SOLUTIONS - CHAPTER 4.1

MATH 548 - SP 2000

1(b). One can assume x is positive - hint is the fact that we need $|x-1|$ which means we are interested in what happens close to 1.

$$|x^2-1| < \frac{1}{10^3} \iff -\frac{1}{10^3} < x^2 - 1 < \frac{1}{10^3} \iff$$

$$\iff 1-\frac{1}{10^3} < x^2 < 1+\frac{1}{10^3} \iff \sqrt{1-\frac{1}{10^3}} < x < \sqrt{1+\frac{1}{10^3}} \iff$$

$$\iff 0.99949987 < x < 1.0004999 \iff -0.00050013 < x-1 < 0.0004999$$

Hence, if we choose $|x-1| < 0.0004$, let's say, the last condition is verified, so by going backwards we get what we need: $|x^2-1| < \frac{1}{10^3}$.

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3.

$$\lim_{x\to c} f(x) = L \iff$$

for any given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $0 < |x-c| < \delta(\epsilon)$ then $|f(x) - L| < \epsilon \iff$ making $x = y + c$, for any given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $|y + c - c| < \delta(\epsilon)$ i.e. $|y - 0| < \delta(\epsilon)$ then $|f(y+c) - L| < \epsilon \iff$

$$\iff \lim_{y\to 0} f(y+c) = L$$

Changing in the last formula the symbol y with x we get the required result.

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6. $|x^2-c^2| = |(x+c)(x-c)| = |x+c||x-c|$. But both $x, c \in (0,a) \Rightarrow x+c > 0$ so we don't need the absolute value sign; more than that, $x+c < a+a = 2a$. Hence, $|x^2-c^2| \leq 2a|x-c|$, which takes care of the first part. Taking now a fixed c, we have: for $\epsilon > 0$, we take $\delta(\epsilon) = \frac{\epsilon}{2a} > 0$ and this gives us, for any x with $|x-c| < \delta(\epsilon)$, $|x^2-c^2| \leq 2a|x-c| < 2a\frac{\epsilon}{2a} = \epsilon \iff$

$$\iff \lim_{x\to c} x^2 = c^2$$

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8. Using same method as in 6, we can assume that our x wanders around some bounded interval that contains c, so we can assume we have $x, c \in$

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$(-M, M)$, for some positive $M \Rightarrow |x| < M, |c| < M$. Then, because $x^3 - c^3 = (x-c)(x^2 + xc + c^2)$ we have $|x^3 - c^3| = |x-c||x^2 + xc + c^2| \leq$
$\leq |x-c|(|x^2| + |x||c| + |c^2| \leq |x-c|(M^2 + M * M + M^2) = 3M^2|x-c|$. Taking now $\epsilon > 0$, and $\delta(\epsilon) = \frac{\epsilon}{3M^2} > 0$ we have that for $|x-c| < \epsilon$, $|x^3 - c^3| \leq 3M^2|x-c| < 3M^2 \frac{\epsilon}{3M^2} = \epsilon \iff$

$$\iff \lim_{x \to c} x^3 = c^3$$

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9. We have two cases: $c = 0$ and $c > 0$. For the first case we have that for $|x - 0| = x < \epsilon^2$, $|\sqrt{x} - 0| = \sqrt{x} < \sqrt{\epsilon^2} = \epsilon$, which takes care of it.

For the second case we use the following: $\sqrt{x} - \sqrt{c} = \frac{x-c}{\sqrt{x}+\sqrt{c}}$ and the fact that, since $c > 0$ we can assume $x, c > r$, for $r > 0$ fixed (obviously small ... but it's important that it exists - notice we couldn't have done it in the first case), so $\sqrt{x} + \sqrt{c} > 2\sqrt{r} \Rightarrow \frac{1}{\sqrt{x}+\sqrt{c}} < \frac{1}{2\sqrt{r}}$, hence $|\sqrt{x} - \sqrt{c}| \leq \frac{1}{2\sqrt{r}}|x-c|$ ... and now for the ϵ we choose $\delta(\epsilon) = 2\sqrt{r} * \epsilon$ ... and we have what we need (I'll leave you the pleasure to write the last details ...).

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10(c). $\epsilon - \delta$: let $\epsilon > 0$, and choose $\delta(\epsilon) = \epsilon \to |x - 0| < \epsilon$ (i.e. $|x| < \epsilon$ ...). Then we have $|\frac{x^2}{|x|} - 0| = |\frac{x^2}{x}| = |x| < \epsilon$ ... done.

sequential: let $(x_n)_{n \in N} \to 0$. We have that $|\frac{{x_n}^2}{|x_n|}| = |\frac{{x_n}^2}{x_n}| = |x_n| \Rightarrow -|x_n| \leq \frac{{x_n}^2}{|n|} \leq |x_n|$, so by Squeeze Theorem, since both $-|x_n|$ and $|x_n| \to 0 \Rightarrow \frac{{x_n}^2}{|x_n|} \to 0$, meaning that for any sequence converging to 0 the new sequence (through the function) converges to 0, so the limit at 0 for the function is 0, which is exactly what we wanted.

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11(c). To show that a limit doesn't exist it's enough to construct a sequence that converges to 0, but which through the function fails to converge. Let's take $x_n = (-1)^n \frac{1}{n}$ (it converges to 0 and alternates the sign).

$x_n \to 0$ (easy to prove), and $x_n + sgn(x_n) = (-1)^n \frac{1}{n} + (-1)^n = (-1)^n(\frac{1}{n} + 1)$. This new sequence fails to converge, since if we take the subsequence x_{2n} (even indexes) we have $x_{2n} = \frac{1}{n} + 1 \to 1$, and if we take the subsequence x_{2n+1} (odd indexes) we have $x_{2n+1} = -(\frac{1}{n} + 1) \to -1$ - if the sequence were convergent, all subsequences must converge to the same value, and this doesn't happen. Hence the limit doesn't exist.

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14. If $f(x) = x \Rightarrow |f(x)| \leq |x|$; if $f(x) = 0 \Rightarrow |f(x)| \leq |x|$. Hence, if we take an $\epsilon > 0$ and choose $\delta(\epsilon) = \epsilon$ we have that for $|x - 0| = |x| < \epsilon$, $|f(x) - 0| = |f(x)| \leq |x| < \epsilon$, hence

$$\lim_{x \to 0} f(x) = 0$$

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For the second part, let's take $c \in R^*$. We can always construct a sequence $(x_n)_{n \in N} \subset Q$ and $(y_n)_{n \in N} \subset R \backslash Q$, both converging to c (if $c \in Q$, for the first sequence just take $(c - \frac{1}{n})$ and for the second $(c - \frac{\sqrt{2}}{n})$; if $c \in R\backslash Q$ take for the first sequence the sequence of approximations - e.g. for π take 3, 3.1, 3.14, 3.141, 3.1415 etc - and for the second $(c - \frac{1}{n})$; easy to check that these sequences verify the above conditions). Then, for (x_n) the limit is going to be c, and for (y_n) the limit is going to be 0 (the function becomes 0 for irrationals, so $f(y_n) = 0$, for all n, and $f(x_n) = x_n$ etc). Since $c \neq 0$ we get that we have 2 sequences which both converge to c, but through f they converge to different values, hence

$$\lim_{x \to c} f(x) DNE^1$$

for $c \neq 0$.

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[1]DNE=does not exist