MIDTERM II, FORM A

MATH 132 WI00

I. Find the derivatives of the following functions.

(a) $y = \ln(2x+1)$ (5 points)

Answer:

$$y' = \frac{1}{2x+1} \cdot 2$$

(b) $y = \ln(\sqrt[4]{\frac{1+x^2}{1-x^2}})$ (10 points)

Answer:

$$y = \ln(\sqrt[4]{\frac{1+x^2}{1-x^2}}) = \ln((\frac{1+x^2}{1-x^2})^{\frac{1}{4}}) =$$

$$= \frac{1}{4}\ln(\frac{1+x^2}{1-x^2}) = \frac{1}{4}(\ln(1+x^2) - \ln(1-x^2))$$

and hence

$$y' = \frac{1}{4}(\frac{1}{1+x^2} \cdot 2x - \frac{1}{1-x^2} \cdot (-2x))$$

(c) $y = e^{5x^2-11x+12}$ (10 points)

Answer:

$$y' = e^{5x^2-11x+12} \cdot (5 \cdot 2x - 11) = e^{5x^2-11x+12}(10x-11)$$

II. $\int (x^3 - \frac{1}{x^4} + 2)\, dx$ (10 points)

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Answer:

$$\int (x^3 - \frac{1}{x^4} + 2)\,dx = \int (x^3 - x^{-4} + 2)\,dx =$$

$$= \frac{x^{3+1}}{3+1} - \frac{x^{-4+1}}{-4+1} + 2x + C = \frac{x^4}{4} - \frac{x^{-3}}{-3} + 2x + C$$

III. Given $y' = -x^2 + 4x + 1$ and $y(3) = 8$, find y. (10 points)

Answer:

$$y = \int (-x^2 + 4x + 1)\,dx = -\frac{x^3}{3} + 4\frac{x^2}{2} + x + C =$$

$$= -\frac{x^3}{3} + 2x^2 + x + C$$

But $y(3) = 8 \Rightarrow$

$$y(3) = -\frac{3^3}{3} + 2 \cdot 3^2 + 3 + C = 8 \Rightarrow -9 + 18 + 3 + C = 8 \Rightarrow C = 8 - 12 = -4$$

Hence $y = -\frac{x^3}{3} + 2x^2 + x - 4$.

IV. Let $f(x) = 2x^3 - 3x^2 - 36x - 50$.
(a) Find its y-intercept. (5 points)

Answer: Plug in 0 for x; $f(0) = 2 \cdot 0^3 - 3 \cdot 0^2 - 36 \cdot 0 - 50 = -50$, so the y-intercept is $(0, -50)$.

(b) Use derivatives to find the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing (5 points)

Answer:

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x-3)(x+2)$$

hence the critical numbers are -2 and 3 (6 is never zero, of course!). If you draw the table and check the sign you should get: (+) for $(-\infty, -2) \cup (3, \infty)$,

$(-)$ for $(-2,3)$, so, in other words, **INCREASING** on $(-\infty,-2)\cup(3,\infty)$ and **DECREASING** on $(-2,3)$.

(c) Use the information obtained in part (b) to find points of relative maxima and relative minima (5 points)

Answer: We only have to check the critical numbers; so, -2 is a relative **maximum**, because the function increases up to it and decreases afterwards, and 3 is a relative minimum, since the function decreases up to it and increases afterward.

(d) Use derivatives to determine the intervals on which graph is concave up and on which it is concave down (5 points)

Answer:For this we need the second derivative:

$$f''(x) = 12x - 6 = 6(2x-1)$$

and $f''(x) = 0$ if $x = \frac{1}{2}$. Draw the table and notice that the second derivative is negative on $(-\infty, \frac{1}{2})$ and positive on $(\frac{1}{2}, \infty)$; in other words, the function f is **concave down** on $(-\infty, \frac{1}{2})$ and **concave up** on $(\frac{1}{2}, \infty)$.

(e) What are its point(s) of inflection? (5 points)

Answer: Since the second derivative has a single zero, we only have to check this one; but we see that the function actually **changes** concavity (goes from concave down to concave up - or the second derivative changes from negative to positive), so $\frac{1}{2}$ indeed is a point of inflection.

V. (a) Use the second derivative test to find relative max and relative min for the function $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2 + 9$ (5 points)

Answer: First we need the critical numbers: $f'(x) = x^3 + x^2 - 2x = x(x^2+x-2) = x(x-1)(x+2) = 0 \Rightarrow x = 0$, $x = 1$ and $x = -2$. Next we test these numbers with the second derivative: $f''(x) = 3x^2 + 2x - 2$.

$$f''(0) = 3 \cdot 0^2 + 2 \cdot 0 - 2 = -2 \Rightarrow f \text{ looks like } (\frown) \Rightarrow 0 \text{ is a relative max}$$

$$f''(1) = 3 \cdot 1^2 + 2 \cdot 1 - 2 = 3 + 2 - 1 = 4 \Rightarrow f \text{ looks like } (\smile) \Rightarrow$$
$$\Rightarrow 1 \text{ is a relative min}$$

$$f''(-2) = 3 \cdot (-2)^2 + 2 \cdot (-2) - 2 = 6 \Rightarrow f \text{ looks like } (\smile) \Rightarrow$$
$$\Rightarrow -2 \text{ is a relative min}$$

(b) Find when absolute max and absolute min occur for the function of part (a) in the interval $[0, 2]$ (5 points)

Answer: Here the approach is different; we need to find the critical numbers (but we have them from above!) and take also the endpoints of the given interval, 0 and 2. Hence we have -2,0,1 and 0, 2. But strike -2, since it's not between 0 and 2, so in summary we only need 0, 1 and 2 (0 was pointed out twice, once for being a critical number and once for being an endpoint). Now we check the abs. max/min with f!

$$f(0) = \frac{0^4}{4} + \frac{0^3}{3} - 0^2 + 9 = 9$$
$$f(1) = \frac{1^4}{4} + \frac{1^3}{3} - 1^2 + 9 = 8\frac{7}{12}$$
$$f(2) = \frac{2^4}{4} + \frac{2^3}{3} - 2^2 + 9 = 11\frac{2}{3}$$

The biggest result is for $x = 2$, so that's the absolute max; the smallest result is for $x = 1$, so that's the absolute min.

VI. The demand equation for a monopolist's product is $p = 2700 - q^2$, where p is the **price per unit** (in dollars), when q units are demanded. (a) Find the value of q for which the revenue is maximum. (b) What is the maximum revenue? (10 points)

Answer: Revenue$=p \times q = (2700 - q^2)q = 2700q - q^3$.

(a) To find the max we need the derivative of revenue: $Revenue' = 2700 - 3q^2$, and we need to find the critical numbers, namely $2700 - 3q^2 = 0 \Rightarrow 3(900 - q^2) = 0 \Rightarrow q^2 = 900 \Rightarrow q = \pm 30$. But we cannot have negative quantities, so we only need to discuss $q = 30$. Draw the table for (Revenue'), and the result is that Revenue increases between 0 and 30 (we only consider q positive) and decreases afterwards (in the table you have positive up to 30 and negative afterwards), so 30 is a relative max. (**always check if the critical number you got - even if it's the only one! - is what the problem asks it to be, max or min**)

(b) Max revenue=Revenue(30) $= 2700 \cdot 30 - 30^3 = 24300$

VII. Given

$$y = \frac{x^2}{x+3}$$
$$y' = \frac{x^2 + 6x}{(x+3)^2}$$
$$y'' = \frac{18}{(x+3)^3}$$

use the derivatives given above to find where the graph of this function is (a) concave up, (b) concave down, (c) where y has points of inflection (10 points)

Answer: For concavity we need the second derivative. Well, we have it (we also have the first derivative, but here it's of no use, so just ignore it). The second derivative has no zeroes (its numerator is 18, so it cannot be zero), but there's a point where it doesn't exist, namely at -3. Draw the table, and notice that the sign of y'' is negative for $(-\infty, -3)$ and positive for $(-3, \infty)$.

(a) concave up = sign of y'' is positive = $(-3, \infty)$.

(b) concave down = sign of y'' is negative = $(-\infty, -3)$.

(c) there's only one candidate for point of inflection, namely -3; but before we jump to the conclusion that this IS a point of inflection we must make sure we can plug in -3 in the original function, y: $y(-3) = \frac{(-3)^2}{-3+3} = \frac{9}{0} = DNE$; so actually we cannot use -3. Since there are no other candidates, the conclusion is that there actually are NO POINTS OF INFLECTION.