SOLUTIONS CHAPTER 7.4

MATH 549 AU00

$\boxed{\text{12a}}$

Proof. The log has problems at 0, but the other points are clear. So we have to compute $\int_a^1 \log(x)\,dx$ and then take lim as $a \to 0$. For the integral use Integration by Parts, taking $u = \log$ and $dv = 1$.

$$\int_a^1 \log(x) * 1\,dx = \log(x) * x|_a^1 - \int_a^1 \frac{1}{x} * x\,dx =$$

$$= (\log(1) * 1 - \log(a) * a) - x|_a^1 = -a\log(a) - (1-a)$$

As $a \to 0 \Rightarrow (1-a) \to 1$, so we only have to care about the first term. Use l'Hospital ($0 * \infty$ case).

$$\lim_{a\to 0} -a\log(a) = \lim_{a\to 0} -\frac{\log(a)}{\frac{1}{a}} = \lim_{a\to 0} -\frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0} a = 0$$

Hence the improper integral exists and is equal to $0 - 1 = -1$.

$\square$

$\boxed{\text{12d}}$

Proof. Again, we have some trouble at 0 ("$\log(0) = -\infty$") so again compute $\int_a^1 x\log(x)\,dx$ and then take lim as $a \to 0$. For the integral use Integration by Parts, taking $u = \log$ and $dv = x$.

$$\int_a^1 \log(x) * x\,dx = \log(x) * \frac{x^2}{2}|_a^1 - \int_a^1 \frac{1}{x} * \frac{x^2}{2}\,dx =$$

$$= (\log(1) * \frac{1}{2} - \log(a) * \frac{a^2}{2}) - \frac{1}{2}\frac{x^2}{2}|_a^1 = -\frac{a^2\log(a)}{2} - \frac{1-a^2}{4}$$

As $a \to 0 \Rightarrow (1-a^2) \to 1$, so we only have to care about the first term. Use l'Hospital ($0 * \infty$ case).

$$\lim_{a\to 0} a^2\log(a) = \lim_{a\to 0} \frac{\log(a)}{\frac{1}{a^2}} = \lim_{a\to 0} \frac{\frac{1}{a}}{-2\frac{1}{a^3}} = \lim_{a\to 0} -\frac{1}{2}a^2 = 0$$

Hence the improper integral exists and is equal to $0 - \frac{1}{4} = -\frac{1}{4}$. $\square$

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$\boxed{\text{14c}}$

Proof. The integral is computed to ∞ .. so it's improper. To compute it we need (parts, $u = \log$, $dv = \frac{1}{x}$)

$$\int_2^M \frac{\log(x)}{x}\,dx = \log(x) * \log(x)|_2^M - \int_2^M \frac{1}{x}\log(x)\,dx \Rightarrow$$

$$\Rightarrow 2 * \int_2^M \frac{\log(x)}{x}\,dx = (\log(x))^2|_2^M \Rightarrow$$

$$\Rightarrow \int_2^M \frac{\log(x)}{x}\,dx = \frac{1}{2}(\log^2(M) - \log^2(2))$$

Taking now lim as $M \to \infty$ we get that the right-hand side diverges! ($\log^2(M) \to \infty$), so the improper integral doesn't exist. □

$\boxed{\text{14d}}$

Proof. Use substitution this time (notice it's improper 'cause of the ∞ ...) and say $\log(x) = u \Rightarrow du = \frac{1}{x}\,dx$ and we have:

$$\int \frac{1}{x\log^2(x)}\,dx = \int \frac{1}{u^2}\,du = \frac{u^{-1}}{-1} = -\frac{1}{\log(x)}$$

hence

$$\int_2^M \frac{1}{x\log^2(x)}\,dx = -\frac{1}{\log(x)}|_2^M = \frac{1}{\log(2)} - \frac{1}{\log(M)}$$

As $M \to \infty$ the second term obviously dies, so the limit of the integral as $M \to \infty$ equals $\frac{1}{\log(2)}$ (that is, the improper integral exists, and that's its value). □

$\boxed{\text{15b}}$

Proof. Take $\int_{-\infty}^0 e^{-x}$ which we compute as follows:

$$\int_{-M}^0 e^{-x}\,dx = -e^{-x}|_{-M}^0 = -e^0 + e^M = e^M - 1$$

As $M \to \infty \Rightarrow -M \to -\infty$ the right-hand side diverges (to ∞). Hence the improper integral diverges.

Note that $\int_0^\infty$ actually exists ... but it's not enough, both must converge). □

$\boxed{\text{15c}}$

Proof. We have problems at both 0 and ∞. Let's compute

$$\int_a^1 \frac{1}{x^2}\,dx = -\frac{1}{x}|_a^1 = -1 + \frac{1}{a}$$

As $a \to 0 \Rightarrow$ the right-hand side diverges (to ∞ again). So (even though $\int_1^\infty \frac{1}{x^2}\,dx$ exists) the improper integral diverges. $\square$