SOLUTION 9.2 # 47

MATH 153 SP01

Prove that a convergent sequence has a unique limit.

Proof: First of all, if we knew already the Summation rule, we would be able to solve this in a minute, since:

$$\lim_{n\to\infty} a_n = L_1$$

and

$$\lim_{n\to\infty} a_n = L_2$$

then

$$\lim_{n\to\infty}(a_n - a_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} a_n) =$$
$$= L_1 - L_2$$

but that means

$$\lim_{n\to\infty} 0 = L_1 - L_2$$

and isnce the constant sequence 0 is only "close" to 0 (any number different from 0 is too far away - try to put this using the ϵ thingie!) it means that $L_1 - L_2 = 0 \Rightarrow L_1 = L_2$, and hence the sequence cannot have two different limits.

Still, we don't need the summation rule. Recall the picture I drew in class ... and it will help you make sense of the following formal proof:

Assume

$$\lim_{n\to\infty} a_n = L_1$$

and

$$\lim_{n\to\infty} a_n = L_2$$

where $L_1 \neq L_2$; for convenience assume $L_1 < L_2$. Let $\epsilon = \frac{L_2 - L_1}{2}$. For this ϵ, since a_n converges to L_1, we have that there exists an index N_1 so that $|a_n - L_1| < \epsilon$ for $n > N_1$. At the same time, a_n converges to L_2, and so there is an index N_2 so that $|a_n - L_2| < \epsilon$ for $n > N_2$. Let's try to summarize this information:

- one of these two indeces is bigger, and call the biggest one N.

- since, if $n > N$ it means that $n > N_1$ and $n > N_2$ at the same time (n is bigger than the bigger of the two, right?)

- hence, $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \epsilon$ at the same time

- what does that mean? it means that the distance from a_n to L_1 is less than ϵ, which, if you recall, is half the distance between L_1 and L_2; at the

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same time, a_n is at distance, again, less than ϵ, but from L_2 this time; what does that mean? imagine the line passing between L_1 and L_2 at half the distance ... it means that a_n is at the same time above that line, and below it - impossible!!

- in conclusion, we cannot have this setup ... and it all started with the sequence having two different limits, so this is impossible.