MIDTERM II, FORM A

MATH 132 WI01

I. Find the derivatives of the following functions.
(a) $y = e^{2+\sqrt{x}}$ (5 points)

Answer:

$$y' = e^{2+\sqrt{x}} \cdot (\frac{1}{2}x^{-\frac{1}{2}})$$

(b) $y = (2x+5)^2 \ln[(3x+5)^5]$ (5 points)

Answer: We have that $\ln[(3x+5)^5] = 5\ln(3x+5)$ so then we have to differentiate

$$y = (2x+5)^2 5 \cdot \ln(3x+5)$$

whose derivative is, hence

$$2(2x+5) \cdot 2 \cdot 5\ln(3x+5) + (2x+5)^2 \cdot 5 \cdot \frac{1}{3x+5} \cdot 3$$

(c) $y = \ln[(x^3+7x-5)^2]$ (5 points)

Answer: Use same trick as before, and take the square outside the ln as a coefficient of 2

$$y' = (2\ln(x^3+7x-5))' = 2\frac{1}{x^3+7x-5} \cdot (3x^2+7)$$

(what, no II ? weird ...)

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III. Given $y' = -4x^3 + 6$ and $y(1) = 2$, find y. (10 points)

Answer:

$$y = \int (-4x^3 + 6)\, dx = -4\frac{x^4}{4} + 6x + C =$$

$$= -x^4 + 6x + C$$

But $y(1) = 2 \Rightarrow$

$$y(1) = -1^4 + 6 \cdot 1 + C = 2 \Rightarrow -1 + 6 + C = 2 \Rightarrow C = 2 - 5 = -3$$

Hence $y = -x^4 + 6x - 3$.

IV. Find the following integral

$$\int (x+1)^2\, dx$$

Answer: $\int x^2 + 2x + 1\, dx = \frac{x^3}{3} + 2\frac{x^2}{2} + x + C$

(or, you can use some newer knowledge, and use subtitution, or the trick I talked about in class, namely the one in which, when having $x + constant$, any constant, you can treat it as if it were just x, only replacing it by $x + constant$ ... in our case, it's as if we need $\int x^2\, dx = \frac{x^3}{3} + C$, so the actual result is $\int (x+1)^2\, dx = \frac{(x+1)^3}{3} + C$)

V. Find the following integral

$$\int (8\sqrt{x} + \frac{1}{x^5} - 11)\, dx$$

Answer: Rewrite, using only powers of x:

$$\int 8x^{\frac{1}{2}} + x^{-5} - 11\, dx = 8\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-5+1}}{-5+1} - 11x + C$$

VI. Let $f(x) = \frac{x^5}{100} - \frac{x^4}{20}$.

Note: before doing anything, let's rewrite the function in a more friendlier way:

$$fx) = \frac{1}{100}x^5 - \frac{1}{20}x^4$$

(a) Find its y-intercept. (3 points)

Answer: Plug in 0 for x; $f(0) = \frac{1}{100}0^5 - \frac{1}{20}0^4 = 0$, so the y-intercept is $(0, 0)$.

(b), (c), (d) critical points, intervals of increase/decrease, rel max and min (5, 5 and 3 points, respectively)

$$\begin{aligned} f'(x) &= (\frac{1}{100}x^5 - \frac{1}{20}x^4)' = \\ &= \frac{1}{100} \cdot 5x^4 - \frac{1}{20} \cdot 4x^3 = \\ &= \frac{1}{20}x^4 - \frac{1}{5}x^3 = \frac{1}{20} \cdot x^3(x-4) \end{aligned}$$

hence the critical numbers are 0 and 4 ($\frac{1}{20}$ is never zero, of course!). The critical **points** are, hence, $(0, f(0)) = (0, 0)$ and $(4, f(4)) = (4, \frac{4^5}{100} - \frac{4^4}{20})$.

If you draw the sign-table and check the sign you should get:

x		0		4	
$f'(x)$	$+$	0	$-$	0	$+$

$\nearrow$ MAX $\searrow$ MIN $\nearrow$

Hence your intervals of increase: $(-\infty, 0) \cup (4, \infty)$ and decrease: $(0, 4)$.

Relative max: $(0, 0)$; relative min: $(4, \frac{4^5}{100} - \frac{4^4}{20})$

(e) Use derivatives to determine the intervals on which graph is concave up and on which it is concave down (5 points)

Answer:For this we need the second derivative:

$$\begin{aligned} f''(x) &= \frac{1}{20} \cdot 4x^3 - \frac{1}{5} \cdot 3x^2 = \\ &= \frac{1}{5}x^3 - \frac{3}{5}x^2 = \frac{1}{5}x^2(x-3) \end{aligned}$$

and $f''(x) = 0$ if $x = 0$ or $x = 3$. Draw the signs-table and get:

x			0		3	
$f''(x)$	$-$		0	$-$	0	$+$
	$\frown$			$\frown$	$infl\_point$	$\smile$

Hence concave up: $(3, \infty)$ and concave down: $(-\infty, 0) \cup (0, 3)$

(f) What are its point(s) of inflection? (3 points)

Answer: in 0 there's no change in concavity - keeps being down both left and right; hence 0 is NOT a point of inflection. 3, on the other hand , is; the point is then $(3, f(3)) = (3, \frac{3^5}{100} - \frac{3^4}{20})$.

(g) Sketch the graph ...

Answer: try it out! (again, you can definitely rely on your calculator)

VII. For the function $f(x) = x^3 - 3x^2 + 10$.
(10 points)
(a) Use the second derivative test to find relative max and relative min occur for this function

Answer: We need, in any case, the first derivative, to compute critical values.

$$f'(x) = 3x^2 - 3 \cdot 2x = 3x^2 - 6x = 3x(x - 2)$$

Hence critical values are 0 and 2.
Use now second derivative test: compute second derivative, plug in 0 and 2 and check the signs.

$$f''(x) = 3 \cdot 2x - 6 = 6x - 6 = 6(x - 1)$$

- $f''(0) = 6(0 - 1) = -6 \rightarrow$ -, hence $\frown$, hence MAX
- $f''(2) = 6(2 - 1) = 6 \rightarrow +$, hence $\smile$, hence MIN

(b) Find when absolute max and absolute min occur for this function in the interval $[0, 4]$ (5 points)

Answer: Here the approach is different; we need to find the critical numbers (but we have them from above!) and take also the endpoints of the given interval, 0 and 4. Hence we have 0,2 and 0, 4. Both 0 and 2 are in our interval (the endpoints surely are), but 0 is repeated, and we'll only use it once. In summary we only need 0, 2 and 4 (again, 0 was pointed out twice, once for being a critical number and once for being an endpoint). Now we check the abs. max/min with f!

$$f(0) = 10$$

$$f(2) = 2^3 - 3 \cdot 2^2 + 10 = 8 - 12 + 10 = 6$$

$$f(4) = 4^3 - 3 \cdot 4^2 + 10 = 64 - 48 + 10 = 26$$

The biggest result is for $x = 4$, so that's the absolute max; the smallest result is for $x = 2$, so that's the absolute min.

VII. The demand equation for a monopolist's product is $p = \frac{200}{\sqrt{q}}$, where p is the **price per unit** (in dollars), when q units are demanded. If the average cost c of producing q units is given by $c = 2 + \frac{80}{q}$

(a) Find the level of production at which profit is maximum

(b) Also find the maximum profit

(10 points)

Answer: Revenue=$p \times q = \frac{200}{\sqrt{q}} \cdot q = 200\sqrt{q}$. Total cost=$c \times q = (2 + \frac{80}{q}) \cdot q = 2q + 80$. We're, hence, looking at Profit=Revenue - Total cost=$200\sqrt{q} - 2q - 80$

(a) To find the max we need the derivative of profit: $Profit' = 200\frac{1}{2}q^{-\frac{1}{2}} - 2 = \frac{100}{\sqrt{q}} - 2$, and we need to find the critical numbers, namely $\frac{100}{\sqrt{q}} - 2 = 0 \Rightarrow \frac{100}{\sqrt{q}} = 2 \Rightarrow \sqrt{q} = \frac{100}{2} = 50 \Rightarrow q = 50^2 = 2500$. Draw the sign table for (Profit'), and the result is that Profit increases between 0 and 2500 (we only consider q positive) and decreases afterwards (in the table you have positive up to 2500 and negative afterwards), so 2500 is a relative max. (**always check if the critical number you got - even if it's the only one! - is what the problem asks it to be, max or min**)

Here's the table:

q		2500	
$profit'$	$+$	0	$-$

(for example plug in 1 to the left and 3600 to the right into Profit'=$\frac{100}{\sqrt{q}} - 2$.)

(b) Max Profit=Profit(2500) $= 200\sqrt{2500} - 2 \cdot 2500 - 80 = 10000 - 5000 - 80 = 4920$

VII. For the functions

$$f(x) = xe^{-x}$$

$$f'(x) = e^{-x} - xe^{-x}$$

(a) use the derivatives given above to determine the interval(s) (if any) on which $f(x)$ is increasing and the interval(s) (if any) on which $f(x)$ is decreasing. (if there are none, please say so)

Answer: set $f'(x) = 0$ and solve: $e^{-x} - xe^{-x} = e^{-x}(1-x) = 0$. But e^{-x} is never zero, moreover, is never negative, always positive; so only $x = 1$ fits the above equation. Draw the sign table (again, keep in mind that exponentials, and in particular e^{-x} are ALWAYS positive ... so the sign changes for us only in $1-x$), and notice that $(-\infty, 1)$ is interval of increase, and $(1, \infty)$ is interval of decrease (use 0 and 2, let's say; $e^{-0}(1-0) = 1 > 0$ and $e^{-2}(1-2) = -\frac{1}{e^2} < 0$).

(b) rel max and rel min

Answer: obviously 1 is a rel max. No other critical values? perfect! no rel min, no other rel max. Done.

(10 points)