SOLUTIONS CHAPTER 4.2

MATH 548 SP 2000

1d

We know that

$$\lim_{x\to 0} x = 0$$

(use $\epsilon - \delta$). Hence, by Th. 4.2.4, we have:

$$\lim_{x\to 0}(x+1) = (\lim_{x\to 0} x) + (\lim_{x\to 0} 1) = 0 + 1 = 1$$

and

$$\lim_{x\to 0} x^2 = \lim_{x\to 0}(x * x) = (\lim_{x\to 0} x)^2 = 0$$

and

$$\lim_{x\to 0}(x^2+2) = (\lim_{x\to 0} x^2) + (\lim_{x\to 0} 2) = 0 + 2 = 2$$

and, finally, since $2 \neq 0$ (that is, the denominator is OK),

$$\lim_{x\to 0} \frac{x+1}{x^2+2} = \frac{1}{2}$$

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2b

As a general rule, when one takes a limit to a point (like in our case $\lim_{x\to 2}$) one always considers $x \neq$ that point ($x \neq 2$); hence in our case we are allowed to cancel $(x-2)$ (if possible - if it's common factor for both numerator and denominator), since it's nonzero. And it must be possible to cancel this factor, since the numerator becomes 0 as well (so it has as a factor $(x-2)$ - common, then, for the numerator and denominator). Let's see how this is done (remember Calculus!!):

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

Hence we have that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = (\lim_{x\to 2} x) + (\lim_{x\to 2} 2) = 2+2$$

where we used Th. 4.2.4 for the sum.

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We notice that if we take limit to 0 for both numerator and denominator (which, based on Th. 4.2.4, consists on just plugging in the value 0 in both

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expressions - well, there's some more work involved in showing it for $\sqrt{1+2x}$ and $\sqrt{1+3x}$, but it can be done, let's say, by $\epsilon - \delta$), we get 0 for both, so for the whole fraction it is impossible to use Th. 4.2.4 (denominator, actually, is the one that ruins the day). So, based on our experience with 2b, let's try to "cancel the 0" (again, remember Calculus - the trick is called "multiply with the conjugate"):

$$\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} = \frac{(\sqrt{1+2x}-\sqrt{1+3x})(\sqrt{1+2x}+\sqrt{1+3x})}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} =$$

$$= \frac{1+2x-(1+3x)}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x}} = \frac{-x}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} =$$

$$= \frac{-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})}$$

Hence (remember the comment from 2b) we have:

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} = \lim_{x\to 0} \frac{-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} = \frac{-1}{1(1+1)} =$$

$$= \frac{-1}{2}$$

where we use again Th. 4.2.4 to get the actual result (plus the bit about the $\sqrt{\cdot}$ thing).

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Case 1: $x = 0 \Rightarrow -0^2 = 0^n = 0^2$, so we're in good shape.

Case 2: we can divide now by x so we have

$$x \neq 0 \Rightarrow -x^2 \leq x^n \leq x^2 \iff -1 \leq x^{n-2} \leq 1$$

(since x^2 is positive, when dividing by it the inequality doesn't change direction).

We need to show that if $-1 \leq x \leq 1 \Rightarrow -1 \leq x^m \leq 1$, for any $m \in \mathbf{N}, m > 0$. Use induction: for $m = 1$ it's clear (it's just the hypothesis all over again). For the induction step:

case $x > 0$: $-1 \leq x^m \leq 1$ true; since $x > 0 \Rightarrow$

$$-1 * x \leq x^m * x \leq 1 * x \iff -x \leq x^{m+1} \leq x$$

but $x \leq 1$ and then $-1 \leq -x \Rightarrow -1 \leq -x \leq x^{m+1} \leq x \leq 1$ done;

case $x < 0$: if $x < 0$ we have

$$-1 * x \geq x^m * x \geq 1 * x \iff x \leq x^{m+1} \leq -x$$

but still $-1 \leq x \leq 1$, so

$$(-1)(-1) \geq -x \geq 1 * (-1) \iff -1 \leq -x \leq 1$$

and using these in the above expression we get again $-1 \leq x \leq x^{m+1} \leq -x \leq 1$. DONE!

Now, using Squeeze Theorem we get that

$$0 = \lim_{x \to 0} -x^2 \le \lim_{x \to 0} x^m \le \lim_{x \to 0} x^2 = 0 \iff$$
$$\iff \lim_{x \to 0} x^m = 0$$

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Classical example of function that doesn't have limit is the **signum** function, defined as follows:

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

This function doesn't have a limit at 0 (best way of seeing this is by using the sequential method - take $(x_n)_{n \in \mathbf{N}} = ((-1)^n \frac{1}{n})_n \to 0$, and plugging this sequence into sgn we get the following sequence: -1, 1, -1, 1, -1, etc; but this sequence doesn't have a limit, because if it had, all subsequences would converge to the same value; but the odd-indexed elements form a sequence -1, -1, -1, -1,etc that converges to -1, and the even-indexed elements form a sequence 1, 1, 1, 1, etc converges to 1)[1]

Take now $f =$sgn and $g = -$sgn; then

$$f + g = sgn - sgn = 0$$

and

$$fg = sgn * (-sgn) = -sgn^2$$

The product has the property that sgn$(x) = -(-1)^2 = -1$ if $x < 0$ and sgn$(x) = -(1)^2 = -1$ if $x > 0$. Since (remember the comment from problem 2b) we don't care about the value sgn^2 has in 0 (which is 0, by the way)

$$\lim_{x \to 0} sgn^2(x) = \lim_{x \to 0} -1 = -1$$

- hence the limit exists. For the sum things are even simpler (since the result is 0, pure and simple).

A more complex (but more simple to use) example would be if one takes the following functions:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{R} \backslash \mathbf{Q} \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Q} \\ 1 & \text{if } x \in \mathbf{R} \backslash \mathbf{Q} \end{cases}$$

These functions **don't have a limit anywhere**! (prove that! -use sequential method, and the fact that one can find either only-rational or only-nonrational, respectively, sequences that converge to a given point; check previous homework). We get $f + g = 1$ and $f * g = 0$ (since a number x is

[1] notice, still, that this function has lateral limits, namely -1 from the left, and 1 from the right!

either rational or nonrational, which means that for one function we get 1 and for the other one we get 0).

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11c

As $x \to 0$ we get that $\frac{1}{x} \to \pm\infty$ (i.e. can have positive and negative values, bigger and bigger). The sin function will then transform these values into values between -1 and 1, but quite likely will take all the values between -1 and 1! which, if passed through sgn, will become, most likely, lots of -1 and 1.

Based on this intuitive analysis, let's try to find a sequence that converges to 0, and which will become a sequence of -1 (for odd-indexed elements) and 1 (for even-indexed ones).

We can get this sequence by going backwards: we get ± 1 when $\sin > 0, \sin < 0$, so there's no harm in considering $\sin(x_n) = \pm 1$. So let's have $\frac{1}{x_{2k+1}} = 2k\pi + \frac{3\pi}{2}$ and $\frac{1}{x_{2k}} = 2k\pi + \frac{\pi}{2} \Rightarrow x_{2k+1} = \frac{1}{2k\pi + \frac{3\pi}{2}}, x_{2k} = \frac{1}{2k\pi + \frac{\pi}{2}}$. As $k \to \infty$ we have that $x_{2k+1}, x_{2k} \to 0$, so $x_n \to 0$. As a result, we get this sequence which converges to 0, but which through sgn(sin) gives us the sequence -1,1,-1,1,... nonconvergent. Hence the limit doesn't exist.

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$\lim_{x\to c} f = L \iff \forall\epsilon\ \exists\delta$ such that $\forall x$ with $|x-c| < \delta, |f(x) - L| < \epsilon$. Let's use now the inequality $||a| - |b|| \leq |a - b| \Rightarrow \forall\epsilon\ \exists\delta$ such that $\forall x$ with $|x - c| < \delta, ||f(x)| - |L|| \leq |f(x) - L| < \epsilon \iff \lim_{x\to c} |f(x)| = |L|$, which is what we needed to prove.