MIDTERMS' ANSWERS

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Midterm 1
1.

$$\frac{dy}{dx} + 3y - y^3 = 0$$

Bernoulli equation, as it has a linear part, and a power of y. Hence the answer is yes, it can be transformed into a linear equation, by the change: $v = y^{1-3}$ (1 minus the power of the y). The equation can be rewritten, as $v = y^{-2} \Rightarrow y = v^{-\frac{1}{2}}$,

$$-\frac{1}{2}v^{-\frac{3}{2}} \cdot v' + 3v^{-\frac{1}{2}} - v^{-\frac{3}{2}} = 0$$

Multiply the equation by $v^{\frac{3}{2}}$ and get:

$$-\frac{1}{2}v' + 3v - 1 = 0$$

Done.

2. It can be viewed as a separable equation:

$$\frac{dy}{dx} = -\frac{y + e^y}{x(1 + e^y)}$$

Separate the x from the y:

$$\frac{1 + e^y}{y + e^y} \cdot dy = -x \cdot dx$$

Integrate both sides, and take in account that the top of the fraction in the left hand side is the derivative of the fraction's denominator (change variables; $z = y + e^y$) and get:

$$\ln |y + e^y| = -\frac{x^2}{2} + C$$

$$y + e^y = K \cdot e^{-\frac{x^2}{2}}$$

This cannot be solved further, so we stop here.

(by the way, this can be done using exact equations, as the initial equation can be checked for exactness etc)

3. Linear, first order differential equation. Rewrite as:

$$x' - 2x = -te^{-2t}$$

Find the integrating factor u:

$$u = e^{\int -2} = e^{-2t}$$

Multiply it both sides, and the left hand side gets reduced to the derivative of the product $u \cdot x$:

$$(e^{-2t} \cdot x)' = -te^{-4t}$$

Integrate both sides, use integration by parts for the right hand side, and get:

$$e^{-2t} \cdot x = \frac{te^{-4t}}{4} + \frac{e^{-4t}}{16} + C$$
$$x = \frac{te^{-2t}}{4} + \frac{e^{-2t}}{16} + C$$

Give the initial condition: $x(0) = 0 + \frac{1}{16} + C = 1 \Rightarrow C = \frac{15}{16}$.

4. Again separable:

$$(y+1)^{-\frac{2}{3}} \cdot dy = (x-3) \cdot dx$$

Integrate both sides:

$$\frac{(y+1)^{\frac{1}{3}}}{\frac{1}{3}} = \frac{x^2}{2} - 3x + C$$
$$3(y+1)^{\frac{1}{3}} = \frac{x^2}{2} - 3x + C$$
$$y+1 = (\frac{x^2}{6} - x + \frac{C}{3})^3$$
$$y = (\frac{x^2}{6} - x + \frac{C}{3})^3 - 1$$

5. (a) Equilibrium solutions are those for which $x' = 0 \Rightarrow (x-1)(2-x)(x-3) = 0 \Rightarrow x = 1$ or $x = 2$ or $x = 3$. Sketching the directions' field (or checking the sign between these values of x: positive below 1, negative between 1 and 2, positive between 2 and 3 and positive from 3 on) we get to say that $x = 1$ is stable, $x = 2$ is unstable, $x = 3$ again stable.

(b) Since the solution starts between 2 and 3, and $x = 3$ is stable, the required limit is 3.

6. The only forces that act on the rocket are gravity and air resistance (and they act against movement, so they are with a negative sign), and combined they are total force, mass times acceleration:

$$-mg - .25 \cdot v = m \cdot v'$$

$$-.98 - .25v = .1v'$$

Reordering terms, we get the following linear, first order differential equation:

$$.1v' + .25v = -.98$$

Write in standard form:

$$v' + 2.5v = -9.8$$

The integrating factor $u = e^{\int 2.5} = e^{2.5t}$, and so we get:

$$(e^{2.5t} \cdot v)' = -9.8 \cdot e^{2.5t}$$

$$e^{2.5t} \cdot v = -\frac{9.8}{2.5} \cdot e^{2.5t} + C$$

$$v = -3.92 + Ce^{-2.5t}$$

The initial speed is 130, so $v(0) = -3.92 + C \cdot 1 = 130 \Rightarrow C = 133.92$.