SAMPLE FINAL EXAM - SOLUTIONS

READ THIS NOTE: I will be using parenthesis "(", ")" and brackets "[", "]" interchangeably (when there are too many parenthesis involved, I will put brackets to clear the situation a bit out, so you can see where one begins and where one ends an expression).

Also, I will be using exclusively the notation y', $f'(x)$, $h'(z)$ etc for the derivative. This doesn't, certainly, mean that notations such as $\frac{dy}{dx}$, $\frac{df}{dx}$ etc are not used, or invalid. If you prefer using the latter notation, kindly replace, without any penalty, accordingly: y' with $\frac{dy}{dx}$, $f'(x)$ with $\frac{df}{dx}$, etc.

Any comments or corrections regarding these solutions should be immediatly directed to me:

cosmin@math.ohio-state.edu

Good luck!

(1) Limits.

(a) Cannot plug in 0 (obtain 0/0; must cancel the common "0").

$$\lim_{x\to 0}\frac{(x+3)^2-9}{x} = \lim_{x\to 0}\frac{x^2+6x+9-9}{x} =$$
$$= \lim_{x\to 0}\frac{x^2+6x}{x} = \lim_{x\to 0}\frac{x(x+6)}{x} = \lim_{x\to 0}(x+6) =$$
$$= 6$$

(b) Cannot plug in 4 (obtain 16/0). We cannot cancel the 0 in the denominator, so it must be an infinity (vertical asymptote ...). To the right of 4 (4^+) plug in 4.5: $16/(-3.75)$ =negative. So

$$\lim_{x\to 4^+}\frac{4x}{16-x^2} = -\infty$$

(c) We CAN plug in 4 (no trouble in the denominator)

$$\lim_{t\to 4}\frac{t^2-2t-8}{9t} = \frac{0}{36} = 0$$

Date: March 15, 2003.

(d) Cannot plug in 0 (obtain 0/0; must cancel the common "0"). Fractions involved, so we need common denominator in the numerator:

$$\lim_{h\to 0}\frac{\frac{4}{5+h}-\frac{4}{5}}{h}=\lim_{h\to 0}\frac{\frac{4\cdot 5}{(5+h)\cdot 5}-\frac{4\cdot(5+h)}{5\cdot(5+h)}}{h}=$$
$$=\lim_{h\to 0}\frac{20-(20+4h)}{h\cdot 5\cdot(5+h)}=\lim_{h\to 0}\frac{-4h}{h\cdot 5\cdot(5+h)}=$$
$$=\lim_{h\to 0}\frac{-4}{5\cdot(5+h)}=-\frac{4}{25}$$

(e) Limit to infinity, so drop lowest powers in the numerator and denominator, respectively:

$$\lim_{x\to\infty}\frac{6x^6-7x+9}{9-8x^6}=\lim_{x\to\infty}\frac{6x^6}{-8x^6}=$$
$$=\lim_{x\to\infty}\frac{6}{-8}=-\frac{6}{8}=-.75$$

(f) Limit to infinity, drop the lowest powers; yet we have a problem! an exponential in the numerator; treat it as a polynomial of huge degree (it is "bigger" than any polynomial).

$$\lim_{t\to-\infty}\frac{9+7e^t}{8-9t}=\lim_{t\to-\infty}\frac{7e^t}{-9t}$$

Since we have a ration between an exponential and t, again, remember that e^t is like a polynomial of huge degree, and so it is much bigger than t, so the ratio grows indefinitely. Pay attention to the sign, though!

$$\lim=-\infty$$

(g) Cannot plug in 1 (obtain 0/0; must cancel common 0). Since we know that 1 makes both polynomials 0, they both factor out with $(x-1)$ as factor:

$$\lim_{x\to 1}\frac{x^2+4x-5}{2x^2+x-3}=\lim_{x\to 1}\frac{(x-1)(x+5)}{(x-1)(2x+3)}=$$
$$=\lim_{x\to 1}\frac{x+5}{2x+3}=\frac{6}{5}=1.2$$

(2) (a) Limit to the right uses only the formula for $x > 1$ ... scan the function, and notice that only the first formula, $3 - 2x$, fits the bill. So:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - 2x) = 3 - 2 \cdot 1 = 1$$

(b) Limit to the left uses the formula for $x < 1$, but not "too far"; the formula for $0 \leq x \leq 1$ is the one to use: x^2.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$

(c) For $x > 1$ the function IS continuous ($(3 - 2x)$ is a polynomial); for $0 < x < 1$ it IS continuous (x^2); for $x < 0$ we may have problems with the denominator, which is 0 for $x = 9$, but 9 is NOT less than 0, so the function IS continuous as well. We only have to check the "articulations", $x = 0$ and $x = 1$, the points where the function glues the separate formulas.
In 1 we already checked the two lateral limits, and they came to be the same value, 1; the VALUE in 1 is given by x^2 ($0 \leq x \leq 1$ for x^2) and is also 1. Since the LATERAL LIMITS and the VALUE are all equal, the function IS continuous in 1.
In 0, same issue. If you check lateral limits (basically plug 0 in both x^2 and $\frac{4x}{x-9}$) they are the same, 0; the value in 0 is what you get for x^2, which is also 0. Same as above, these three values are equal so the function IS continuous in 0.
Conclusion: the function is continuous EVERYWHERE.

(3) Differentiate both left and right, treat y as a function; mind the many products, where you have to use PRODUCT RULE, and the power of y, where you have to use the chain rule (and a last note: dy/dx is the same thing as y', and I'm going to use the latter notation):

$$(5x^4 - 6x^5y + 8y^7)' = (17)'$$

$$(5x^4)' - (6x^5y)' + (8y^7)' = 0$$

$$(5x^4)' - (6x^5y)' + (8y^7)' = 0$$

$$5 \cdot 4x^3 - (6 \cdot 5x^4 \cdot y + 6x^5 \cdot y') + 8 \cdot 7y^6 \cdot y' = 0$$

$$20x^3 - 30x^4 \cdot y - 6x^5 \cdot y' + 56y^6 \cdot y' = 0$$

$$-6x^5 \cdot y' + 56y^6 \cdot y' = -20x^3 + 30x^4 \cdot y$$

$$(-6x^5 + 56y^6)y' = -20x^3 + 30x^4 \cdot y$$

$$y' = \frac{-20x^3 + 30x^4 \cdot y}{-6x^5 + 56y^6}$$

(4) As a rule of thumb, "DO NOT SIMPLIFY" means to stop when there are no more derivatives to compute (that is, you needn't distribute, multiply etc)

(a) Product rule (and a power rule on the way):

$$f'(u) = [(7u^5-1)^8(u^2+9u+1)]' = [(7u^5-1)^8]'(u^2+9u+1) + (7u^5-1)^8(u^2+9u+1)' =$$
$$= 8(7u^5-1)^7 \cdot (7 \cdot 5u^4) \cdot (u^2+9u+1) + (7u^5-1)^8 \cdot (2u+9)$$

(b) Quotient rule:

$$y' = (\frac{x+7}{x^5-8})' = \frac{(x+7)'(x^5-8)-(x^5-8)'(x+7)}{(x^5-8)^2} =$$
$$= \frac{(x^5-8)-5x^4 \cdot (x+7)}{(x^5-8)^2}$$

(c) Product rule, and exponential derivation:

$$y' = (x^5 \cdot e^{-9x})' = (x^5)' \cdot e^{-9x} + x^5 \cdot (e^{-9x})' =$$
$$= 5x^4 \cdot e^{-9x} + x^5 \cdot e^{-9x} \cdot (-9)$$

(d) First simplify the formula, by using properties of ln:

$$y = \ln[(7x+6)^6(4x-3)^9e^{8x}] = \ln[(7x+6)^6] + \ln[(4x-3)^9] + \ln[e^{8x}] =$$
$$= 6\ln(7x+6) + 9\ln(4x-3) + 8x\ln(e) = 6\ln(7x+6) + 9\ln(4x-3) + 8x$$

Now differentiate:

$$y' = (6\ln(7x+6) + 9\ln(4x-3) + 8x)' = (6\ln(7x+6))' + (9\ln(4x-3)' + (8x)' =$$
$$= 6 \cdot \frac{1}{7x+6} \cdot 7 + 9 \cdot \frac{1}{4x-3} \cdot 4 + 8$$

(e) Power rule for the first term (mind the fact that both the 5 and the x are INSIDE the power) and simple power differentiation for the second term (5 is just a coefficient):

$$y' = [(5x)^{9/5} + 5(x)^{9/5}]' = [(5x)^{9/5}]' + [5(x)^{9/5}]' =$$
$$= \frac{9}{5}(5x)^{\frac{9}{5}-1} \cdot 5 + 5 \cdot \frac{9}{5}x^{\frac{9}{5}-1}$$

(f) the first term can be simplified, by using properties of ln; the second one, though, cannot, since the power of 9 applies to the ln itself, is NOT INSIDE the ln

$$f(x) = \ln(4x^9) + [\ln(4x)]^9 = \ln(4) + \ln(x^9) + (\ln(4x))^9 =$$
$$= \ln(4) + 9\ln(x) + (\ln(4x))^9$$

Differentiate now:

$$f'(x) = [\ln(4) + 9\ln(x) + (\ln(4x))^9]' = (\ln(4))' + (9\ln(x))' + [(\ln(4x))^9]' =$$
$$= 0 + 9 \cdot \frac{1}{x} + 9 \cdot (\ln(4x))^8 \cdot \frac{1}{4x} \cdot 4$$

(The last term is a combination of 3 functions: $x \to 4x = y \to \ln(y) = z \to z^9 = f(x)$, and the derivative must take all of them into account, first the power, then the ln, and lastly the coefficient)

(g) Again, use properties of ln:

$$y = \ln[x^{(7x-5)}] = (7x-5) \cdot \ln(x)$$

Differentiate (product rule):

$$y' = [(7x-5) \cdot \ln(x)]' = (7x-5)' \cdot \ln(x) + (7x-5) \cdot (\ln(x))' =$$
$$= 7\ln(x) + (7x-5) \cdot \frac{1}{x}$$

(5) The fraction can change sign when the numerator OR denominator is 0: 5, -4 and 9.

Draw a table, and plug in values "outside" and between values (for instance, you can use: -5, 0, 6, 10):

x		-4		5		9	
sign	$-$	0	$+$	0	$-$	DNE	$+$

We need negative values, so we use the intervals $(-\infty, -4)$ and $(5, 9)$. Since it's an "less than OR EQUAL" look at endpoints as well: -4 and 5 are OK, 9 is not. Answer: (change the parenthesis into bracket at the appropriate endpoints) $(-\infty, -4]$ and $[5, 9)$.

(6)

$$y' = e^{6x+4} \cdot 6 = 6e^{6x+4}$$
$$y'' = 6(e^{6x+4} \cdot 6) = 36e^{6x+4}$$

(7)

$$\ln(y) = \ln[\frac{(8x^5 - 9x + 5)^5 \cdot (x^4 - 2x + 1)^7}{(x^8 - 6x + 1)}]$$
$$\ln(y) = \ln[(8x^5 - 9x + 5)^5] + \ln[(x^4 - 2x + 1)^7] - \ln(x^8 - 6x + 1)$$
$$\ln(y) = 5\ln(8x^5 - 9x + 5) + 7\ln(x^4 - 2x + 1) - \ln(x^8 - 6x + 1)$$

Differentiate both sides:

$$[\ln(y)]' = [5\ln(8x^5 - 9x + 5) + 7\ln(x^4 - 2x + 1) - \ln(x^8 - 6x + 1)]'$$
$$\frac{1}{y} \cdot y' = 5\frac{1}{8x^5 - 9x + 5} \cdot (8 \cdot 5x^4 - 9) + 7\frac{1}{x^4 - 2x + 1} \cdot (4x^3 - 2) - \frac{1}{x^8 - 6x + 1} \cdot (8x^7 - 6)$$

Multiply by $y = \frac{(8x^5-9x+5)^5 \cdot (x^4-2x+1)^7}{(x^8-6x+1)}$ both sides (in fact, actually multiply by y the left hand side, as to cancel the y in the denominator, and with the expression in x the right hand side):

$$y' = \frac{(8x^5 - 9x + 5)^5 \cdot (x^4 - 2x + 1)^7}{(x^8 - 6x + 1)} \cdot [5\frac{1}{8x^5 - 9x + 5} \cdot (8 \cdot 5x^4 - 9) +$$
$$+ 7\frac{1}{x^4 - 2x + 1} \cdot (4x^3 - 2) - \frac{1}{x^8 - 6x + 1} \cdot (8x^7 - 6)]$$

(8) (a) Slope of the tangent is given by the derivative of $y = x^6 - 5x + 4$ at the point $(1, 0)$: $y' = 6x^5 - 5$; plug in the x-value, 1: $m = y'(1) = 6 \cdot 1^5 - 5 = 6 - 5 = 1$.

(b) we have all the elements, the slope and the point: $m = 1$, $(x, y) = (1, 0)$ (we can completely ignore the function originally used in (a)):

$$y - 0 = 1 \cdot (x - 1)$$

(9) $f'(x) = 2 \cdot 3x^2 - 3 \cdot 2x - 36 = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-2)(x-3)$; $f'(x) = 0 \Rightarrow x = 2$ or $x = 3$. These are the critical numbers. Attach to these the endpoints: -5 and 5. One last check: both 2 and 3 are inside the interval $[-5, 5]$ - yes!

$$f(-5) = -250 - 75 + 180 + 8 = -137$$
$$f(2) = 16 - 12 - 72 + 8 = -60$$
$$f(3) = 54 - 27 - 108 + 8 = -73$$
$$f(5) = 250 - 75 - 180 + 8 = 3$$

The biggest value is 3, at $x = 5$; smallest is -137, at $x = -5$.

Comment: that 2^{nd} derivative test hint is kind of bogus, ignore it. If you see something like this in the exam, ask the proctor if that's really needed to be used.

(10) We need to compute the second derivative, and only check the sign FOR THAT.

$$y' = -\frac{1}{4}\cdot 4x^3 + \frac{9}{2}\cdot 2x + 2 = -x^3 + 9x + 2$$
$$y'' = -3x^2 + 9 = -3(x^2 - 3)$$

$y'' = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$.

Draw the table, plug in values outside, between these roots (for example, -2, 0, 2) into y'' and only look at the sign

x		$-\sqrt{3}$		$\sqrt{3}$	
y''	$-$	0	$+$	0	$-$
concavity	$\frown$		$\smile$		$\frown$

The sign for the second derivative changes at both $x = -\sqrt{3}$ and $x = \sqrt{3}$; since you can plug these numbers back into the original y, they are on the graph, and so they're both points of inflection.

(11) (a) y-intercept: $f(0) = -4/-1 = 4$

(b) x-intercepts - we need the x where the NUMERATOR is 0: $x^2 - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

(c) For horizontal asymptotes we need

$$\lim_{x\to\pm\infty} \frac{x^2 - 4}{x^2 - 1} =$$
$$= \lim_{x\to\pm\infty} \frac{x^2}{x^2} = \lim_{x\to\pm\infty} 1 = 1$$

H.A. is $y = 1$

(d) For vertical asymptotes we need the x where the DENOMINATOR is 0: $x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$.

(e) Note that the derivative IS ALREADY COMPUTED! read the problems for such simplifying informations!

The denominator is squared, so we only check the sign of the numerator, and we draw a table for it - plug in -2 and 2 into y' and only write the sign (note that the numerator is 0 only when $x = 0$):

x		0	
y'	$-$	0	$+$
	$\searrow$		$\nearrow$

Decrease on $(-\infty, 0)$ and increase on $(0, \infty)$.

(f) as you can see, we only have a relative minimum, at $x = 0$ (and $f(0) = 4$), and no relative max.

(g) Same as for the first derivative, the second derivative is already computed. For this one, however, the denominator is NOT squared, and so it must be taken into account! The numerator of the second derivative is never 0, but the denominator is 0 in $x = \pm 1$. Plug in -2, 0 and 2 into y'' and only write the sign in the following table:

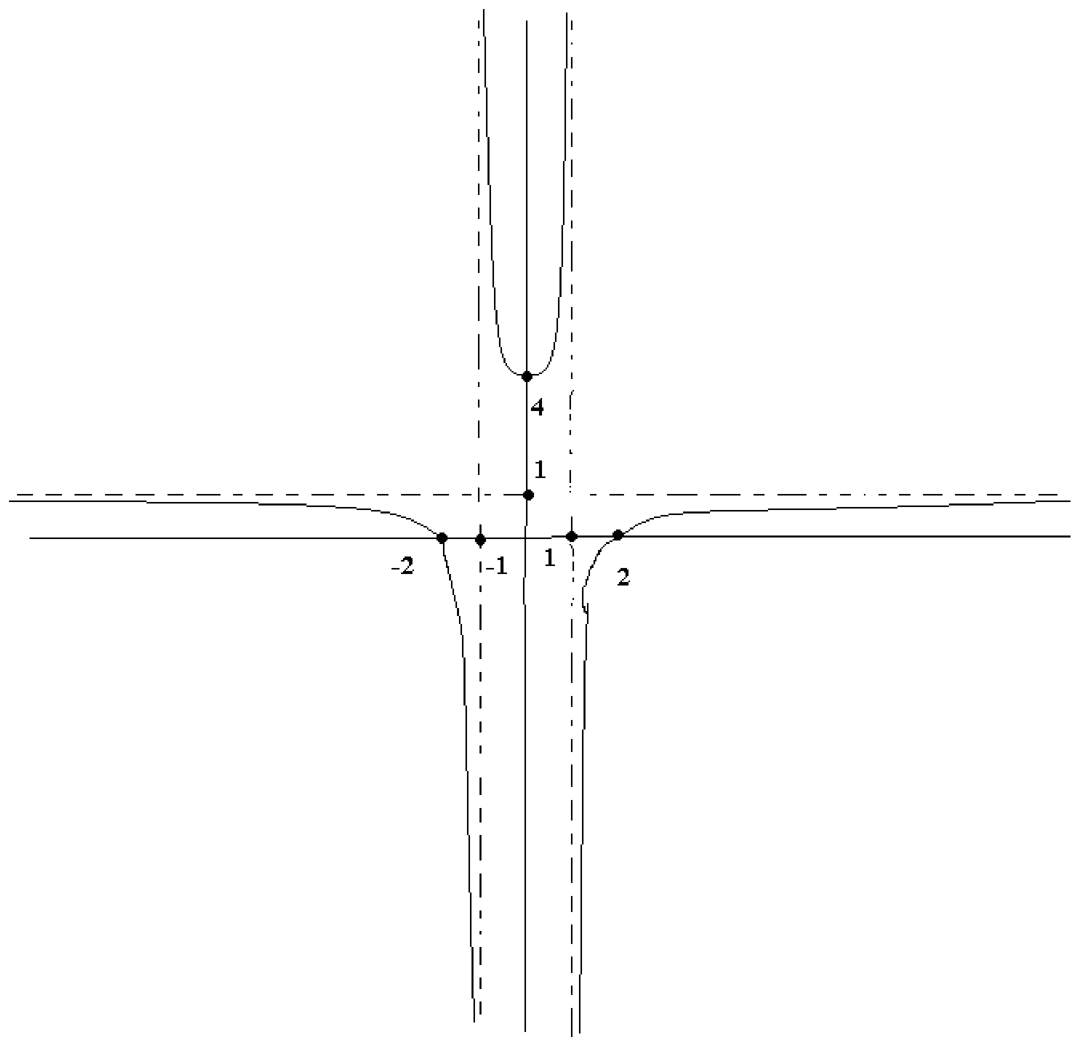
x		-1		1	
y''	$-$	0	$+$	0	$-$
	$\frown$		$\smile$		$\frown$

(h) change of sign only in $x = \pm 1$. But try plugging in ± 1 into f! it is not valid! so we have NO POINT OF INFLECTION!

(i) The above tables can be combined into a bigger one:

x		-1		0		1	
y'	$-$	$-$	$-$	0	$+$	$+$	$+$
y''	$-$	0	$+$	$+$	$+$	0	$-$
	$\frown\searrow$		$\searrow\smile$		$\smile\nearrow$		$\nearrow\frown$

Here a sketch:



(12) Total cost c is the product between the average cost (cost-per-unit) and the quantity:

$$c = \overline{c} \cdot q$$

So, the total cost function looks like this for our problem:

$$c = (2q^2 - 36q + 210 - \frac{210}{q}) \cdot q = 2q^3 - 36q^2 + 210q - 200$$

We have to find the minimum for the total cost, in the interval $[2, 10]$. We have the function, we have the endpoints.

Let's go for critical numbers:

$$c' = 2 \cdot 3q^2 - 36 \cdot 2q + 210 = 6q^2 - 72q + 210 = 6(q^2 - 12q + 35) = 6(q-5)(q-7)$$

$c' = 0 \Rightarrow q = 5$ or $q = 7$. Both are inside the interval $[2, 10]$, so both are usable.

The candidates: 5, 7, 2, 10 (critical numbers and endpoints)

$$c(2) = 16 - 144 + 420 - 200 = 92$$
$$c(5) = 250 - 900 + 1050 - 200 = 200$$
$$c(7) = 686 - 1764 + 1470 - 200 = 192$$
$$c(10) = 2000 - 3600 + 2100 - 200 = 300$$

Smallest value: 92. Production should be fixed at $q = 2$.

(13) We have two unknowns. Call them x and y. The relation between them is that their sum is 60, and so $x + y = 60$. We have to find the maximum for their product, $P = xy$. We can maximize only if we have a single variable provided; we do need to express one of the variables in terms of the other: $y = 60 - x$ for example.

$$P(x) = x \cdot (60 - x) = 60x - x^2$$

We have the function; do we have any endpoints? certainly $x \geq 0$ (the numbers are supposed to be non negative). Also, $x \leq 60$ (if it's bigger, then y will become negative!)

Endpoints are, hence, 0 and 60.

Critical numbers: $P' = 0 \Rightarrow 60 - 2x = 0 \Rightarrow x = 30$. We have 0, 30 and 60 as candidates.

$$P(0) = 0$$
$$P(30) = 900$$
$$P(60) = 0$$

Biggest value is 900 - the two numbers are $x = 30$ and $y = 60 - x = 60 - 30 = 30$.