SOLUTION 9.3 # 40

MATH 153 SP01

Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)$.
(a) using the definition of ln show that

$$\ln(n+1) - \ln(n) \geq \frac{1}{n+1}$$

proof: by definition

$$\ln(n) = \int_1^n \frac{1}{x}\,dx$$

also

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx$$

hence

$$\begin{aligned}\ln(n+1) - \ln(n) &= \int_1^{n+1} \frac{1}{x}\,dx - \int_1^n \frac{1}{x}\,dx = \\ &= \int_n^{n+1} \frac{1}{x}\,dx\end{aligned}$$

Keep now n fixed. Let's prove something more. We know that $\frac{1}{x}$ decreases, and so we know that $\frac{1}{n} \geq \frac{1}{x} \geq \frac{1}{n+1}$ for all $n \leq x \leq n+1$. Hence, integrating $\frac{1}{x}$ from n to $n+1$ will give us a bigger area than integrating $\frac{1}{n+1}$ on the same n to $n+1$ interval, and a smaller area than integrating $\frac{1}{n}$ on the same interval. Let's write this down:

$$\int_n^{n+1} \frac{1}{n}\,dx \geq \int_n^{n+1} \frac{1}{x}\,dx \geq \int_n^{n+1} \frac{1}{n+1}\,dx$$

but this means, looking at the meaning of the middle integral, and computing the extremes (by the way, the extremes are just the integrals of constants, since n is fixed)

$$\frac{1}{n}x|_n^{n+1} = \frac{1}{n} \geq \ln(n+1) - \ln(n) \geq \frac{1}{n+1}x|_n^{n+1} = \frac{1}{n+1}$$

done.

(b) skip this one, is not so important (well, it is - but ...) but mind the fact that the sequence decreases, which, together with the result in (c), that says that the sum of fractions is always bigger than the ln, says that the difference between the sum of fractions and the ln becomes smaller and smaller

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(c) show that $\ln(n) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

proof: use the additional inequality we produced in (a)

$$\ln(n) - \ln(n-1) \leq \frac{1}{n-1}$$
$$\ln(n-1) - \ln(n-2) \leq \frac{1}{n-2}$$
$$\cdots$$
$$\ln(2) - \ln(1) \leq \frac{1}{1}$$

and add the left hand sides together, and same for the right hand sides; we get

telescopic sum on the left, and the sum of fractions on the right

$$\ln(n)-\ln(n-1)+\ln(n-1)-\ln(n-2)+\cdots+\ln(2)-\ln(1) \leq \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-1}$$
$$\ln(n) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(d) now, since a_n, the difference between the sum of fractions and the ln, is positive, and becoming smaller and smaller, it's bound to stop somewhere ... that is, it converges! the book even gives you an estimate of the limit (it's not zero, but not far from it).