SOLUTIONS CHAPTER 7.3

MATH 549 AU00

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Proof. If you remember problem # 3, last section, we know that a function ϕ defined as: $\phi(x) = 0$ when $x \in [0, \frac{1}{2}]$ and $\phi(x) = 1$ when $x \in (\frac{1}{2}, 1]$ is integrable. In a similar fashion we can prove that the following two functions, defined on $[-1, 1]$, are integrable: ϕ_1 defined as $\phi_1(x) = -1$ when $x \in [-1, 0)$ and $\phi_1(x) = 0$ when $x \in [0, 1]$; ϕ_2 defined as $\phi_2(x) = 0$ when $x \in [-1, 0]$ and $\phi_2(x) = 1$ when $x \in (0, 1]$ are integrable. By the summation property of integrable functions we know that $\phi_1 + \phi_2$ is integrable as well. But notice that $\text{sgn} = \phi_1 + \phi_2$ (when x is negative ϕ_1 kicks in, while ϕ_2 is zero, etc).

If sgn had an antiderivative $\Rightarrow$ there exists a function Φ such that $\Phi' = \text{sgn}$. Let's try to see how this Φ should look like: since for $x < 0$ sgn$=-1 \Rightarrow$ by Fundamental Theorem of Calculus (since Φ is differentiable it must also be continuous!) for $x < 0$ we get:

$$\int_{-1}^{x} \text{sgn} = \Phi(x) - \Phi(-1) \Rightarrow \Phi(x) = \int_{-1}^{x} (-1) + \Phi(-1)$$

$$\Phi(x) = -x + 1 + \Phi(-1)$$

Similarly we get

$$\int_{x}^{1} \text{sgn} = \Phi(1) - \Phi(x) \Rightarrow \Phi(x) = \Phi(1) - (\int_{x}^{1} 1)$$

$$\Phi(x) = \Phi(1) - (1 - x) = x + \Phi(1) - 1$$

for $x > 0$.We need now to patch these formulas together in 0 - and it has to be a continuous, and actually **differentiable** patch! by continuity we get $\Phi(0) = -0 + 1 + \Phi(-1) = 0 + \Phi(1) - 1 \Rightarrow \Phi(1) - \Phi(-1) = 2$. Assuming that this condition is checked let's see how differentiality fares: $\Phi' = -1$ to the left of 0 and $\Phi' = 1$ to the right of 0 (differentiate the respective expressions of Φ). Approaching to 0 these values should get closer too, but they don't! (if you compute the derivative using definition - with limits - you'll see the necessity to compute the limit from the left and from the right; but these two limits do not coincide) so you can only infer that the derivative in 0 does not exist - and this contradicts the assumption that Φ were differentiable everywhere. Hence the Φ does not exist.

Obviously $H' = \text{sgn}$ (when $x < 0$ then $H(x) = |x| = -x$ etc) for $x \neq 0$

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As for

$$\int_{-1}^{1} \text{sgn} = H(1) - H(-1)$$

we use the fact that $\int_{-1}^{0} \text{sgn} = 1$ and $\int_{0}^{1} \text{sgn} = 1$ while $H(1) - H(-1) = 1 - (-1) = 1 + 1 = 2$ so the above equality (also based on Theorem 7.2.4, which allows us to compute an integral on subintervals, and then add them back together) holds. Note that it's a coincidence (taking $G(x)$ defined as $G(x) = -x - 10$ when $x < 0$ and $G(x) = x + \pi$ when $x \geq 0$, $G' = \text{sgn}$ for $x \neq 0$, but the integral does not equal $G(1) - G(-1)$ - check this). It will hold, however, if we have the aditional assumption that H be continuous. □

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Proof. a) sgn is integrable on $[-1, 1]$ hence it is also on $[0, b]$; we have that H is continuous, and indeed $H' = \text{sgn}$ for $x \in (0, b)$. Since the conditions in the First Form of the Fundamental Theorem of Calculus are checked, we can use its conclusion, which in our case states:

$$\int_{0}^{b} \text{sgn} = H(b) - H(0) = b$$

b) We can do the same thing on the interval $[a, 0]$ (H is continuous, $H' = \text{sgn}$, sgn which is integrable) so 1st Form of FTC gives us

$$\int_{a}^{0} \text{sgn} = H(0) - H(a) = -|a| = -(-a) = a$$

(remember that $a < 0$). By Theorem 7.2.4 (we used it above too) we have that

$$\int_{a}^{b} \text{sgn} = \int_{a}^{0} \text{sgn} + \int_{0}^{b} \text{sgn} = a + b$$

which concludes our proof.

Note that it was essential that H were continuous - if it weren't, FTC could not have been used. □

5

Proof. $x \neq 0$:

$$G'(x) = 2x * \sin(\frac{\pi}{x^2}) + x^2 * \cos(\frac{\pi}{x^2}) * (-2)\pi x^{-3} = 2x * \sin(\frac{\pi}{x^2}) + \frac{\cos(\frac{\pi}{x^2}) * (-2)\pi}{x}$$

$x = 0$ - use limits:

$$\lim_{x \to 0} \frac{G(x) - G(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(\frac{\pi}{x^2}) - 0}{x} = \lim_{x \to 0} x \sin(\frac{\pi}{x^2}) = 0$$

because the sin keeps between -1 and 1, so it's bounded, and $x \to 0$ (if you want to give it more details, go for the Squeeze Theorem, and "squeeze" the above formula between $-|x|$ and $|x|$). Hence $g = G'$ exists everywhere. The fact that g is not bounded we can check by looking at the above expression:

the part with the sin will go to 0, since it has an x appended to it, while the other term has an x in the denominator ... so making x arbitrarily small will kill the first term, but quite probably will blow up the second; the best way to prove this is by finding a sequence $\{x_n\}_{n\in\mathbb{N}} \to 0$ and which will allow the above expression to sport the above commented behaviour. Let's make the sin equal 0 and the cos equal 1 (to make the matter simpler): $\frac{\pi}{x^2} = 2n\pi \Rightarrow x^2 = \frac{1}{n} \Rightarrow x = \frac{1}{\sqrt{n}}$. We notice that the sequence $x_n = \frac{1}{\sqrt{n}}$ converges to 0. The sin dissapears ($\sin(2n\pi) = 0$) and cos is 1 ($\cos(2n\pi) = 1$) so the above expression becomes $\frac{-2\pi}{\frac{1}{\sqrt{n}}} = -2\sqrt{n}\pi \to \infty$.

As $a > 0$ we have that g is continuous and hence bounded on $[a, 1]$ (g is actually computed above!) so it's integrable on this interval, so by the FTC we have:

$$\int_a^1 g = G(1) - G(a) = \sin(\pi) - a^2\sin(\frac{\pi}{a^2}) = -a^2\sin(\frac{\pi}{a^2})$$

Taking now limit we have that the sin is bounded, and that, as $a \to 0$, $a^2 \to 0$ as well, hence the limit is 0 (again, for more details use Squeeze Theorem). So the limit does exist, indeed. □

10

Proof. Obviously F is continuous on any interval $(n-1, n)$. The problem is, what happens at the endpoints (mind the fact that to the right of n it's the interval $[n, n+1) = [(n-1)+1, n+1)$, so in the formula of F you have to increment all occurences of n for $x > n$)

$$\lim_{x\to n^-} F(x) = \lim_{x\to n^-} (n-1)x - \frac{(n-1)n}{2} = (n-1)n - \frac{(n-1)n}{2} = \frac{(n-1)n}{2}$$

$$\lim_{x\to n^+} F(x) = \lim_{x\to n^+} nx - \frac{n(n+1)}{2} = n * n - \frac{n^2+n}{2} = \frac{2n^2}{2} - \frac{n^2+n}{2} =$$

$$= \frac{2n^2 - n^2 - n}{2} = \frac{n^2 - n}{2} = \frac{(n-1)n}{2}$$

Since the two limits coincide, and they actually equal the value of F in n, F is continuous at n, hence at all n's, hence continuous everywhere.

Now ... on each interval $[n, n+1]$ F is continuous, on $(n, n+1)$ it's differentiable (it's linear, in fact) and moreover $F' = n \Rightarrow$ for $n < x < n+1$ $F' = [x]$. Hence

$$\int_a^b [x] = F(b) - F(a)$$

as long as $n < a \le b < n+1$.

If $0 \le a < b$, we can of course divide the interval $[a, b]$ into $[a, k], [k, k+1], [k+1, k+2], \dots, [k+n, b]$, where $k < k+1 < \cdots < k+n$ are all the integers between a and b (of course, there might be none - above case - or some amount of them; we can have $b = k+n$ but we assume that if a is an integer, we start with $a+1$, in order to be consistent with the definition of F). We can use Theorem 7.2.4 again, and compute integral on each subinterval (we can, since the FTC on each is satisfied), so

$$\int_a^b [x] = \int_a^k [x] + \int_k^{k+1} [x] \dots \int_{k+n}^b [x] =$$

$$= (F(k) - F(a)) + (F(k+1) - F(k)) + \cdots + (F(b) - F(k+n)) = F(b) - F(a)$$

where you can of course find formulas for $F(a)$ and $F(b)$ also:

$$k - 1 \leq a < k \Rightarrow F(a) = (k-1)a - \frac{(k-1)k}{2}$$

and

$$k + n \leq b < k + n + 1 \Rightarrow F(b) = (k+n)b - \frac{(k+n)(k+n+1)}{2}$$

□

13a

Proof. Call the antiderivative $\int \sin(t^2) = S(t)$. We have by FTC:

$$F(x) = S(x) - S(0) \Rightarrow F'(x) = (S(x) - S(0))' = (S(x))' - 0 = S'(x)$$

but S is the antiderivative of $\sin(t^2)$ hence $S' = \sin(t^2) \Rightarrow$

$$F'(x) = \sin(x^2)$$

□

20a

Proof. Use substitution (I'll leave as a job for you to do the justification - that is, point out each Theorem/Corollary/Proposition that is used) $1+t^2 = u \Rightarrow 2t\,dt = du$ hence the integral becomes

$$\int t\sqrt{1+t^2}\,dt = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{(1+t^2)^{\frac{3}{2}}}{\frac{3}{2}}$$

hence the definite integral is

$$\int_0^1 t\sqrt{1+t^2}\,dt = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}|_0^1 = \frac{2}{3}$$

□

21b

Proof. Use again substitution, $2t + 3 = u \Rightarrow 2\,dt = du$ but also $t = \frac{u-3}{2}$. Hence the integral becomes

$$\int t\sqrt{2t+3}\,dt = \int \frac{u-3}{2} * \sqrt{u} * \frac{1}{2}\,du = \frac{1}{4}\int u\sqrt{u} - 3\sqrt{u}\,du =$$

$$\frac{1}{4}\int u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\,du$$

and I'll leave you the pleasure to finish it off (get the antiderivative, put back the variable t, use the FTC then to compute the actual definite integral) □

22b

Proof. Use substitution, but in another sense: $t = (x-1)^2 \Rightarrow dt = 2(x-1)\,dx$, but also $\sqrt{t} = x - 1 \Rightarrow x = \sqrt{t}+1$) hence the integral becomes:

$$\int \frac{\sqrt{t}}{1+\sqrt{t}}\,dt = \int \frac{x-1}{1+x-1} 2(x-1)\,dx = 2\int \frac{x^2-2x+1}{x}\,dx =$$
$$= 2\int (\frac{x^2}{x} - \frac{2x}{x} + \frac{1}{x})\,dx = 2\int (x - 2 + \frac{1}{x})\,dx =$$
$$= x^2 - 4x + 2\ln(x) = (\sqrt{t}+1)^2 - 4(\sqrt{t}+1) + 2\ln(\sqrt{t}+1)$$

Use now FTC and you're done. □