SOLUTIONS CHAPTER 7.2

MATH 549 AU00

$\boxed{2}$

Proof. Let's prove the first inequality - the second one is dual to the first, and proof is similar.

Taking a partition P_n we obviously have that $L(P_n, f) \leq L(P_n, g)$. Taking now the sup on the right-hand side (which will yield a bigger value) we get $L(P_n, f) \leq L(g)$. Hence we get that, for any partition P_n, $L(P_n, f)$ is less than the FIXED number $L(g)$. Hence, the sup of $L(P_n, f)$ cannot be larger than this number (in short, because that would imply the existence of partitions Q_m for which $L(Q_m, f)$ is bigger, absurd!), hence $L(f) \leq L(g)$. Done! □

$\boxed{7}$

Proof. By the definition of $L(f)$ we have that it's the sup on ALL POSSIBLE partitions (in particular, these include the special partitions that contain c). Since we're taking for $L(f)$ the sup on a larger set (hence we have a bigger variety of values, which means we can have bigger values - the lower values don't interest us for sup) it means that $L(f) \geq \sup_{P \in P_c} L(P, f)$.

Take now any partition P_n. A refined partition Q_m will yield $L(Q_m, f)$ that is greater than $L(P_n, f)$. So, let's refine P_n by putting c into it (well, c might be already point of P_n, that's true - but then instead of $\geq$ we have $=$, which is still OK). Hence we have that $L(P_n, f) \leq L((P_n \cup \{c\}), f)$; but the right-hand side is less than the sup on partitions that contain c, hence $L(P_n, f) \leq \sup_{P \in P_c} L(P, f)$. The right-hand side is now a FIXED number, so we can take the sup in the left-hand side as well, and there it will yield $L(f) \Rightarrow L(f) \leq \sup_{P \in P_c} L(P, f)$.

Combining these two inequalities (observe that one is one way and the other is the other way) we get equality. Which is what we wanted.

□

$\boxed{11}$

Proof. A simple example would be f defined as follows: $f(x) = x$ for $x > 0$; $f(0) = 1$. This function is integrable (you can even try to show this using Riemann sums!). As for $\frac{1}{f}$ it's obvious it's not bounded (as x approaches 0, $\frac{1}{x}$

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approaches ∞), so it's not integrable (the definition of Riemann integrability involves boundedness!).

$\square$

$\boxed{12}$

Proof. Take a variation of the Dirichlet function: $g(x) = 1$ if $x \in \mathbb{Q}$; $g(x) = -1$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $|g| = 1$, which is obviously integrable (you already know that the Dirichlet function is not - hence neither is g).

$\square$

$\boxed{15}$

Proof. The most trivial example (which is also quite general, at the same time) would be $f = 0$ and g any non-integrable function (something like the Dirichlet function, let's say). $fg = 0$, so that's that.

$\square$

$\boxed{17}$

Proof. We have:

$$m \leq f \leq M \iff$$
$$\iff m^2 \leq f^2 \leq M^2 \Rightarrow$$
$$\Rightarrow \int_a^b m^2 \leq \int_a^b f^2 \leq \int_a^b M^2 \iff$$
$$\iff (b-a)m^2 \leq \int_a^b f^2 \leq (b-a)M^2 \iff$$
$$\iff m^2 \leq \frac{1}{b-a}\int_a^b f^2 \leq M^2 \iff$$
$$\iff m \leq [\frac{1}{b-a}\int_a^b f^2]^{\frac{1}{2}} \leq M$$

(we essentially used the fact that $0 \leq m, M, f(x)\ \forall x$).

$\square$