FINAL EXAM

MATH 132 WINTER 2000

1. Compute the following limits (leave the answers in fractions)

(a) $$\lim_{x\to 2} \frac{x^2-4x+4}{x^2+x-6}$$ (8 points)

Proof. plug in $x = 2$ ... we get $\frac{0}{0}$! so we need to simplify it first:

$$\frac{x^2-4x+4}{x^2+x-6} = \frac{(x-2)(x-2)}{(x-2)(x+3)} = \frac{x-2}{x+3}$$

so the limit becomes

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2+x-6} = \lim_{x\to 2} \frac{x-2}{x+3} = \frac{2-2}{2+3} = 0$$

□

(b) $$\lim_{x\to -\infty} \frac{x^2-9x+3}{x-3-7^2}$$ (8 points)

Proof. it's limit to $-\infty$ so we consider only the biggest power in the numerator and the biggest power in the denominator:

$$\lim_{\to -\infty} \frac{x^2-9x+3}{x-3-7x^2} = \lim_{x\to -\infty} \frac{x^2}{-7x^2} =$$
$$= \lim_{x\to -\infty} \frac{1}{-7} = -\frac{1}{7}$$

□

2. Find the derivatives of the following function (do not simplify)

(a) $$f(t) = (5t^7+3t^2-5)(8t^5-5t+2)$$ (8 points)

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Proof. use product rule:

$$[(5t^7 + 3t^2 - 5)(8t^5 - 5t + 2)]' =$$
$$= (5t^7 + 3t^2 - 5)' \cdot (8t^5 - 5t + 2) + (5t^7 + 3t^2 - 5) \cdot (8t^5 - 5t + 2)' =$$
$$= (5 \cdot 7t^6 + 3 \cdot 2t)(8t^5 - 5t + 2) + (5t^7 + 3t^2 - 5)(8 \cdot 5t^4 - 5)$$

□

(b) $$y = \frac{4 - x^3}{5x^2 + 7}$$ (8 points)

Proof. use quotient rule:

$$[\frac{4 - x^3}{5x^2 + 7}]' = \frac{(4 - x^3)' \cdot (5x^2 + 7) - (4 - x^3) \cdot (5x^2 + 7)'}{(5x^2 + 7)^2} =$$
$$= \frac{(-3x^2)(5x^2 + 7) - (4 - x^3)(5 \cdot 2x)}{(5x^2 + 7)^2}$$

□

(c) $$f(x) = x^7 \ln(5x + 3)$$ (8 points)

Proof. again product rule, but for the ln also use chain rule:

$$[x^7 \ln(5x + 3)]' = 7x^6 \cdot \ln(5x + 3) + x^7 \cdot \frac{1}{5x + 3} \cdot 5$$

□

(d) $$f(x) = (\frac{x - 2}{x - 7})^7$$ (8 points)

Proof. chain rule and then quotient rule for the inside function:

$$[(\frac{x - 2}{x - 7})^7]' = 7(\frac{x - 2}{x - 7})^6 \cdot \frac{1 \cdot (x - 7) - (x - 2) \cdot 1}{(x - 7)^2}$$

□

3. Let $f(x) = 11x^2 + 5$. Find

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

by finding $f(x + h)$, substitute and simplify. (8 points)

Proof.

$f(x+h) = 11(x+h)^2 + 5 = 11(x^2 + 2xh + h^2) + 5 = 11x^2 + 22xh + 11h^2 + 5$

and so

$$\frac{f(x+h) - f(x)}{h} = \frac{(11x^2 + 22xh + 11h^2 + 5) - (11x^2 + 5)}{h} =$$

$$= \frac{11x^2 + 22xh + 11h^2 + 5 - 11x^2 - 5}{h} = \frac{22xh + 11h^2}{h} = \frac{h(22x + 11h)}{h} =$$

$$= 22x + 11h$$

Hence

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}(22x + 11h) = 22x$$

□

4. Let

$$f(x) = \begin{cases} \frac{5x-1}{2x-3}, \text{if} & x \leq 0 \\ \frac{7x}{9x-8}, \text{if} & x > 0 \end{cases}$$

Find:

(a) $\lim_{x\to 0^+} f(x)$ (4 points)

Proof. since $x \to 0^+$ we only have $x > 0$, which means we only use the second case:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{7x}{9x-8} = \frac{0}{-8} = 0$$

□

(b) $\lim_{x\to 0^-} f(x)$ (4 points)

Proof. same trick ... only now we have $x \to 0^-$ and so $x < 0$ - first case:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{5x-1}{2x-3} = \frac{-1}{-3} = \frac{1}{3}$$

□

4. did you notice that there are two "number 4" problems? weird :)

Find the equation of the tangent line to the graph of $y = -11x^3 + 9x + 8$ at the point $(0, 8)$ (8 points)

Proof. The considered point is $(0, 8)$, which means that $x = 0$ and $y = 8$. Find slope of tangent line in this point, by computing $y' = -11 \cdot 3x^2 + 9 = -33x^2 + 9$ and plugging in $x = 0$: *slope* $= 9$. Hence the equation is:

$$y - 8 = slope(x - 0) = 9x \iff y = 9x + 8$$

□

5. The total profit function for a product is given by

$$P(x) = 95 + 10x^2 - .5x^3$$

where x is the number of units sold.

(a) What function gives the marginal profit? (6 points)

Proof. it's simply the derivative of P, namely:

$$Marginal\_Profit = P'(x) = 10 \cdot 2x - .5 \cdot 3x^2 = 20x - 1.5x^2$$

□

(b) What it the marginal profit when 100 units are sold? (2 points)

Proof. plug 100 in the above formula; get $20 \cdot 100 - 1.5 \cdot 100^2 = 2000 - 15000 = -13000$ (a good question is now, how come the answer is negative? well, in case the formula given for P above is correct, it just means that at 100 the company producing those units looses at this point with this rate ... bankruptcy?) □

6. Use derivatives to find absolute extrema of the function $y = f(x) = (x + 1)^3$ in the interval $[-2, 3]$ (8 points)

Proof. We need critical values and endpoints. Start with the later: -2 and 3. For the critical values we need the derivative $y' = 3(x + 1)^2$ which gives $y' = 0 \Rightarrow 3(x + 1)^2 = 0 \Rightarrow x = -1$ (worthy to mention the fact that -1 is indeed between -2 and 3!)

Let's check out the values f for these 3 numbers:

- $f(-2) = (-2 + 1)^3 = (-1)^3 = -1$
- $f(-1) = (-1 + 1)^3 = 0^3 = 0$
- $f(3) = (3 + 1)^3 = 4^3 = 64$

Biggest value (and hence ABSOLUTE MAX) is 64 (for $x = 3$), while smallest value (ABSOLUTE MIN) is -1 (for $x = -1$). □

7. For the graph of the function $f(x) = x^3 + 3x^2 - 9x + 15$

(a) Find its y-intercept (3 points)

Proof. plug in $x = 0$: $y = f(0) = 15$ $\square$

(b) use derivatives to find the interval(s) where $f(x)$ is increasing and where $f(x)$ is decreasing (5 points)

Proof. $f' = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3)$. Draw the sign table, and point out -3 and 1, where f' becomes 0. You get:

x		-3		1	
$f'(x)$	$+$	0	$-$	0	$+$
	$\nearrow$	MAX	$\searrow$	MIN	$\nearrow$

so intervals of increase are $(-\infty, -3) \cup (1, \infty)$ and interval of decrease is $(-3, 1)$. $\square$

(c) Use information obtained in part (b) to find its point(s) of rel max and rel min (3 points)

Proof. as pointed out in the above table, rel max is at $x = -3$ ($f(-3) = -27 + 3 \cdot 9 - 9 \cdot (-3) + 15 = 33$) and rel min is at $x = 1$ ($f(1) = 1 + 3 - 9 + 15 = 10$) $\square$

(d) use derivatives to determine the interval(s) where it is concave up and where it is concave down (5 points)

Proof. same as for increase/decrease, we need now compute $f'' = 6x + 6 = 6(x + 1)$ and find its sign; draw another sign table, pointing out -1, where f'' becomes zero.

x		-1	
$f'(x)$	$-$	0	$+$
		$\frown$	$INFLECTION\_POINT$ $\smile$

Interval of concavity upwards is $(-1, \infty)$, concavity downwards is $(-\infty, -1)$. $\square$

(e) what are its point(s) of inflection? (3 points)

Proof. since at $x = -1$ the function changes concavity it means that at $x = -1$ (and $y = f(-1) = -1 + 3 + 9 + 15 = 26$) we have an inflection point (as stated in the sign table) $\square$

(f) sketch a graph of the function $f(x)$ using information obtained in parts (a)-(e) above (5 points)

Proof. well ... give it a try (you can, of course, rely on what the calculator shows you, too!) □

8. Find $f(x)$ given $f'(x) = 7x^4 - 9x + 5$ and $f(0) = 5$ (8 points)

Proof.

$$f(x) = \int f'(x)\,dx = \int 7x^4 - 9x + 5\,dx = 7\frac{x^5}{5} - 9\frac{x^2}{2} + 5x + C$$

we need to find the C, so use the second part of the hypothesis, namely $f(0) = 5$

$$7\frac{0^5}{5} - 9\frac{0^2}{2} + 5 \cdot 0 + C = 5 \Rightarrow C = 5$$

so the function f is actually

$$f(x) = 7\frac{x^5}{5} - 9\frac{x^2}{2} + 5x + 5$$

□

9. Evaluate the following integrals and simplify

(a) $$\int \frac{5\,dx}{9\sqrt{x^5}}$$ (8 points)

Proof. rewrite the integral:

$$\int \frac{5\,dx}{9\sqrt{x^5}} = \frac{5}{9}\int \frac{dx}{x^{\frac{5}{2}}} =$$

$$= \frac{5}{9}\int x^{-\frac{5}{2}}\,dx = \frac{5}{9}\frac{x^{-\frac{5}{2}+1}}{-\frac{5}{2}+1} + C =$$

$$= \frac{5}{9}\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + C$$

□

(b) $$\int \frac{\ln(x)}{x}$$ (12 points)

Proof. since it's a combination on a polynomial with ln - we have to use substitution, and the candidate is $u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx$ - which actually suits our problem, rewritten as:

$$\int x \cdot \frac{1}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln(x))^2}{2} + C$$

□

10. Evaluate

$$\int_1^3 \frac{x^2 - 7x - 5}{x}\,dx \qquad\qquad (8points)$$

Proof. compute the antiderivative first:

$$\int \frac{x^2 - 7x - 5}{x}\,dx = \int x - 7 - \frac{5}{x}\,dx = \frac{x^2}{2} - 7x - 5\ln(x)$$

and so the definite integral is:

$$\int_1^3 \frac{x^2 - 7x - 5}{x}\,dx = (\frac{x^2}{2} - 7x - 5\ln(x))|_1^3 =$$

$$= (\frac{3^2}{2} - 7 \cdot 3 - 5\ln(3)) - (\frac{1^2}{2} - 7 \cdot 1 - 5\ln(1)) =$$

$$= 4.5 - 21 - 5\ln(3) - 0.5 + 7 + 5 \cdot 0 = -10 - 5\ln(3)$$

□

11. Evaluate (using tables) $\int \sqrt{4x^2 + 9}\,dx$. Write down the formula number you are using for this question (12 points)

Proof. in the book (Appendix C) the formula needed is #23; in the list given in the sample, it's #10. The only thing is, we need to arrange it a bit (it needs $\sqrt{u^2 + a^2}$ while we have $\sqrt{4x^2 + 9}$. One can do it in two ways: substitution or algebra.

Substitution approach: $u^2 = 4x^2 \Rightarrow u = 2x; du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$, so then we have

$$\int \sqrt{4x^2 + 9}\,dx = \int \sqrt{u^2 + 9}\frac{1}{2}\,du = \frac{1}{2}\int \sqrt{u^2 + 9}\,du =$$

$$= \frac{1}{2} \cdot \frac{1}{2}(u\sqrt{u^2 + 9} + 9\ln(|u + \sqrt{u^2 + 9}|) + C =$$

$$= \frac{1}{2} \cdot \frac{1}{2}(2x\sqrt{4x^2 + 9} + 9\ln(|2x + \sqrt{4x^2 + 9}|) + C$$

Algebra approach: $\sqrt{4x^2+9} = \sqrt{4(x^2+\frac{9}{4}} = \sqrt{4}\sqrt{x^2+\frac{9}{4}} = 2\sqrt{x^2+\frac{9}{4}}$. Hence we have:

$$\int \sqrt{4x^2+9}\,dx = \int 2\sqrt{x^2+\frac{9}{4}}\,dx = 2\int \sqrt{x^2+\frac{9}{4}}\,dx =$$
$$= 2\cdot\frac{1}{2}(x\sqrt{x^2+\frac{9}{4}}+\frac{9}{4}\ln(|x+\sqrt{x^2+\frac{9}{4}}|)+C$$

□

12. Solve the inequality

$$\frac{(2x-1)(3-x)}{(5+x)} \leq 0 \qquad \text{(8 points)}$$

Proof. draw the sign table, pointing out the x's for which either the numerator or denominator is zero (-5, $\frac{1}{2} = 0.5$ and 3):

$$\begin{array}{c|ccccccc} x & & -5 & & 0.5 & & 3 & \\ fraction & + & DNE & - & 0 & + & 0 & - \end{array}$$

where to find the sign you can plug in:

- $-6 : \frac{(-12-1)(3+6)}{5-6} = \frac{-13\cdot 9}{-1} \rightarrow +$
- $0 : \frac{(-1)\cdot 3}{5} \rightarrow -$
- $1 : \frac{(2-1)(3-1)}{5+1} = \frac{1\cdot 2}{6} \rightarrow +$
- $4 : \frac{(8-1)(3-4)}{5+4} = \frac{7\cdot(-1)}{9} \rightarrow -$

The solution is hence: $(-5, 0.5] \cup [3, \infty)$ (-5 must be ignored, since the fraction doesn't exist for $x = -5$, but all the other endpoints have to be considered, since the it's a $\leq$) □

13. Find the area of the region bounded by $y = x^3 - x$ and the x-axis (12 points)

Proof. Find where the graph of $x^3 - x$ intersects the x-axis - that is, solve $y = 0$.

$$x^3 - x = 0 \iff x(x^2-1) = 0 \iff x(x-1)(x+1) = 0 \Rightarrow$$

$\Rightarrow x = 0$, $x = -1$ and $x = 1$. The smallest value is where we start integrating, the biggest value is where we stop, and the intermediate values (if any) will just break the interval, and we compute the integrals on each part. For our problem we have: integrate from -1 to 1, and break this interval into two parts, $[-1, 0]$ and $[0, 1]$. Let's compute the integrals now:

$$\int_{-1}^{0} x^3 - x\,dx = \frac{x^4}{4} - \frac{x^2}{2}|_{-1}^{0} = 0 - (\frac{1}{4} - \frac{1}{2}) = 0.25$$

$$\int_{0}^{1} x^3 - x\,dx = \frac{x^4}{4} - \frac{x^2}{2}|_{0}^{1} = (\frac{1}{4} - \frac{1}{2}) - 0 = -0.25$$

The second result is negative, since the graph must be **below** the x-axis; hence we just remove the negative sign, and add the two numbers. We get:

$$Total\_Area = 0.25 + 0.25 = 0.5$$

□

14. Evaluate the integral

$$\int \frac{x+1}{(x^2+2x+2)^3}\,dx \qquad \text{(8 points)}$$

Proof. substitution, and take u the polynomial with the biggest power, namely $u = x^2+2x+2 \Rightarrow du = 2x+2\,dx = 2(x+1)\,dx \Rightarrow (x+1)\,dx = \frac{1}{2}\,du$. Replace now everything with u in the integral and get:

$$\int \frac{1}{u^3} \cdot \frac{1}{2}\,du = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\frac{u^{-2}}{-2} + C =$$

$$= \frac{1}{2}\frac{(x^2+2x+2)^{-2}}{-2} + C$$

□

15. If the demand equation for a product is given by $p = q^2 - 10q + 25$ and the supply equation is $p = q^2 + q + 3$

(12 points)

(a) find the equilibrium point

Proof. it's a regular setup - p in terms of q - so draw the graphs accordingly. For the equilibrium we have:

$$\begin{array}{rcl} q^2 - 10q + 25 & = & q^2 + q + 3 \\ \underline{-q^2} & & \underline{-q^2} \\ \\ -10q + 25 & = & q + 3 \\ \underline{+10q} & & \underline{10q} \\ \\ 25 & = & 11q + 3 \\ \underline{-3} & & \underline{-3} \\ \\ 22 & = & 11q \\ \underline{\frac{1}{11}} & & \underline{\frac{1}{11}} \\ \\ 2 & = & q \end{array}$$

Hence $q = 2 \Rightarrow p = q^2 - 10q + 25 = 4 - 20 + 25 = 9$ □

(b) find the consumer's surplus

Proof. The Consumer's Surplus the "highest" area, hence (**look at the graph at this point!** it's Integral, from **0** to the equilibrium **q**, of the **Demand function** minus equilibrium **p**; so:

$$\int_0^2 (q^2 - 10q + 25) - 9\,dq = \int_0^2 q^2 - 10q + 16\,dq = \frac{q^3}{3} - 10\frac{q^2}{2} + 16q|_0^2 =$$

$$= (\frac{2^3}{3} - 5 \cdot 2^2 + 16 \cdot 2) - 0 = 2.66 - 20 + 32 = 14.66$$

□