SOLUTIONS CHAPTER 8.2

MATH 549 AU00

$\boxed{2}$

Proof. It's quite clear that the sequence is made out of continuous functions and that the limit itself is continuous. To check that the limit is right, namely that

$$\lim_{n\to\infty} f_n = f = 0$$

notice that for a fixed x, if $n > \frac{2}{x}$ then $f_n(x) = 0$ for all following n-s; so the limit is clearly 0. About non-uniformity ... as in the previous chapter we just need a value that's keeping f_n fixed, or (which will go better here) which will bring $f_n(x_n)$ even farther away from its limit of 0 (the only thing that should not happen is that $f_n(x_n)$ gets closer to 0!). Notice that $x_n = \frac{1}{n}$ does the trick, since $f_n(x_n) = n$ which definitely goes away from 0, and hence kills the uniform norm, by not allowing it to approach 0. $\square$

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Proof. We have to prove the following assertion:

$$\forall \epsilon > 0, \exists N = N(\epsilon) \text{such that} \, \forall n > N, |f_n(x_n) - f(x_0)| < \epsilon$$

and we use a very common trick, namely we try to break the difference in a sum of differences, which each, independently, can be made less than a fraction of ϵ, and by adding them back together it will give us the desired result. We have to use the fact that $f_n \to^u f$ and that f_n is continuous, $\forall n$.

$$|f_n(x_n) - f(x_0)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x_0)| \leq$$
$$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$$

Since $f_n \to^u f \Rightarrow \exists N^* = N^*(\epsilon)$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{4}, \forall\, n \geq N^* \Rightarrow$

$$|f_n(x_n) - f(x_n)| < \frac{\epsilon}{4}, \forall n \geq N^*$$

Using again the above assertion we get that

$$|f(x_n) - f(x_0)| = |f(x_n) - f_{N^*}(x_n) + f_{N^*}(x_n) - f_{N^*}(x_0) + f_{N^*}(x_0) - f(x_0)| \leq$$
$$\leq |f(x_n) - f_{N^*}(x_n)| + |f_{N^*}(x_n) - f_{N^*}(x_0)| + |f_{N^*}(x_0) - f(x_0)| \leq$$
$$\leq \frac{\epsilon}{4} + |f_{N^*}(x_n) - f_{N^*}(x_0)| + \frac{\epsilon}{4}$$

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But f_{N^*} is continuous, hence, since $x_n \to x_0$, $\exists N^{**} = N^{**}(\epsilon)$ such that $|f_{N^*}(x_n) - f_{N^*}(x_0)| < \frac{\epsilon}{4}$, $\forall n \geq N^{**}$.

Take now $N = N(\epsilon) = \max(N^*, N^{**})(= \max(N^*(\epsilon), N^{**}(\epsilon)))$. Combining the above inequalities we get:

$$|f(x_n) - f(x_0)| \leq \frac{\epsilon}{4} + |f_{N^*}(x_n) - f_{N^*}(x_0)| + \frac{\epsilon}{4} \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} = \frac{3\epsilon}{4}$$

which, if we now combine it with the very first inequality, will give us:

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| \leq$$

$$\leq \frac{\epsilon}{4} + \frac{3\epsilon}{4} = \epsilon, \forall n \geq N$$

Notice that N depends only on ϵ - which is the way it should be! (should not depend on a particular n, e.g.). Hence $f_n(x_n) \to f(x_0)$ (compare with the proof of **Theorem** 8.2.2).

□

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Proof. Since f is uniformly continuous it means that $\forall \epsilon > 0\ \exists \delta = \delta(\epsilon) > 0$ such that, $\forall x, y \in \mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$ (in other words, closeness of two values of f is insured by closeness of the two inputs ... counterexample would be $f = \frac{1}{x}$, e.g. - for this function, if you're close enough to 0, even if distance between x and y is really small you can have huge distance between $f(x)$ and $f(y)$!)

Let now $\epsilon > 0$ and we have to show that $\exists N = N(\epsilon)$ such that $|f_n(x) - f(x)| < \epsilon\ \forall n > N \iff |f(x + \frac{1}{n}) - f(x)| < \epsilon\ \forall n > N$. But if we choose $N = N(\epsilon)$ such that $\frac{1}{N} < \delta(\epsilon) \iff N > \frac{1}{\delta}$ we have that $|x + \frac{1}{n} - x| = \frac{1}{n} \leq \frac{1}{N} < \delta$ hence $|f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \epsilon$ (by uniform continuity) $\forall n > N$. Done! □

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Proof. $\frac{\sin(nx)}{nx} \to^u 0$ on $[a, \pi]$, since $|\frac{\sin(nx)}{nx}| \leq \frac{1}{nx} \leq \frac{1}{na} \to 0$. So we can interchange, by **Theorem** 8.2.4 the limit with the integral, and we get:

$$\lim \int \frac{\sin(nx)}{nx}\, dx = \int (\lim \frac{\sin(nx)}{nx})\, dx = \int 0\, dx = 0$$

If $a = 0$ the limit is no longer uniform: the actual limit is going to be

$$f = \begin{cases} 0, & x > 0 \\ 1, & x = 0 \end{cases}$$

(since $f_n(0) = \lim_{x \to 0} \frac{\sin(nx)}{nx} = \lim_{nx \to 0} \frac{\sin(nx)}{nx} = 1$).

Choose now a value less than 1, let's say $\frac{1}{2}$:

$$\frac{\sin(nx)}{nx} = \frac{1}{2} \Rightarrow \sin(nx) = \frac{nx}{2}$$

which has a certain solution x_n between 0 and $\frac{\pi}{n}$ (why? take the function $g = \sin(nx) - \frac{nx}{2}$; $g' = n(\cos(nx) - \frac{1}{2})$ is positive up to $x = \frac{\pi}{3n}$ and negative up to $\frac{\pi}{n}$, hence g is increasing first, from $g(0) = 0$ up to $g(\frac{\pi}{3n}) = \sin(\frac{\pi}{3}) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} > 0.1 > 0$, and then decreases up to $g(\frac{\pi}{n}) = \sin(\pi) - \frac{\pi}{2} = -\frac{\pi}{2}$; but g is continuous, so it must pass through zero, and we call that value x_n). Therefore the limit is no longer uniform. In this case we cannot use the above mentioned theorem.

We can try to use the boundedness of $\frac{\sin(nx)}{nx}$: take $a = \frac{\epsilon}{2}$; since $-1 \leq \frac{\sin(nx)}{nx} \leq 1 \Rightarrow$

$$(-1) * (a - 0) \leq \int_0^a \frac{\sin(nx)}{nx}\, dx \leq 1 * (a - 0) \iff$$

$$\iff -\frac{\epsilon}{2} \leq \int_0^a \frac{\sin(nx)}{nx} \leq \frac{\epsilon}{2} \iff$$

$$\iff |\int_0^a \frac{\sin(nx)}{nx}\, dx| < \frac{\epsilon}{2}$$

Since we already know that for our a we get that

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin(nx)}{nx}\, dx = 0 \Rightarrow$$

$\Rightarrow \exists N = N(\epsilon)$ such that $|\int_a^\pi \frac{\sin(nx)}{nx}\, dx| < \frac{\epsilon}{2}\ \forall n > N$.

Hence, if we take $n > N$ we have that

$$|\int_0^\pi \frac{\sin(nx)}{nx}\, dx| < |\int_0^a \frac{\sin(nx)}{nx}\, dx| + |\int_a^\pi \frac{\sin(nx)}{nx}\, dx| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

hence

$$\lim \int_0^\pi \frac{\sin(nx)}{nx}\, dx = 0$$

still ... but not due to uniform convergence, which is not present here. □

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Proof. If $x = 0 \Rightarrow f_n(0) = 0 \Rightarrow f(0) = 0$

If $x \neq 0 \Rightarrow f_n(x) = \frac{nx}{1+nx} = \frac{1}{\frac{1}{nx}+1} \to 1$. Hence we have

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

It's clear that f is integrable, and that $\int_0^1 f(x)\, dx = 1$.

$$\int_0^1 f_n(x)\, dx = \int_0^1 \frac{nx}{1+nx}\, dx = \int_0^1 (1 - \frac{1}{1+nx})\, dx =$$

$$= 1 - \int_0^1 \frac{1}{1+nx}\, dx = 1 - \frac{1}{n} \ln(1+nx)|_0^1 = 1 - \frac{\ln(n+1)}{n}$$

But $\lim \frac{\ln(n+1)}{n} =^{\text{l'Hospital}} \lim \frac{\frac{1}{n+1}}{1} = 0$ hence

$$\lim \int_0^1 f_n(x)\,dx = 1 - 0 = 1 = \int_0^1 f(x)\,dx$$

Again, we have equality, but not due to uniform convergence - again, not applicable here - and it's not actually a bad idea to look upon this equality as a coincidence, rather. $\square$