SOLUTIONS CHAPTER 9.2

MATH 549 AU00

1b

Proof. **Limit Comparison Test** against $\Sigma\frac{1}{n}$:

$$\lim \frac{\frac{n}{(n+1)(n+2)}}{\frac{1}{n}} = \lim \frac{n^2}{(n+1)(n+2)} = \lim \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} = 1$$

hence the series is divergent, since the series $\Sigma\frac{1}{n}$ is. $\square$

2b

Proof. **Limit Comparison Test** against the series $\Sigma\frac{1}{n^{\frac{3}{2}}}$:

$$\lim \frac{\frac{1}{(n^2(n+1))^{\frac{1}{2}}}}{\frac{1}{n^{\frac{3}{2}}}} = \lim \frac{n\sqrt{n}}{n\sqrt{n+1}} =$$

$$= \lim \sqrt{\frac{n}{n+1}} = \lim \sqrt{\frac{1}{1+\frac{1}{n}}} = 1$$

Since the power at which n appears in the series $\Sigma\frac{1}{n^{\frac{3}{2}}}$ is bigger than 1, it means it's convergent, hence so is our series. $\square$

3a

Proof. **Limit Comparison Test** against $\Sigma\frac{1}{n}$:

$$\lim \frac{\frac{1}{(\log(n))^p}}{\frac{1}{n}} = (\lim \frac{n^{\frac{1}{p}}}{\log(n)})^p =^{\text{l'Hospital}} (\lim \frac{\frac{1}{p}n^{\frac{1}{p}-1}}{\frac{1}{n}})^p = (\lim \frac{1}{p}n^{\frac{1}{p}})^p =$$

$$= \infty$$

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hence the top series is bigger than the bottom series; but since the bottom series diverges ($\Sigma\frac{1}{n}$), so does the top; hence our series is divergent.

Another way you can prove this is by using **Cauchy's Condensation Test** ... try it out! ($\lim \frac{2^n}{n^p} = \infty$) □

3b

Proof. Try **Root Test**:

$$\lim_{n\to\infty} \sqrt[n]{\log(n)^{-n}} = \lim_{n\to\infty} \frac{1}{\log(n)} = 0$$

hence it's convergent. □

3c

Proof. Use **Cauchy Condensation Test**:

$$\Sigma 2^n \frac{1}{\log(2^n)^{\log(2^n)}} = \Sigma \frac{2^n}{n^n} = \Sigma(\frac{2}{n})^n$$

but since $\frac{2}{n} \leq \frac{2}{3}$ for $n \geq 3$ we have that, by **Comparison Test**, that the above series converges:

$$\Sigma 2^n \frac{1}{\log(2^n)^{\log(2^n)}} < 2 + 1 + \Sigma_{n\geq 3}(\frac{2}{3})^n < \infty$$

so the original series is also convergent. □

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Proof.

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \cdots < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < \infty$$

(we know that the right-hand side is convergent)

For the two tests we need the general formula for the series' term: $a_{2k} = \frac{1}{(2k)^3}$ and $a_{2k+1} = \frac{1}{(2k+1)^2}$.

The **Ratio Test** applied here gives two cases:

$$\lim_{k\to\infty} \frac{\frac{1}{(2k)^3}}{\frac{1}{(2k+1)^2}} = \lim \frac{(2k+1)^2}{8k^3} = 0$$

and

$$\lim_{k\to\infty} \frac{\frac{1}{(2k+1)^2}}{\frac{1}{(2k)^3}} = \lim \frac{8k^3}{(2k+1)^2} = \infty$$

Since we have two subsequences, one that is in the first case of the Ratio Test (the condition of being less than r) and the other one is in the case when it's bigger than 1, we cannot apply this test here.

The **Root Test** applied here also needs to cases to be looked at:

$$\lim_{k\to\infty} \sqrt[2k]{\frac{1}{(2k)^3}} = \lim(\frac{1}{(2k)^{\frac{1}{2k}}})^3 = 1^3 = 1$$

and

$$\lim_{k\to\infty} \sqrt[2k+1]{\frac{1}{(2k+1)^2}} = \lim(\frac{1}{(2k+1)^{\frac{1}{2k+1}}})^2 = 1^2 = 1$$

and both $\sqrt[2k]{\frac{1}{(2k)^3}}$ and $\sqrt[2k+1]{\frac{1}{(2k+1)^2}}$ are less than 1. Hence the Root Test tells us nothing here (we cannot find the r like in the Theorem 9.2.3) □

7a

Proof.

$$\frac{1}{2*1+1} < \frac{1}{2}$$
$$\frac{2}{2*2+1} < \frac{2}{2*2} = \frac{1}{2}$$
$$\vdots$$
$$\frac{n}{2n+1} < \frac{n}{2n} = \frac{1}{2}$$

Multiply all and we get:

$$\frac{1*2*3*\cdots*n}{3*5*7*\cdots*(2n+1)} < \frac{1}{2}*\frac{1}{2}*\cdots*\frac{1}{2} = \frac{1}{2^n} \iff$$
$$\iff \frac{n!}{3*\cdots*(2n+1)} < \frac{1}{2^n}$$

so by **Comparison Test** we get that the series with the above general term is convergent. □

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Proof. We have that:

$$|x_{n+1}| \le r|x_n|$$
$$|x_{n+2}| \le r|x_{n+1}| \le r^2|x_n|$$
$$\vdots$$
$$|x_{n+k}| \le r|x_{n+k-1}| \le \cdots \le r^k|x_n|$$

Since

$$|s - s_n| = |x_{n+1} + x_{n+2} + \dots| \leq |x_{n+1}| + |x_{n+2}| + \dots \leq$$
$$\leq r|x_n| + r^2|x_n| + \dots = |x_n|(r + r^2 + \dots) = |x_n| r \frac{1}{1-r}$$

done. □

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Proof. First of all notice that if $p > q$ then the general term is always bigger than 1

$$\frac{p+1}{q+1} > 1$$
$$\frac{p+2}{p+2} > 1$$

and so on, and hence

$$\frac{(p+1)(p+2)\dots(p+n)}{(q+1)(q+2)\dots(q+n)} > 1$$

so the series diverges obviously. Same if $p = q$ ... the general term equals 1, and $\Sigma 1$ is divergent too.

Let's restrict our attention now to $p < q$. Let's use **Raabe's Test**:

$$\lim(n(1 - \frac{\frac{(p+1)(p+2)\dots(p+n+1)}{(q+1)(q+2)\dots(q+n+1)}}{\frac{(p+1)(p+2)\dots(p+n)}{(q+1)(q+2)\dots(q+n)}})) =$$

$$\lim(n(1 - \frac{p+n+1}{q+n+1})) = \lim(n(\frac{q-p}{q+n+1})) = (q-p)\lim\frac{n}{q+n+1} = q - p$$

We made sure that $q - p > 0$, so everything is nice and dandy. As by the test, when $q - p > 1 \iff q > p + 1$ the series is convergent (well, it says absolutely convergent, but when we talk about positive terms series it's equivalent) and when $q - p < 1 \iff q < p + 1$ it's divergent. There's only one more case to discuss, namely $q = p + 1$:

$$\frac{(p+1)(p+2)\dots(p+n)}{(p+2)(p+3)\dots(p+n)(p+n+1)} = \frac{p+1}{p+n+1}$$

but by Limit Comparison Test we have that, since

$$\lim_{n\to\infty} \frac{\frac{p+1}{p+n+1}}{\frac{1}{n}} = \lim\frac{n(p+1)}{p+n+1} = p + 1$$

the series is behaving same as $\Sigma\frac{1}{n}$ which is divergent. So this case also gives us divergence.

Cumulating the results now gives: $q > p + 1$ convergence, $q \leq p + 1$ divergence. □