SOLUTIONS CHAPTER 4.3

MATH 548, SP 2000

4

$\lim_{x\to c} f = \infty \iff \forall\alpha > 0\exists\delta$ such that for all $x \in (0,\infty)$ with $|x-c| < \delta$ then $|f(x)| > \alpha > 0 \iff \forall\alpha > 0\exists\delta$ such that for all $x \in (0,\infty)$ with $|x-c| < \delta$ then $\frac{1}{|f(x)|} < \frac{1}{\alpha}$; taking α big enough makes $\frac{1}{\alpha}$ very small, so it might be a good idea to call $\frac{1}{\alpha}$ "ϵ", so we can get exactly the definition for the limit; so let's rewrite the last statement:$\forall\epsilon > 0\exists\delta$ such that for all $x \in (0,\infty)$ with $|x-c| < \delta$ then $|\frac{1}{f(x)}| < \epsilon \iff \lim_{x\to c}\frac{1}{f(x)} = 0$.

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5c

$$\lim_{x\to 0^+}\frac{x+2}{\sqrt{x}} = \lim_{x\to 0^+}(\frac{x}{\sqrt{x}} + \frac{2}{\sqrt{x}}) = \lim_{x\to 0^+}(\sqrt{x} + \frac{2}{\sqrt{x}})$$

But $\sqrt{x} \to 0$ as $x \to 0^+$, so we have to see what happens to the second term. Take $\alpha > 0$, and construct $\delta(\alpha) = \frac{4}{\alpha^2}$. Then, if $x = |x| = |x-0| < \frac{4}{\alpha^2} \Rightarrow |\frac{2}{\sqrt{x}}| > \frac{2}{\sqrt{\frac{4}{\alpha^2}}} = \frac{2}{\frac{2}{\alpha}} = \alpha \Rightarrow$ (by definition)

$$\lim_{x\to 0^+}\frac{2}{\sqrt{x}} = \infty$$

hence the whole limit is $0 + \infty = \infty$.

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5h

Divide top and bottom by x:

$$\frac{\sqrt{x}-x}{\sqrt{x}+x} = \frac{\frac{\sqrt{x}-x}{x}}{\frac{\sqrt{x}+x}{x}} = \frac{\frac{1}{\sqrt{x}}-1}{\frac{1}{\sqrt{x}}+1}$$

Taking now numerator and denominator separately we get:

$$\lim_{x\to\infty}(\frac{1}{\sqrt{x}} - 1) = (\lim_{x\to\infty}\frac{1}{\sqrt{x}}) - 1 = 0 - 1 = -1$$

and

$$\lim_{x\to\infty}(\frac{1}{\sqrt{x}} + 1) = (\lim_{x\to\infty}\frac{1}{\sqrt{x}}) + 1 = 0 + 1 = +1$$

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where we used $\lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0 \iff \forall\epsilon, \exists\delta = \delta(\epsilon) = \frac{1}{\epsilon^2}$ such that, if $x > \delta = \frac{1}{\epsilon^2}, |f(x) - 0| = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$.

Hence,

$$\lim_{x\to\infty}\frac{\sqrt{x}-x}{\sqrt{x}+x} = \lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}-1}{\frac{1}{\sqrt{x}}+1} = \frac{\lim_{x\to\infty}(\frac{1}{\sqrt{x}}-1)}{\lim_{x\to\infty}(\frac{1}{\sqrt{x}}+1)}$$

(by Th. 4.2.4 - since the limit of the bottom, namely 1, is not zero), so we end up getting

$$= \frac{-1}{1} = -1$$

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7

If $\lim_{x\to\infty} g = \infty$ we're done (since ∞ is "bigger" than any number).

Let now the limit be finite, $\lim_{x\to\infty} g = L$. Proof by contradiction: assume $\lim_{x\to\infty} f(x) > L \Rightarrow \exists\ \zeta_0 > 0$ such that, $\forall\ \delta\ \exists\ x_\delta$ with $x_\delta > \delta$ and $f(x_\delta) > L + \zeta_0$ [1]

But then, since $g(x) \to L \Rightarrow \exists\ \delta$ such that $|g(x) - L| < \frac{\zeta_0}{2}, \forall\ x > \delta \Rightarrow g(x) < L + \frac{\zeta_0}{2} < L + \zeta_0, \forall\ x > \delta \Rightarrow g(x_\delta) < L + \zeta_0 < f(x_\delta)$, for the particular $x_\delta > \delta$ given by the above affirmation, which leads to the contradiction ($f(x)$ is supposed to be $\leq g(x)$ for all x. Hence

$$\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x)$$

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9

$$\lim_{x\to\infty} xf(x) = L \iff$$

$\forall\ \epsilon, \exists\ \delta = \delta(\epsilon)$, such that if $x > \delta$ then $|xf(x) - L| < \epsilon$, which inequality is equivalent to (note $x \neq 0$ and $x > 0$) $L - \epsilon < xf(x) < L + \epsilon \iff \frac{L-\epsilon}{x} < f(x) < \frac{L+\epsilon}{x}$.

Let now $\delta_1(\epsilon) = \max(\delta(\epsilon), \frac{L+\epsilon}{\epsilon}, -\frac{L-\epsilon}{\epsilon})$. If $x > \delta_1 \Rightarrow$
first: $x > \delta$ the above inequality is true
second $x > \frac{L+\epsilon}{\epsilon} \Rightarrow x\epsilon > L + \epsilon \Rightarrow \frac{L+\epsilon}{x} < \epsilon$;
third $x > -\frac{L-\epsilon}{\epsilon} \Rightarrow -x\epsilon < L - \epsilon \Rightarrow -\epsilon < \frac{L-\epsilon}{x}$.

Putting all inequalities together (you notice that the second inequality patches the right-hand side, and the third the left-hand side), and considering **in what conditions** they happen, we get:

$$\forall\ \epsilon, \exists\ \delta_1(\epsilon)\ \textbf{such that}\ x > \delta_1\ \textbf{implies}\ |f(x)| < \epsilon \iff$$

$$\iff \lim_{x\to\infty} f(x) = 0$$

[1] Note that the opposite affirmation would have stated that, for every ζ there exists a value δ such that for all $x > \delta$ we get $f(x) < L + \zeta$ - which means that the limit, if it exists, is less that $L + \zeta,\ \forall\ \zeta > 0$; but this would imply that the limit is $\leq L$ - imagine making ζ smaller and smaller

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12

Since $\lim f(x) = L > 0 \Rightarrow \exists\ \delta_0 = \delta_0(L)$ such that $|f(x) - L| < \frac{L}{2}$ which inequality is equivalent to $\frac{L}{2} < f(x) < \frac{3L}{2}$, for all $x > \delta_0$.

Let $\alpha > 0$; we know that $\exists\ \delta_1 = \delta_1(\alpha)$ such that $g(x) > \frac{2\alpha}{L}$ for all $x > \delta_1 \Rightarrow f(x)g(x) > \frac{L}{2}\frac{2\alpha}{L} = \alpha$.

Let's write down again the conditions and the final inequality: $\forall\ \alpha > 0, \exists\ \delta = \delta(\alpha)$ =max$(\delta_0(L), \delta_1(\alpha))$ such that $f(x)g(x) > \alpha$ for all $x > \delta$ (remember, L is fixed! hence δ_0 is fixed ... so the last δ is just a function of α) $\Longleftrightarrow$

$$\Longleftrightarrow \lim_{x\to\infty} f(x)g(x) = \infty$$

As a counterexample for the case $L = 0$ (by the way, did you see where we used the fact that $L \neq 0$? .. there's a fraction involving L as a denominator at some point ...) just take $g(x) = x$ and $f(x) = \frac{1}{x}$. $g(x) = x \to \infty$ and $f(x) \to 0$ as $x \to \infty$, but $f(x)g(x) = 1 \to 1$ as $x \to \infty$.

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13

If we could find such functions we would have the following:

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{f - g + g}{g} = \lim_{x\to\infty} \frac{f - g}{g} + \lim_{x\to\infty} \frac{g}{g} =$$

$$= \lim_{x\to\infty} \frac{f - g}{g} + 1 = 0 \iff \lim_{x\to\infty} \frac{f - g}{g} = -1$$

But $g(x) \to \infty \Rightarrow \frac{1}{g} \to 0$ and $(f - g)(x) \to 0$ hence their product (by Th. 4.2.4) has the property

$$\lim_{x\to\infty} \frac{f - g}{g} = \lim_{x\to\infty} (f - g)\frac{1}{g} = (\lim_{x\to\infty} (f - g))(\lim_{x\to\infty} \frac{1}{g}) = 0$$

so it can never be -1! the answer is **NO**.

As a last observation: how does one get the above result for $\frac{1}{g}$? We have that $\forall\epsilon$, there exists $\delta = \delta(\epsilon) = \delta(\frac{1}{\epsilon})$ such that for $x > \delta$ we get $g(x) > \frac{1}{\epsilon} \Rightarrow \frac{1}{g(x)} < \epsilon$ ... (look at problem 4!!).

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