SOLUTIONS CHAPTER 9.1

MATH 549 AU00

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Proof. Let Σa_k be the series. The trick is to consider the two series, the positive and negative terms' series, as follows:

$$a_k^+ = \begin{cases} a_k, & a_k \geq 0 \\ 0, & a_k < 0 \end{cases}$$

and

$$a_k^- = \begin{cases} 0, & a_k \geq 0 \\ a_k, & a_k < 0 \end{cases}$$

As you can see these two series now are "complementary": their sum is the full series, yet they contain only the positive terms or only the negative terms, with zeroes to fill in the blanks.

Assume one of the two series is convergent - let's say the positive terms (if it's the other one just consider the "minus" series). Then, since the series is convergent (conditionally, true, but still convergent), and since we know that the sum/difference of two convergent series is also convergent, we get that the negative terms' series (equal to the difference between the full series and the positive terms series) is also convergent. But then, if we take the following series: $\Sigma a_k^+ - \Sigma a_k^-$, it is going to be convergent (being the difference of two convergent series), but if you take a closer look you'll notice that it's exactly $\Sigma|a_k|$!! because the positive terms keep their sign, while the negative terms get a negative sign, which transforms them in their absolute value. This is now a contradiction, since we assumed that the series was conditionally convergent, that is, the series is NOT absolutely convergent. Hence both the positive terms and the negative terms' series must be divergent. □

6b

Proof. We have:

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} =$$

$$= \frac{A(n+1)(n+2) + Bn(n+2) + Cn(n+1)}{n(n+1)(n+2)} \Rightarrow$$

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$$\Rightarrow 1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1)$$

Plug in $n = 0$, $n = -1$ and $n = -2$ respectively; we get:

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

$$1 = -B \Rightarrow B = -1$$

and

$$1 = 2C \Rightarrow C = \frac{1}{2}$$

hence

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\frac{1}{n+2}$$

Let's write down a few of these equalities, and add them up to see what we get:

$$\frac{1}{1*2*3} = \frac{1}{2}\frac{1}{1} - \frac{1}{2} + \frac{1}{2}\frac{1}{3}$$

$$\frac{1}{2*3*4} = \frac{1}{2}\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\frac{1}{4}$$

$$\frac{1}{3*4*5} = \frac{1}{2}\frac{1}{3} - \frac{1}{4} + \frac{1}{2}\frac{1}{5}$$

$$\vdots$$

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\frac{1}{n+2}$$

and we get - notice the cancelling pattern: $\frac{1}{1}$ does not get cancelled at all, $\frac{1}{2}$ only partially, and then $\frac{1}{3}$ completely, same for $\frac{1}{4}$ etc, up to $\frac{1}{n+1}$ which is again only partially cancelled, while $\frac{1}{n+2}$ isn't cancelled at all

$$\Sigma\frac{1}{k(k+1)k+2)} = \frac{1}{2}\frac{1}{1} - \frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{n+1} + \frac{1}{2}\frac{1}{n+2} =$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \rightarrow_{n\to\infty} \frac{1}{4}$$

□

7

Proof. Not necessarily; $\Sigma(-1)^n\frac{1}{\sqrt{n}}$ kills this assertion, since itself it's convergent (alternating series), while $\Sigma((-1)^n\frac{1}{\sqrt{n}})^2 = \Sigma\frac{1}{n}$ is divergent. But if we put the additional assumption that it's a positive series (all terms are positive) then it's true (why? since the series converges, it means that the general term goes to zero, hence from some point on it's less than 1; we can consider the series starting at that point only - and then we have $\Sigma a_k \geq \Sigma(a_k)^2$, so the second series converges as well).

For the second part the answer is again NO. Take $\Sigma\frac{1}{n^2}$ - it's convergent, but the series $\Sigma\frac{1}{n}$ is divergent. □

10

Proof. Use **Cauchy's Criterion**; let $\epsilon > 0$:

$$|a_m b_m + a_{m+1} b_{m+1} + \cdots + a_n b_n| \leq |a_m b_m| + |a_{m+1} b_{m+1}| + \cdots + |a_n b_n|$$

But $\{b_n\}_n$ is bounded, hence $\exists M$ such that $|b_k| \leq M$. Also, since the series Σa_k is absolutely convergent it means that for our ϵ there exists $N = N(\epsilon)$ such that

$$|a_m| + |a_{m+1}| + \cdots + |a_n| \leq \frac{\epsilon}{M},\ \forall m, n > N$$

Getting back to the above expression we have, for $m, n > N$, that

$$|a_m b_m + a_{m+1} b_{m+1} + \cdots + a_n b_n| \leq |a_m||b_m| + |a_{m+1}||b_{m+1}| + \cdots + |a_n||b_n| \leq$$

$$\leq M(|a_m| + \cdots + |a_n| \leq M\frac{\epsilon}{M} = \epsilon$$

hence, by **Cauchy's Criterion** the series $\Sigma a_k b_k$ is convergent. □

11

Proof. We have to prove an "if and only if" assertion, hence we have to prove that:

-if $\Sigma_1^\infty(a_n)$ converges then $\Sigma_1^\infty(2^n a_{2^n})$ converges

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One fact to remember: since we're talking about positive terms' series, there are only two cases with respect to convergency - either the series converges and hence it's finite, or it's divergent to infinity (so to test convergency it's enough to show it's finite, like it's less than some number). Let's assume that $\Sigma_1^\infty(2^n a_{2^n})$ converges. Since a_n decreases (very important assumption - if this doesn't happen, neither the Cauchy Condensatio Test won't happen) we have:

$$a_1 \leq a_1$$

(they're actually equal, but we're only interested in the inequality)

and then

$$a_2 \leq a_2$$
$$a_3 \leq a_2$$

and then

$$a_4 \leq a_4$$
$$a_5 \leq a_4$$
$$a_6 \leq a_4$$
$$a_7 \leq a_4$$

and so on (see the pattern? starting at a_{2^k} all higher index terms are less than a_{2^k}, and we write this until the $a_{2^{k+1}}$ - not included). Now add all these inequalities (and group terms) and we get:

$$a_1 + a_2 + \cdots \leq$$

$$\leq (2^1 - 2^0)a_{1=2^0} + (2^2 - 2^1)a_{2^1} + (2^3 - 2^2)a_{2^2} + \cdots + (2^{k+1} - 2^k)a_{2^k} + \cdots =$$

$$= 2^0 a_{2^0} + 2^1 a_{2^1} + \dots + 2^k a_{2^k} + \dots$$

but since the right-hand side is convergent (hence strictly less than ∞) the left-hand side is too (comparison test, let's say).

Let's assume now that $\Sigma_1^\infty(a_n)$ is convergent (so we need some reverse inequality; of course, we cannot expect to have the above inequality reversed, since this would mean that the two series are equal, which is not true - we ignore way too many terms of $\Sigma_1^\infty(a_n)$ in $\Sigma_1^\infty(2^n a_{2^n})$ for this to happen). Let's see:

$$a_2 \le a_1$$

and then

$$a_4 \le a_2$$

$$a_4 \le a_3$$

and then

$$a_8 \le a_4$$

$$a_8 \le a_5$$

$$a_8 \le a_6$$

$$a_8 \le a_7$$

and so on (the pattern: a_{2^k} is less than all terms starting at $a_{2^{k-1}}$ and up to itself, not included) which gives after summing all inequalities:

$$(2^1 - 2^0)a_{2^1} + (2^2 - 2^1)a_{2^2} + (2^3 - 2^2)a_{2^3} + \dots \le a_1 + a_2 + \dots \iff$$

$$\iff 2^0 a_{2^1} + 2^1 a_{2_2} + 2^2 a_{2^3} + \dots \le a_1 + a_2 + \dots$$

from which we get

$$\frac{1}{2}(2^1 a_{2^1} + 2^2 a_{2^2} + \dots) \le a_1 + a_2 + \dots \iff$$

$$\iff \frac{1}{2}(2^0 a_{2^0} + 2^1 a_{2^1} + 2^2 a_{2^2} + \dots) \le \frac{a_1}{2} + a_1 + a_2 + \dots \Rightarrow$$

$$2^0 a_{2^0} + 2^1 a_{2^1} + 2^2 a_{2^2} + \dots \le 2(\frac{a_1}{2} + a_1 + a_2 + \dots)$$

but the right-hand side is finite (since the series is convergent) hence the left-hand side is finite (therefore convergent) too. □

13a

Proof. By the above property we can check convergency by looking at the following series:

$$\Sigma 2^n a_{2^n} = \Sigma 2^n \frac{1}{2^n \log(2^n)} = \Sigma \frac{1}{n \log(2)} = \frac{1}{\log(2)} \Sigma \frac{1}{n}$$

but this series diverges, hence the original one does the same. □

14a

Proof. Pretty much the same thing:

$$\Sigma 2^n a_{2^n} = \Sigma 2^n \frac{1}{2^n (\log(2^n))^c} = \Sigma \frac{1}{n^c \log^c(2)} = \frac{1}{\log^c(2)} \Sigma \frac{1}{n^c}$$

and we know that this series converges only when $c > 1$. □

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Proof. Multiply by the conjugate:

$$\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \frac{1}{\sqrt{n^2+n} + n} > \frac{1}{n+1+n+1} = \frac{1}{2} \frac{1}{n+1}$$

which is divergent, hence the original series is divergent. (the inequality $\sqrt{n^2+n} + n < n+1+n+1$ you can prove as follows: $\sqrt{n^2+n} + n < \sqrt{n^2+n+n+1} + n + 1 = \sqrt{(n+1)^2} + n + 1 = n+1+n+1$) □