (b) above
plot $(-2, 0)$, $(2, 0)$, $(0, -.88)$ and draw the vertical line through $x = -9$; connect these (also, make sure the graph stays below the x-axis to the left of -9, as we have no x-intercepts there. (I'm including a very approximate graph here - I'm lacking the dexterity to draw these things with the mouse, bear with me ... make corrections as necessary



(4) (a) use the second derivative test to find point(s) of relative max and relative min for the function

$$f(x) = 2x^3 + 5x^2 - 4x + 15$$

find critical numbers: $f'(x) = 6x^2 + 10x - 4 = 2(3x^2 + 5x - 2) = 2(3x - 1)(x + 2)$; $f'(x) = 0$ implies $x = 1/3$ or $x = -2$
compute second derivative: $f''(x) = 12x + 10$ and now test the two critical numbers against this second derivative:
$f''(.333) = 4 + 10 = 14 = + \Rightarrow$ "$\smile$" $\Rightarrow$ relative minimum
$f''(-2) = -24 + 10 = -14 = - \Rightarrow$ "$\frown$" $\Rightarrow$ relative maximum

(b) find the absolute max and absolute min that occur for the function in part (a) ($f(x) = 2x^3 + 5x^2 - 4x + 15$) in the interval $[0, 3]$.

we already have the critical numbers, namely .333 and -2; yet only .333 is INSIDE the interval given, so we ignore -2.
we have three numbers to analyze: 0, .333, 3; plug these numbers into the actual function (remember, ABSOLUTE MAX/MIN problems are DIFFERENT from relative max/min problems; here after we find the critical numbers - and the endpoints - we plug them into f, not the derivatives)
$f(0) = 15$, $f(.333) = .074 + .555 - 1.333 + 15 = 14.296$; $f(3) = 54 + 45 - 12 + 15 = 102$
absolute max is of course 102, for $x = 3$; absolute min is 14.296 for $x = .333$

(5) (a) use derivatives only to determine the interval(s) where the graph of

$$f(x) = x^4 - 4x^3 + 15$$

is concave up and where it is concave down.

we need the second derivative: $f'(x) = 4x^3 - 12x^2$;

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

$f''(x) = 0$ implies either $x = 0$ or $x = 2$; let's draw the sign table:

x		0		2	
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$		$\frown$		$\smile$

(b) use the information obtained in part (a) to find its points of inflection

change of sign: both in 0 and in 2, both are points of inflection (since f is NOT a fraction, there is no doubt these point ARE on the graph)