# SOLUTIONS CHAPTER 5.1

MATH 548 -SP00

$\boxed{3}$

*Proof.* Take $x \in [a, b]$; we distinguish 3 cases:

1) $x < c \Rightarrow x \in [a, c]$ on which interval $h = f$, hence $h$ is continuous in $x$, since $f$ is;

2) $x > c \Rightarrow x \in [c, b]$ on which interval $h = g$, hence $h$ is continuous in $x$, since $g$ is;

3) $x = c$; since $f$ is continuous on $[a, b]$ it means it is continuous "to the left" of $c \Rightarrow$ for any neighbourhood $V_\epsilon(f(c))$ there exists $V_\delta^{\leftarrow}(c)$ such that, if $x \in V_\delta^{\leftarrow}(c) \cap [a, c]$ then $f(x) \in V_\epsilon(f(c))$; similarly, since $g$ is continuous "to the right" of $c \Rightarrow$ for the same neighbourhood $V_\epsilon(f(c))$ there exists $V_\delta^{\rightarrow}(c)$ such that, if $x \in V_\delta^{\rightarrow}(c) \cap [c, b]$ then $f(x) \in V_\epsilon(f(c))$. Take now $V_\delta(c) = V_\delta^{\leftarrow} \cap V_\delta^{\rightarrow}$ and this will do the trick (since any $x \in V_\delta(c)$ is either in the $\leftarrow$ neighbourhood or in the $\rightarrow$ one).

□

$\boxed{4c}$

*Proof.* Let's analyse this function. First of all, it's periodic, so we can restrict our attention to only the interval $[0, 2\pi)$.

1) $x \in [0, \frac{\pi}{2}) \Rightarrow \sin(x) \in [0, 1) \Rightarrow [\sin(x)] = 0$

2) $x = \frac{\pi}{2} \Rightarrow [\sin(x)] = 1$

3) $x \in (\frac{\pi}{2}, \pi] \Rightarrow [\sin(x)] = 0$

4) $x \in (\pi, 2\pi) \Rightarrow [\sin(x)] = -1$

It's simpler to give the answer to this problem by pointing out the points at which the function **fails** to be continuous: $\frac{\pi}{2} + 2k\pi$, $\pi + 2k\pi$ and ... $0 + 2k\pi$ ($k \in \mathbf{Z}$) (the last one because to the right of 0 one has 0 as an output - see behaviour for $[0, \frac{\pi}{2})$ - and to the left it has -1 as output - see behaviour for $(\pi, 2\pi)$, which can be shifted by $2\pi$ to the left).

□

$\boxed{6}$

*Proof.* Let $\epsilon > 0$. Since $f$ is continuous at $c \Rightarrow$ there exists $V_\delta(c)$ such that, for any $x \in V_\delta(c), |f(x) - f(c)| < \frac{\epsilon}{2}$. Take now $x, y \in V_\delta(c)$;

$|f(x) - f(y)| = |f(x) - f(c) + f(c) - f(y)| = |(f(x) - f(c)) + (f(c) - f(y))| <$

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$$< |f(x) - f(c)| + |f(c) - f(y)| = |f(x) - f(c)| + |f(y) - f(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(triangle inequality for the critical part)

□

$\boxed{8}$

*Proof.* Since $f$ is continuous in $\mathbf{R}$ $\Rightarrow$ it's continuous in $x$ as well. Since $x_n \to x \Rightarrow f(x_n) \to f(x)$ (continuity) $\Rightarrow$ the sequence formed of 0 (zeros) converges to $f(x)$; but it converges to 0 too, and since it cannot have 2 limits $\Rightarrow f(x) = 0 \Rightarrow x \in S$.

□

$\boxed{9}$

*Proof.* (a) using sequential criterion (which says that if $x_n \to c$ then $f(x_n) \to f(c)$, for any $(x_n)_{n\in\mathbf{N}}, x_n \in B$) we get that for those particular $x_n$s which consist of elements of A, if $x_n \to c \Rightarrow f(x_n) \to f(c) \iff g(x_n) \to g(c) \Rightarrow g$ is continuous at $c$.

(b) Taking $f$ to be such that $f(x) = 0$ if $x \in \mathbf{Q}$ and $f(x) = 1$ if $x \in \mathbf{R}\backslash\mathbf{Q}$, we know that f is not continuous at $c$, for any $c \in \mathbf{R}$. But taking $g$ to be $f$'s restriction to $\mathbf{Q}$, then $g = 0$. Hence $g$ is continuous, let's say at 0, even though $f$ isn't (the way one should think about this phenomenon is that since $f$ is defined is a bigger set - B - the variety of possible sequnces convergent to a point is richer than for $g$, which is defined on a smaller set, and hence allows sequences a lesser freedom of choice; so even though the restricted kind of sequences get you the continuity property, the more general sequences most probably won't).

□

$\boxed{12}$

*Proof.* Any $c \in \mathbf{R}$ has at least one sequence of rational numbers $(x_n)_{n\in\mathbf{N}}$ which converges to it ($x_n \to c$)(the best example is the sequence of approximations with some finite number of decimals - e.g. for $\pi$ take 3, 3.1, 3.14, 3.141, 3.1415, 3.14159 etc). Since $f$ is continuous we have that $f(x_n) \to f(c) \Rightarrow (0)_n \to f(c) \Rightarrow f(c) = 0$ for any $c \in \mathbf{R}\backslash\mathbf{Q}$.

□

$\boxed{13}$

*Proof.* Let $c \in \mathbf{Q} \Rightarrow$ take $x_n = c$ (rational numbers) and $y_n = c + \frac{\pi}{n}$ (irrational numbers - mind the definition for $g$); both these sequences converge to $c$ ... hence if we want to have continuity we must have both limits of $g(x_n)$ and $g(y_n)$ equal, and equal to $g(c)$: $\lim_{n\to\infty} g(x_n) = \lim 2c = 2c$ and $\lim_{n\to\infty} g(y_n) = \lim c + \frac{\pi}{n} + 3 = c + 3 \Rightarrow 2c = c + 3 \Rightarrow c = 3$. If we choose now $d \in \mathbf{R}\backslash\mathbf{Q}$ and have $x_n =$ sequence of approximations (as in the previous problem) and $y_n = d$ we get the same thing as for $c$ ... problem is that $d = 3$ doesn't cope with $d$ being irrational, hence there's no such $d$.

$\square$