SOLUTIONS CHAPTER 6.2

MATH 548 SP00

$\boxed{\text{2a}}$

Proof. Compute the derivative:

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

The sign of this function is as follows: "+" in $(-\infty, -1)$, "−" in $(-1, 0)$ and $(0, 1)$, "+" in $(1, \infty)$ (mind the fact that the denominator is **always** positive). Hence we have f increasing up to -1, decreasing up to 1 (in the sense that to the left of 0 it decreases to $-\infty$ and then to the right it starts from ∞ and decreases further) and increasing afterwards; hence we have a relative max in -1, not defined (of course!) in 0 and relative minimum in 1. □

$\boxed{\text{2b}}$

Proof. Compute the derivative:

$$\frac{(x^2+1) - x * 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

The sign of this derivative is (again, the denominator is positive!): "−" in $(-\infty, -1)$, "+" in $(-1, 1)$ and "−" in $(1, \infty)$; hence f is decreasing up to -1, then increasing up to 1, then decreasing afterwards, with -1 being relative minimum and 1 being relative maximum. □

$\boxed{\text{3a}}$

Proof. By the definition of absolute value, we have that f is: $x^2 - 1$ if $x \in [-4, -1] \cup [1, 4]$; $1 - x^2$ for $x \in (-1, 1)$. Hence f' is:$2x$ if $x \in [-4, -1] \cup [1, 4]$; $-2x$ for $x \in (-1, 1)$. We notice that f' doesn't exist for $x = \pm 1$. Hence critical numbers are ± 1 and 0 (that's where the derivative is 0) [1]. Let's see what happens to the sign of f' ... which is best to find relative extrema: f' is "−" for $x \in (-4, -1)$, then "+" for $x \in (-1, 0)$, "−" for $x \in (0, 1)$ and finally "+" for $x \in (1, 4)$. So we see that -1 and 1 are relative minima, and 0 is a relative maximum (why? f decreases up to -1, increases afterwards up to 0, decreases again up to 1, and decreases afterwards up to 4 ... following the behaviour of f'-'s sign. □

Date: 05/17/2000.

[1] remember that critical number means the x-value where f' is 0 or doesn't exist!

$\boxed{4}$

Proof. Let's find the derivative of f:

$$f' = \sum_{i=1}^{n} 2(a_i - x) * (-1) = -2a_1 + 2x - 2a_2 + 2x - \cdots - 2a_n + 2x =$$

$$= -2(a_1 + a_2 + \cdots + a_n) + 2nx$$

The zero of f' is $\frac{2(a_1+a_2+\cdots+a_n)}{2n} = \frac{a_1+a_2+\cdots+a_n}{n}$ and the sign of this function is as follows:"$-$" for $x < \frac{a_1+a_2+\cdots+a_n}{n}$ and "+" for $x > \frac{a_1+a_2+\cdots+a_n}{n}$, so $\frac{a_1+a_2+\cdots+a_n}{n}$ is a relative minimum (f decreases up to it, and increases afterwards). $\square$

$\boxed{6}$

Proof. By MVTh we have that, since sin is continuous and differentiable actually everywhere, for any interval $[x, y]$ there exists a $c \in (x, y)$ such that $\sin y - \sin x = \sin' c * (y - x) \iff \sin y - \sin x = \cos c * (y - x) \Rightarrow |\sin y - \sin x| = |\cos c| * |(y - x)|$. But $|\cos c| \leq 1$ for any c, hence

$$|\sin y - \sin x| = |\cos c| * |(y - x)| \leq 1 * |(y - x)| \Rightarrow$$

$$\Rightarrow |\sin y - \sin x| \leq |y - x|$$

$\square$

$\boxed{7}$

Proof. Since log is continuous and differentiable for $x \geq 1$ we have (by MVTh) that there's a $c \in (1, x)$ sucht that $\log x - \log 1 = \log' c(x - 1) \iff \log x - \log 1 = \frac{1}{c}(x - 1)$ Use now the fact that $c > 1$ (and tacitly the fact that $\log 1 = 0$):

$$\log x = \frac{x-1}{c} < \frac{x-1}{1} = x - 1$$

which is the second inequality that needed to be proved; secondly, $c < x$ so we have:

$$\log x = \frac{x-1}{c} > \frac{x-1}{x}$$

which is the first inequality. $\square$

$\boxed{10}$

Proof. To compute $g'(0)$ we have to rely on the definition; why? let's take a look at the derivative elsewhere:

$$g' = 1 + 4x \sin\frac{1}{x} + 2x^2 * \cos\frac{1}{x}(-\frac{1}{x^2}) = 1 + 4x \sin\frac{1}{x} - 2\cos\frac{1}{x}$$

and as you can see getting close to zero meeses up the third term (first is just 1, and second gets tamed by the x, which will make the second term go to 0 ... nothing helps the third term though).

$$g'(0) = \lim_{x \to 0} \frac{x + 2x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} 1 + 2x \sin \frac{1}{x} = 1$$

(again, the x tames the wild $\sin \frac{1}{x}$). As for the multitude of signs around 0, check the following two sequences: $x_n = \frac{2}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$; both converge to 0 (hence we get as close to 0 as we want, and $g'(x_n) = 1 + \frac{1}{2n\pi} \sin \frac{1}{\frac{1}{2n\pi}} - 2 \cos \frac{1}{\frac{1}{2n\pi}} = 1 + \frac{1}{2n\pi} \sin 2n\pi - 2 \cos 2n\pi = 1 - 2 = -1$ while $g'(y_n) = 1 + 2 = 3$, so one is negative while the second one is positive $\Rightarrow$ we have both signs in any neighbourhood of 0. □

$\boxed{12}$

Proof. If such an f existed, it would be differentiable (hence continuous) hence we could use the MVTh ... but then we would have, for $x > 0$:

$$f(x) - f(0) = f'(c)(x - 0) = x \Rightarrow f(x) = x + f(0)$$

for some $0 < c < x$. Also, for $x < 0$ we sould have:

$$f(0) - f(x) = f'(d)(0 - x) = 0 * (0 - x) = 0 \Rightarrow f(x) = f(0)$$

for some $x < d < 0$. So our function should look like this: $f(x) = x + f(0)$ for $x > 0$ and $f(x) = f(0)$ for $x < 0$. This function is continuous alright ... but it's not diffferentiable in 0!! so we reach a dead end: there's no such function.

As an example for two functions ... just make the negative part differ by a constant, while keeping the positive part fixed: $g(x) = 0$ for $x < 0$ and $g(x) = x$ for $x \geq 0$ and $h(x) = 1$ for $x < 0$ and $h(x) = x$ for $x \geq 0$ □

$\boxed{13}$

Proof. Just use MVTh: have $x < y$, $x, y \in I$, and since f is differentiable (hence continuous) in I, it is also in the interval $[x, y]$; we get:

$$f(y) - f(x) = f'(c)(y - x)$$

for some $c \in (x, y)$; but $f'(c) > 0$, $y - x > 0$ hence

$$f(y) - f(x) > 0 \Rightarrow f(y) > f(x)$$

hence we get that f is strictly increasing ($f(x) = f(y)$ would mean that there's a c such that $f'(c) = 0$, which is not true, f' being **strictly** bigger than 0). □