SOLUTIONS CHAPTER 14.3

MATH 132 WI01

12. $y = x^4 - 8x^2 - 6$. We need the sign of the second derivative.

$$y' = 4x^3 - 8 * 2x = 4x^3 - 16x$$

$$y'' = 4 * 3x^2 - 16 = 12x^2 - 16$$

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$y'' = 0 \to 12x^2 - 16 = 0 \to 12x^2 = 16 \to x^2 = \frac{16}{12} \to x^2 = \frac{4}{3} \to x = \pm\sqrt{\frac{4}{3}} = \pm 1.1547$. Check sign of y'' between these two values and outside:

plug in -2 in y'', and we get $12 * (-2)^2 - 16 = 12 * 4 - 16 = 48 - 16 = 32$, positive, so y is ($\smile$) **concave up** in the interval $(-\infty, -1.1547)$;

plug in 0, and we get $12 * 0^2 - 16 = -16$, negative, hence y is ($\frown$) **concave down** in the interval $(-1.1547, 1.1547)$;

plug in 2, and we get $12 * 2^2 - 16 = 48 - 16 = 32$, positive, hence y is ($\smile$) **concave up** again. Since at ± 1.1547 the sign changes it means that they are the points of inflection.

y'':	$-\infty$	$(+)$	-1.1547	$(-)$	1.1547	$(+)$	∞.
y:	$-\infty$	$\smile$	-1.1547	$\frown$	1.1547	$\smile$	∞.

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21. $y = \frac{1}{30}x^6 - \frac{7}{12}x^4 + 5x^2 + 2x - 1 \to y' = \frac{1}{30} * 6x^5 - \frac{7}{12} * 4x^3 + 5 * 2x + 2 = \frac{1}{5}x^5 - \frac{7}{3}x^3 + 10x + 2 \to y'' = \frac{1}{5} * 5x^4 - \frac{7}{3}3x^2 + 10 = x^4 - 7x^2 + 10 = (x^2 - 5)(x^2 - 2)$. Solve $y'' = 0$; it's either $x^2 - 5 = 0 \to x^2 = 5 \to x = \pm\sqrt{5} \to x = \pm 2.236$ or $x^2 - 2 = 0 \to x^2 = 2 \to x = \pm\sqrt{2} \to x = \pm 1.4142$. We'll have to check sign of y'' in (-∞,-2.236), (-2.236,-1.4142), (-1.4142,1.4142), (1.4142,2.236), (2.236,∞). Best values would be, respectively, -3, -2, 0, 2, 3, and they will give us:

$y''(-3) = ((-3)^2 - 5)((-3)^2 - 2) = (9-5)(9-2) = 4*7 = 28 \to \textit{positive} \to (\smile)$;

$y''(-2) = ((-2)^2 - 5)((-2)^2 - 2) = (4-5)(4-2) = (-1)*2 = -2 \to \textit{negative} \to (\frown)$;

$y''(0) = (0-5)(0-2) = (-5)*(-2) = 10 \to \textit{positive} \to (\smile)$;

$y''(2) = (4-5)(4-2) = -2 \to \textit{negative} \to (\frown)$;

$y''(3) = (9-5)(9-2) = 28 \to \textit{positive} \to (\smile)$.

y'':	$-\infty$	$(+)$	-2.236	(-)	-1.4142	$(+)$	1.4142	(-)	2.236	$(+)$	∞
y:	$-\infty$	$\smile$	-2.236	$\frown$	-1.4142	$\smile$	1.4142	$\frown$	2.236	$\smile$	∞

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27. $y = \frac{21x+40}{6(x+3)^2} = \frac{1}{6} * \frac{21x+40}{(x+3)^2}$.

$$y' = \frac{1}{6} * \frac{21*(x+3)^2 - (21x+40)*2(x+3)*1}{((x+3)^2)^2}$$

$$y' = \frac{1}{6} * \frac{21(x^2+6x+9) - (42x^2+126x+80x+240)}{(x+3)^4}$$

$$y' = \frac{1}{6} * \frac{-21x^2-80x-51}{(x+3)^4}$$

$$y'' = \frac{1}{6} * \frac{(-42x-80)*(x+3)^4 - (-21x^2-80x-51)*4(x+3)^3*1}{((x+3)^4)^2)}$$

$$y'' = \frac{1}{6} * \frac{(x+3)^3 * [(-42x-80)*(x+3)+84x^2+320x+204]}{(x+3)^8}$$

$$y'' = \frac{1}{6} * \frac{(x+3)^3 * (-42x^2-126x-80x-240+84x^2+320x+204)}{(x+3)^8}$$

$$y'' = \frac{1}{6} * \frac{(x+3)^3 * (42x^2+114x-36)}{(x+3)^8}$$

$$y'' = \frac{1}{6} * \frac{(x+3)^3 * 6 * (7x^2+19x-6)}{(x+3)^8}$$

$$y'' = \frac{(x+3)^3 * (7x-2)(x+3)}{(x+3)^8} = \frac{(x+3)^4 * (7x-2)}{(x+3)^8}$$

Check the sign of y'' - so first find zeroes: $x+3=0 \to x=-3$ and $7x-2 = 0 \to 7x = 2 \to x = \frac{2}{7}$. But ...$y$ is NOT DEFINED in -3, so only candidate for inflexion point is $\frac{2}{7}$. Notice, at the same time, that the denominator is a square (see the pattern?), but also the $(x+3)$ that appears in the numerator is at an even power, so it is positive as well, regardless of what x is. Hence ... let's check sign of y'', which just means checking sign for $7x-2$.

If $x < \frac{2}{7} \to$ plug 0 in $y'' \to y''(0) = -2 \to$ *negative* $\to (\frown)$.

If $x > \frac{2}{7} \to$ plug 1 in $y'' \to y''(1) = 7-2 = 5 \to$ *positive* $\to (\smile)$. We notice that $\frac{2}{7}$ **is an inflexion point**, since we have a change from concave down to concave up.

y'':	$-\infty$	$(-)$	$\frac{2}{7}$	$(+)$	∞
y:	$-\infty$	$\frown$	$\frac{2}{7}$	$\smile$	∞

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55. $y = 4x^2 - x^4$. Let's find intercepts: x-intercepts we get from $4x^2 - x^4 = 0 \to x^2(4-x^2) = 0 \to x^2(2-x)(2+x) = 0 \to x = 0, x = \pm 2$; y-intercepts we get by pluging 0 in y, $y(0) = 4*0^2 - 0^4 = 0$. So our graph must pass through $(0,0), (-2,0), (2,0)$.

I. intervals of increase, decrease; relative extrema: $y' = 4*2x - 4x^3 = 8x - 4x^3 = 4x(2-x^2)$. Zeroes for y' are 0, $2 - x^2 = 0 \to 2 = x^2 \to x = \pm\sqrt{2} = \pm 1.4142$. By plugging in values (I suggest -2, -1,1 and 2) in y' we get:

y':	$-\infty$	$(+)$	-1.4142	$(-)$	0	$(+)$	1.4142	$(-)$	∞
y:	$-\infty$	$\nearrow$	-1.4142	$\searrow$	0	$\nearrow$	1.4142	$\searrow$	∞

As we can see, ± 1.4142 and 0 are all relative extrema - ± 1.4142 being relative maxima and 0 being relative minimum.

II. intervals of concavity upwards, downwards; inflexion points: $y'' = 8 - 4 * 3x^2 = 8 - 12x^2$. Zeroes for y'' are $8 - 12x^2 = 0 \rightarrow 8 = 12x^2 \rightarrow x^2 = \frac{8}{12} \rightarrow x^2 = \frac{2}{3} \rightarrow x = \pm\sqrt{\frac{2}{3}} = \pm 0.8164$. By plugging in values (again, let me suggest - -1, 0, 1) in y'' we get:

y'': $-\infty$	$(-)$	-0.8164	$(+)$	0.8164	$(-)$	∞
y: $-\infty$	$\frown$	-0.8164	$\smile$	0.8164	$\frown$	∞

Both ± 0.8164 are inflexion points, since concavity changes at these.

Now, before doing the last step, that is sketching the graph, let's compute the value of y at all the important points we found (relative extrema, inflexion points).

$y(-1.4142) = 4$

$y(1.4142) = 4$

$y(0) = 0$

$y(-0.8164) = 0.8888$

$y(0.8164) = 0.8888$

Recall $y(-2) = y(2) = 0$ (x-intercepts).

Now armed with all this data, let's draw the graph.

58. $y = (x-1)^2(x+2)^2$. Find x-intercepts: $y = 0 \rightarrow (x-1)^2(x+2)^2 = 0 \rightarrow x = 1, x = -2$. Find y-intercepts: $y(0) = (-1)^2(2)^2 = 4$. So our graph must pass through $(1,0), (-2,0), (0,4)$.

I. intervals of increase, decrease; relative extrema: $y' = 2(x-1) * (x+2)^2 + (x-1)^2 * 2(x+2) = (x-1)(x+2)(2x+4+2x-2) = (x-1)(x+2)(4x+2)$. Zeroes for y' - critical numbers that is - are $x - 1 = 0 \rightarrow x = 1$, $x + 2 = 0 \rightarrow x = -2$, $4x + 2 = 0 \rightarrow 4x = -2 \rightarrow x = -\frac{2}{4} = -0.5$. By plugging in numbers (suggestion: -3, 0, 0.75) we get:

y':	$-\infty$	$(-)$	-2	$(+)$	-0.5	$(-)$	1	$(+)$	∞
y:	$-\infty$	$\searrow$	-2	$\nearrow$	-0.5	$\searrow$	1	$\nearrow$	∞

We see that -2 and 1 are relative minima, and -0.5 is a relative maximum.

II. intervals of concavity upwards, downwards; inflexion points: $y'' = [(x-1)(x+2)(4x+2)]' = [(x^2+x-2)(4x+2)]' = (2x+1)*(4x+2)+(x^2+x-2)*4 = 8x^2+4x+4x+2+4x^2+4x-8 = 12x^2+12x-6 = 6(2x^2+2x-1)$. Zeroes for y'' will be then obtained from $2x^2+2x-1=0 \to x = \frac{-2\pm\sqrt{4-4*2*(-1)}}{2*2} = \frac{-2\pm\sqrt{12}}{4}$ which gives us $x = 0.366$ and $x = -1.366$. By plugging in numbers (suggest: -2, 0, 1) in y'' we get:

y'':	$-\infty$	$(+)$	-1.366	$(-)$	0.366	$(+)$	∞
y:	$-\infty$	$\smile$	-1.366	$\frown$	0.366	$\smile$	∞

We see that both numbers are inflexion points, since they both experience change of concavity.

Compute the values of y in all the important numbers we've got:

$y(1) = y(-2) = 0$ (they are also the x-intercepts, remember).

$y(-0.5) = (-0.5-1)^2(-0.5+2)^2 = 5.0625$

$y(-1.366) = 2.25$

$y(0.366) = 2.25$

Having now all this data, let's draw the graph.