SOLUTIONS CHAPTER 6.3

MATH 548 SP00

$\boxed{2}$

Proof. Note that we **cannot** use L'Hospital's rule! or 6.3.1 for that purpose, since $\lim_{x\to c} f(x) = A \neq 0$ so we have to rely on something different - so let's simply use the definiton of the limit.

Let $M > 0$. We have that there exists $\delta' > 0$ such that $f(x) > \frac{A}{2}$ for all x such that $|x - c| < \delta'$. At the same time, since $\lim_{x\to c} g(x) = 0 \Rightarrow$ there exists a δ'' such that $g(x) = |g(x)| < \frac{A}{2M}$ (take in account that $g(x) > 0$). Let now $\delta = \min(\delta', \delta'')$ and we have that for x such that $|x - c| < \delta$

$$\frac{f(x)}{g(x)} > \frac{\frac{A}{2}}{\frac{A}{2M}} = M$$

which means that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$$

As for the case when $A < 0$ we have that, if solving the problem in the above manner for $-f$ and g,

$$\lim_{x\to c} \frac{-f(x)}{g(x)} = \infty \Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$$

□

$\boxed{4}$

Proof. Using Theorem 6.3.1 (possible since: $f(0) = g(0) = 0, g(x) \neq 0$ for $0 < x < \frac{\pi}{2}$ - since we need limit at 0 we can restrict our attention to a small interval with 0 as an endpoint - and since g is clearly differentiable actually everywhere, and f is differentiable in 0 - not so clearly, but if one computes $\lim_{x\to 0} \frac{f(x)}{x}$ we can use Squeeze Theorem to prove it's 0 - **and** $g'(0) = \cos 0 = 1 \neq 0$) we get

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \frac{0}{1} = 0$$

Using now symmetry of f and (anti)symmetry of sin we get that

$$lim_{x\to 0^-} \frac{f(x)}{g(x)} = 0$$

too. Hence the limit is 0.

Date: 05/23/2000.

We cannot use Theorem 6.3.3 since one key feature of f should be continuity - but you can see that f is **not** continuous but in 0, and this is not enough (it's required to have continuity on a full interval). $\square$

The following problems involve also a short check whether one can use the mentioned theorem (like continuity, differentiability and so on). I will leave this for you as an (useful) exercise - helps remember the theorems, e.g.

$\boxed{\text{6b}}$

Proof. Using 6.3.3 (and, again, we don't need to have the full interval $(0, \frac{\pi}{2})$ at our disposal, but merely $(0, \frac{\pi}{3})$ will be more than enough - so 6.3.3 can be used without any problems, even though tan is not defined in $\frac{\pi}{2}$!)we get:

$$\lim_{x \to 0^+} \frac{\tan x}{x} = \lim_{x \to 0^+} \frac{\frac{1}{\sec^2 x}}{1} = \frac{1}{\sec^2 0} = 1$$

$\square$

$\boxed{\text{7b}}$

Proof. We need to rearrange the expression a bit (based on $x > 0$ among others):

$$\frac{1}{x(\log x)^2} = (\frac{1}{\sqrt{x}\log x})^2 = (\frac{\frac{1}{\sqrt{x}}}{\log x})^2$$

and we need to compute just the limit of the expression inside the "square", using 6.3.6:

$$\lim_{x \to 0} \frac{\frac{1}{\sqrt{x}}}{\log x} = \lim_{x \to 0} \frac{-\frac{1}{2x^{\frac{3}{2}}}}{\frac{1}{x}} = -\frac{1}{2} \lim_{x \to 0} \frac{1}{x^{\frac{1}{2}}} = \infty$$

hence the square limit is also ∞. $\square$

$\boxed{\text{8d}}$

Proof. Using the modified Theorem 6.3.6 (not written down in the textbook, but merely mentioned - page 212, just before Example 6.3.7)we get:

$$\lim_{x \to \infty} \frac{x + \log x}{x \log x} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{\log(x) + 1} = \frac{1}{\infty} = 0$$

$\square$

$\boxed{\text{9c}}$

Proof. We start by taking the ln of the expression (case 1^∞) and we go for the following limit:

$$\lim_{x \to \infty} x \ln(1 + \frac{3}{x})$$

Modifying it a bit we get, by 6.3.4:

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}}\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} = 3 \lim_{x\to\infty} \frac{1}{1+\frac{3}{x}} = 3$$

Finally, we have to get back to our original limit, which is exp of the above result: $\lim = e^3$ □

$\boxed{\text{10d}}$

Proof. Rewrite (using the formulas of sec and tan):

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1-\sin x}{\cos x}$$

Using now a symmetric form of 6.3.6 (concerning left limits, and not right limits)we get:

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{\sin x} = \frac{-0}{1} = 0$$

□