SOLUTIONS CHAPTER 8.3

MATH 549 AU00

$\boxed{3}$

Proof. The first inequality is clear: for a fixed x we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

As for the second part (again, fixed x):

$$e^x - (1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}) = \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots =$$

$$= \frac{x^n}{n!}(1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \dots) \leq \frac{x^n}{n!}(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots) \leq$$

$$\leq \frac{x^n}{n!}(1 + \frac{a}{1!} + \frac{a^2}{2!} + \dots) = \frac{e^a x^n}{n!}$$

□

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Proof. We use the fact that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots$$

so we have:

$$0 < e - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!})$$

(a finite sum is less than the whole series, when the series has only positive terms) so we get

$$0 = 0 * n! < en! - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!})n!$$

which solves the first inequality.

For the second we have:

$$e - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}) < \frac{1}{(n+1)!} + \dots$$

$$< \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots) <$$

$$< \frac{1}{n!}\frac{1}{n+1}(1 + \frac{1}{1!} + \frac{1}{2!} + \dots) = \frac{1}{n!}\frac{e}{n+1}$$

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hence

$$en! - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!})n! < \frac{e}{n+1}$$

The above (strict) inequalities are actually true for all n.

For the third inequality:

$$3 - e = 3 - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots = \frac{1}{2} - \frac{1}{3!} - \ldots$$

but we have the following obvious inequality for $n \geq 2$:

$$\frac{1}{n(n+1)} \geq \frac{1}{(n+1)!} \iff \frac{1}{n} - \frac{1}{(n+1)!} > \frac{1}{n+1}$$

which gives us:

$$\frac{1}{2} - \frac{1}{3!} \geq \frac{1}{3}$$
$$\frac{1}{3} - \frac{1}{4!} \geq \frac{1}{4}$$
$$\vdots$$

add all the above and we get that

$$\frac{1}{2} - \frac{1}{3!} - \frac{1}{4!} - \cdots - \frac{1}{n!} > \frac{1}{n} > 0$$

for all n, hence

$$3 - e > 0$$

hence

$$\frac{e}{n+1} < 1$$

for $n \geq 2$.

Assume now that e is rational $\Rightarrow e = \frac{m}{n} \Rightarrow en! = m(n-1)(n-2)\ldots 1 \in \mathbb{Z}$; but we have that $(1+1+\frac{1}{2!}+\ldots \frac{1}{n!})n! \in \mathbb{Z}$ as well; their difference is also an integer. But look at the inequality we just proved: it says that this particular difference is less than 1! so we have an integer, which is positive, but less than 1, impossible ... so the assumption that e was rational is false. $\square$

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Proof. We have that:

$$(x+1)(1 - x + x^2 - x^3 + \cdots + (-x)^{n-1}) =$$
$$= x + 1 - x^2 - x + x^3 + x^2 - x^4 - x^3 + \cdots + x(-x)^{n-1} + (-x)^{n-1} =$$
$$= 1 - (-x)^n$$

and just rearrange the equation ... By **Theorem** 8.3.9 we have that the log is the antiderivative of $\frac{1}{x}$; use this and we have:

$$\int_0^x \frac{1}{t+1}\,dt = \int_0^x (1 - t + t^2 - t^3 + \cdots + (-t)^{n-1} + \frac{(-t)^n}{1+t})\,dt \iff$$

$$\log(x+1)-\log(1)=(x-\frac{x^2}{2}+\cdots+(-1)^{n-1}\frac{x^n}{n})-0+\int_0^x\frac{(-t)^n}{1+t}\,dt \iff$$
$$\log(1+x)=x-\frac{x^2}{2}+\cdots+(-1)^{n-1}\frac{x^n}{n}+\int_0^x\frac{(-t)^n}{1+t}\,dt$$

From here we get:

$$|\log(1+x)-(x-\frac{x^2}{2}+\cdots+(-1)^{n-1}\frac{x^n}{n})|=|\int_0^x\frac{(-t)^n}{1+t}\,dt| \iff$$
$$\iff |\log(1+x)-(x-\frac{x^2}{2}+\cdots+(-1)^{n-1}\frac{x^n}{n})|\le\int_0^x|\frac{(-t)^n}{1+t}|\,dt\le\int_0^x t^n\,dt=$$
$$=\frac{x^{n+1}}{n+1}-0$$

(since $t+1>1$) and the inequality is proved. □

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Proof. Let $g(x)=\frac{f(x)}{e^x}$; let's compute the derivative of this function:

$$g'(x)=\frac{f'(x)e^x-f(x)e^x}{(e^x)^2}=\frac{f(x)e^x-f(x)e^x}{e^{2x}}=0$$

Hence the function g must be a constant, $g(x)=K$, $\forall x\in\mathbb{R}$. So $\frac{f(x)}{e^x}=K \iff f(x)=Ke^x$. □

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Proof. Actually the inequality $1+x\le e^x$ is true for all x: taking $h(x)=e^x-x-1$ we have that $h'(x)=e^x-1$ which is negative up to 0 and then positive from 0, which means that 0 is a local (and actually a global) minimum; but $h(0)=e^0-0-1=1-1=0$ hence the lowest posible value for $h(x)$ is 0, so $e^x-x-1\ge 0\Rightarrow e^x\ge x+1$.

Computing all the values x_k, some will be negative - but we just observed that it's OK to plug in even negative numbers into the inequality that we have to use. What we get is:

$$\frac{a_k}{A}-1+1\le e^{\frac{a_k}{A}-1} \iff \frac{a_k}{A}\le e^{\frac{a_k}{A}-1}$$

We also notice that after pluging the values x_k in the inequality we actually get all positive numbers - and it's safe to multiply:

$$\frac{a_1}{A}\frac{a_2}{A}\cdots\frac{a_n}{A}\le e^{\frac{a_1}{A}-1}e^{\frac{a_2}{A}-1}\cdots e^{\frac{a_n}{A}-1} \iff$$
$$\iff \frac{a_1a_2\ldots a_n}{A^n}\le e^{\frac{a_1}{A}+\frac{a_2}{A}+\cdots+\frac{a_n}{A}-n} \iff$$
$$\iff \frac{a_1a_2\ldots a_n}{A^n}\le e^{\frac{a_1+a_2+\cdots+a_n}{A}-n}=e^{\frac{a_1+a_2+\cdots+a_n}{\frac{a_1+a_2+\cdots+a_n}{n}}-n} \iff$$

$$\iff \frac{a_1 a_2 \dots a_n}{A^n} \leq e^{\frac{1}{\frac{1}{n}} - n} = e^{n-n} = 1 \iff$$

$$\iff a_1 a_2 \dots a_n \leq A^n \iff (a_1 a_2 \dots a_n)^{\frac{1}{n}} \leq \frac{1}{n}(a_1 + a_2 + \dots + a_n)$$

Equality is obtained when all values x_k provide us with equality when pluged into the inequality (remember, 0 was a global minimum) - hence $\frac{a_k}{A} - 1 = 0 \Rightarrow a_k = A$, $\forall k$; but then $a_1 = a_2 = \dots = a_n = A$. Done. $\square$