SOLUTIONS CHAPTER 5.5

MATH 548 - SP00

$\boxed{3}$

Proof. Both functions are increasing since, if $x_1 < x_2 \Rightarrow f(x_1) = x_1 < x_2 = f(x_2)$ and $x_1 < x_2 \Rightarrow g(x_1) = x_1 - 1 < x_2 - 1 = g(x_2)$. Taking now their product, we have that $fg(0) = 0 * (-1) = 0$, but $fg(\frac{1}{2}) = \frac{1}{2} * (-\frac{1}{2}) = -\frac{1}{4}$ hence, even though $0 < \frac{1}{2}$ we have $fg(0) > fg(\frac{1}{2})$ (reason? well, g has negative values! hence it reverses the inequality ...) □

$\boxed{6}$

Proof. Let's show "$\Rightarrow$": if f is continuous at c, then for any sequence $(x_n)_{n\in\mathbf{N}}$ converging to c we have $f(x_n) \to f(c)$; in particular, this will happen for the kind of sequence described in the problem, too! so let's find one: $x_n = c + (-1)^n \frac{1}{n}$

"$\Leftarrow$":we have to show that if f is continuous for the sequence (x_n) as defined, it's actually continuous for any sequence (sequential definition of continuity, that is). But then, take any sequence $(y_n)_{n\in\mathbf{N}}$ with $y_n \to c$. We notice the following: since $f(x_n) \to f(c)$ for any ϵ there exists N_ϵ such that $|f(x_n) - f(c)| < \epsilon$ for any $n > N_\epsilon$; but that means that we have an even indexed x (let's take x_{2N_ϵ}) and an odd indexed x ($x_{2N_\epsilon+1}$) after N_ϵ, with the above equality (involving ϵ) satisfied. We have that $y_n \to c \Rightarrow$ for some M_ϵ $y_n \in (x_{2N_\epsilon+1}, x_{2N_\epsilon})$ for any $n > M_\epsilon$; but then, since f is increasing, it means that $f(x_{2N_\epsilon+1}) < f(y_n) < f(x_{2N_\epsilon}) \Rightarrow f(x_{2N_\epsilon+1}) - f(c) < f(y_n) - f(c) < f(x_{2N_\epsilon}) - f(c) \Rightarrow -\epsilon < f(y_n) - f(c) < \epsilon$; since ϵ is arbitrary, we have that $f(y_n) \to f(c)$, hence, since the sequence y_n was arbitrary, f is continuous at c.

□

$\boxed{9}$

Proof. Let $x \neq y$ and assume $f(x) = f(y)$; then we have either $x = y$ (both rational), false; $x = 1 - y \Rightarrow y = 1 - x$ (first rational, second irrational; but that would imply, since x is rational, that y is too), false; $1 - x = y$ (first irratinal, second rational; same reason) false; $1 - x = 1 - y \Rightarrow x = y$ (both irrational), false. Hence we it's absurd to have equality, hence $f(x) \neq f(y)$, hence f is injective.

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For the next part observe that if x is rational, $f(x) = x$ is same, obviously; and if x is irrational, then $f(x) = 1 - x$ is also irrational. So, when taking $f(f(x))$ we either have $f(f(x)) = f(x) = x$ when x is rational, or $f(f(x)) = f(1-x) = 1-(1-x) = x$ when x is irrational.

For the last part, since for any number there always is a rational and an irrational sequence converging to it, to have continuity we must have $x = 1 - x$ for that particular x where f is continuous (we did something similar some chapters ago). But that means $x = \frac{1}{2}$ ... and it's easy to show (based on the property of $\frac{1}{2}$ of satisfying the above equality) that f is continuous there.

□

$\boxed{11}$

Proof. f increases from 0 to 1, and from 1 to 2 - so the only problem would be if $f(1)$ is not less than ANY $f(x)$ with $1 < x \leq 2$; but if $1 < x \leq 2 \Rightarrow 2 < f(x) = x + 1 \leq 3$, hence $f(1) = 1 < 2 < f(x)$, so it's OK.

The inverse function's domain is $[0,1] \cup (2,3]$ (it's the direct function's range!). We have that $f^{-1}(x) = x$ for $x \in [0,1]$ and $f^{-1}(x) = x - 1$ for $x \in (2,3]$ (draw this function's graph!). It's increasing on both intervals, and again, the only problem could be if $f^{-1}(1)$ would be greater than some value $f^{-1}(x)$ with $2 < x \leq 3$; but this doesn't happen, since $2 < x \leq 3 \Rightarrow 1 = 2-1 < f^{-1}(x) = x - 1 < 3 - 1 = 2 \Rightarrow f^{-1}(1) = 1 < f^{-1}(x)$. Hence f^{-1} is also increasing (you could've said this also because of the fact of being the inverse of a increasing function).

As for continuity, f is continuous everywhere on its domain, except at 1 (obviously); as for f^{-1} it's continuous everywhere (the horizontal gap doesn't count as discontinuity for f, but merely as a **discontinuity for the domain**!)

□

$\boxed{12}$

Proof. Assume, by contradiction, that f is not increasing $\Rightarrow$ there exist $x < y$ with $f(x) > f(y)$. We have two cases (we can ignore the cases when $x = 0$ or $y = 1$ due to the fact that, since f is continuous, we can go slightly to the right of 0 or slightly to the left of 1, and find x or y respectively, which still satisfy the inequality!):

case 1: $f(x) > f(0)$; since f is continuous, f takes all values between $f(0)$ and $f(x)$ AND all values between $f(x)$ and $f(y)$ (remember that $f(x) > f(y)$!) - but that means that f takes all values between $\max(f(0), f(y))$ and $f(x)$ (at least) twice! (at least once to the left of x and at least once to the right of x), which is unheard of ... uh ... no, contradicts the hypothesis :)

case 2: $f(x) < f(0)$ (cannot have equality, right?) but that means that $f(y) < f(x) < f(0) < f(1)$, hence we are in the symmetric setting, when

$f(x) > f(y)$ and $f(y) < f(1)$ ... and we'll get that f takes at least twice all the values between $f(y)$ and $\min(f(x), f(1))$, contradiction.

Hence, since we have contradiction for all possible cases, it means that our asssumption is wrong, hence f is increasing.

□

$\boxed{13}$

Proof. f is continuous on a compact interval, hence it attains its absolute maximum; since it must have same value EXACTLY twice, we must have this maximum EXACTLY twice, and let's call these to numbers at which we have the max c_1 and c_2, $c_1 < c_2$. If $c_1 \neq 0 \Rightarrow$ we get all values of f in points slightly to the left of c_1 (and slightly to the right of c_1) THREE times: twice around c_1, and once slightly to the left of c_2! (it's not a totally rigurous proof - if you get annyoed by the "slightly" you can work with δ neighbourhoods ... but I think it's rigurous enough for our purpose). Same will happen if c_2 is not 1 - we'll get three times the values slightly around c_2, because of what happens slightly to the right of c_1. Hence $c_1 = 0$ and $c_2 = 1$.

Taking now the absolute minimum, we have $d_1 < d_2$ the two numbers where we have it ... and again, if these are not 0 and 1, we get same values at least thrice (uh, three times) ... but since the place is already taken by the absolute maximums, the only thing that we can have is to have constant function (only for it we can have abs max=abs min) ... but constant function has same value infinite number of times, so it's not good, either. Hence ... no such function exists.

□