SOLUTIONS CHAPTER 6.1

MATH 548 SP00

$\boxed{\text{1c}}$

Proof.
$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}*\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\lim_{x\to a}\frac{x-a}{x-a}\frac{1}{\sqrt{x}+\sqrt{a}}=$$
$$=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$
□

$\boxed{4}$

Proof. We have to compute
$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}$$
which equals $\frac{x^2}{x}=x$ for x rational and $\frac{0}{x}=0$ for x irrational. But then we have
$$-|x|\le\frac{f(x)}{x}\le|x|$$
and by the Squeeze Theorem (limits to 0 for both $-|x|$ and $|x|$ are 0)
$$\lim_{x\to 0}\frac{f(x)}{x}=0$$
By definition, since the limit exists it means f is differentiable, and more than that, $f'(0)=0$.

Note: f is not differentiable anywhere else, since first of all f is not continuous anywhere else! □

$\boxed{\text{5d}}$

Proof. **Chain Rule**:
$$k'(x)=\tan'(x^2)*(x^2)'=\sec^2(x^2)*2x$$
□

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$\boxed{\text{8b}}$

Proof. $g = 2x + |x| \Rightarrow$ since $2x$ is differentiable everywhere we have to see where $|x|$ is not differentiable ... and the only point where this happens is $x = 0$ (to the left of 0 its derivative is -1, to the right it's 1 ... but in 0 it doesn't exist - check using the definition, and taking limit to the left **and** to the right, and you'll see you get different values).

Now, to find this derivative, we base our computation on dividing **R** into two parts:

$(-\infty, 0)$: $g(x) = 2x + (-x) = 2x - x = x \Rightarrow g'(x) = 1$

$(0, \infty)$: $g(x) = 2x + x = 3x \Rightarrow g'(x) = 3$

□

$\boxed{10}$

Proof. Everywhere but in 0 we can compute the derivative of g using the rules we know (product, chain, etc):

$$g' = 2x * \sin\left(\frac{1}{x^2}\right) + x^2 * \cos\left(\frac{1}{x^2}\right)(-2)x^{-3} = 2x * \sin\left(\frac{1}{x^2}\right) - 2\frac{1}{x} * \cos\left(\frac{1}{x^2}\right)$$

For 0 though it doesn't work ... (try pluging in 0 in the above formula ... it does **not** work). So ... we have to use the definition:

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x^2}\right)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$$

We have that

$$-|x| \leq x \sin\left(\frac{1}{x^2}\right) \leq |x|$$

(since the sine function ranges from -1 to 1 etc) so by Squeeze Theorem we get that

$$g'(0) = \lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right) = 0$$

hence the derivative exists everywhere, and is defined as above ... separately for $x \neq 0$ and $x = 0$.

As for the boundedness of g' just notice that the first term of its formula is always bounded (the x tames the fluctuating sine) but the second part is too wild, since $\frac{1}{x^2}$ goes to infinity, and the cosine fluctuates between -1 and 1. Let's find a sequence that converges to 0, and which plugged into g' goes to infinity; for that let's have the value through cosine a fixed value, let's say 1 ... this happens for $2k\pi$ for $k \in \mathbf{N}$ ... we want hence to have

$$\frac{1}{x^2} = 2k\pi \Rightarrow x = \frac{1}{\sqrt{2\pi}\sqrt{k}}$$

Let's see if this one accomodates us: the first term actually becomes 0! (since $\sin 2k\pi = 0$), and the second term becomes $-\sqrt{2\pi}\sqrt{k}$, which, as $k \to \infty$ goes to $-\infty$ ... proving that g' is not bounded (if it were, it would not be able to go beyond certain values ... but here it actually goes beyond any value!).

□

$\boxed{\text{11b}}$

Proof. **chain rule**:
$g' = 3(L(x^2))^2 * (L(x^2))' = 3(L(x^2))^2 * L'(x^2) * (x^2)' = 3(L(x^2))^2 * L'(x^2) * 2x$
and let's use $L' = \frac{1}{x}$:

$$g' = 3(L(x^2))^2 * \frac{1}{x^2} * 2x = 6\frac{L^2(x^2)}{x}$$

□

$\boxed{13}$

Proof. Since f is differentiable it means that

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

But the limit can be viewed also in the spirit of the sequential definition of a limit ... in the sense that any sequence $(h_n)_{n\in\mathbf{N}}$ with $h_n \to_{(n\to\infty)} 0$ will make $\frac{f(c+h_n)-f(c)}{h_n} \to_{(n\to\infty)} f'(c)$; taking the particular sequence $h_n = \frac{1}{n}$ we get

$$f'(c) = \lim_{n\to\infty} \frac{f(c+\frac{1}{n}) - f(c)}{\frac{1}{n}} = \lim_{n\to\infty} n(f(c+\frac{1}{n}) - f(c))$$

Note: it does **not** go both ways! you can have the above limit to exist even if the derivative of f does not (think of that function defined separately for rationals and irrationals!) □

$\boxed{14}$

Proof. The points corresponding to $x = 0, 1, -1$ are, respectively: $y = 0^3 + 2*0+1 = 1, 1^3+2*1+1 = 4, (-1)^3+2(-1)+1 = -2$. Also, $h' = 3x^2+2$, and based on that we have

$$(h^{-1})'(y) = \frac{1}{h'(h^{-1}(y))} = \frac{1}{h'(x)}$$

where y and x are corresponding ($y = h(x) \iff x = h^{-1}(y)$)

We need to compute then:

$$(h^{-1})'(1) = \frac{1}{3*0^2+2} = \frac{1}{2}$$
$$(h^{-1})'(4) = \frac{1}{3*1^2+2} = \frac{1}{5}$$
$$(h^{-1})'(-2) = \frac{1}{3*(-1)^2+2} = \frac{1}{5}$$

□