Research Statement

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1 Introduction

My mathematical research is in the general areas of geometric group theory and geometric topology, more specifically, in spaces of non-positive curvature and link homotopy.

The concept of Riemannian manifolds was first introduced by Gauss and then formalized by Riemann in the mid 19th century. It defined the notion of curvature for manifolds, in a way that was intrinsic to the manifold and not dependent upon its embedding in higher-dimensional spaces. Since then it has been studied and used by a lot of mathematicians and physicists. In fact, Einstein used the theory of Riemannian manifolds to develop his general theory of relativity. The theory of Riemannian manifolds brings together the notions of geometry and topology. In particular, Riemannian manifolds with non-positive sectional curvature have been interesting to study for the rich interplay between their geometric, topological and dynamical properties, and powerful local-global properties like the Cartan-Hadamard theorem.

Gromov defined a notion of non-positive curvature for the larger class of geodesic metric spaces. A geodesic metric space is said to have non-positive curvature if for every point, there is a neighborhood such that the geodesic triangles in the neighborhood are "no fatter" than Euclidean triangles. These spaces satisfy results that are analogues of results for non-positively curved Riemannian manifolds.

I have been interested in understanding the relationship between these two notions of curvature for manifolds. In particular, if M is a closed manifold with a locally CAT(0) metric, we want to know if this manifold can support a Riemannian metric with non-positive sectional curvature. In Section 2, I describe this problem in detail giving the work done in this direction till now, and then explain my contribution.

The study of this question and the techniques used in solving it prompted another question about links in 3-sphere which has led to a few interesting results. In particular, I study the relationship between the equivalence relations on links: link homotopy and link isotopy. I discuss my results regarding these in Section 3.

Finally, in Section 4, I lay out various questions and research directions that I plan to pursue in the future.

2 Obstruction for Riemannian metric of non-positive sectional curvature

2.1 Background

Gromov generalized the notion of non-positive curvature to the class of geodesic metric spaces, which can be defined in the following way.

Definition 2.1.1. Let (X, d) be a metric space. A path in X, $\gamma : [0, L] \to X$ is a geodesic path if $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t \in [0, L]$. If every two points x, y in X can be joined by a geodesic segment, then (X, d) is said to be a *geodesic metric space*.

Now, let Δ be any geodesic triangle in the geodesic metric space X (i.e. all three sides are geodesic segments in X connecting the vertices). A triangle Δ' in $\mathbb{R}^2$ with sides of the same length as the sides of Δ is called a *comparison triangle*. If the distances between points on Δ are less than or equal to the distances between corresponding points on Δ', then X is said to be a *CAT(0) space*.

A geodesic metric space X is said to be of curvature $\leqslant 0$ (or *non-positively curved*) if it is locally a CAT(0) space, i.e. for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$, endowed with the induced metric, is a CAT(0) space.

Let M be any closed manifold with locally CAT(0) metric. We are interested in knowing if this manifold can support a Riemannian metric with non-positive sectional curvature.

In low dimensions, the class of manifolds that support a locally CAT(0) metric is the same as the class of manifolds that support a Riemannian metric of non-positive curvature. In dimension= 2, this follows from the classification of surfaces, which says that every closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a finite connected sum of the projective plane. In dimension = 3, this is a consequence of Thurston's geometrization conjecture, which is now a theorem by Perelman and others.

For dimensions $\geqslant 5$, Davis and Januszkiewicz [6] constructed examples of locally CAT(0) manifolds that do not support Riemannian metric of non-positive sectional curvature. In fact, they showed that for each $n \geqslant 5$, there is a piecewise flat, non-positively curved closed manifold M^n whose universal cover $\tilde{M}^n$ is not simply connected at infinity. They applied hyperbolization techniques to certain non-PL triangulations of S^n to prove this result. This, in particular, means that the universal cover is not homeomorphic to $\mathbb{R}^n$, and by the Cartan-Hadamard theorem, the manifold cannot have a smooth non-positively curved Riemannian metric.

Recently, Davis, Januszkiewicz and Lafont [7] dealt with the remaining case of dimension 4. Their techniques are different from the dimension $\geqslant 5$ case, and in fact, their example has universal cover $\tilde{M}$ diffeomorphic to $\mathbb{R}^4$. Further, they show that if K is any locally CAT(0) manifold, then $M \times K$ is a locally CAT(0) manifold which does not support any Riemannian metric of non-positive sectional curvature. To prove this, they construct a "knottedness" in the boundary of the manifold which gives the obstruction to non-positively curved Riemannian smoothing.

I extend their method to provide new examples of such locally CAT(0) 4-manifolds which do not support a non-positively curved Riemannian metric. I show that linking in the boundary of the manifold is another obstruction and I prove the following:

Theorem 2.1.1. *There exists a 4-dimensional closed manifold M with the following properties:*

1. *M supports a locally CAT(0)-metric.*
2. *The boundary of its universal cover $\tilde{M}$ is homeomorphic to S^3, and $\tilde{M}$ is diffeomorphic to $\mathbb{R}^4$.*
3. *The maximal dimension of flats in $\tilde{M}$ is 2, and the boundary of every $\mathbb{Z}^2$-periodic 2-flat is a circle that is unknotted in $\partial^\infty \tilde{M}$.*
4. *$\pi_1(M)$ is not isomorphic to the fundamental group of any Riemannian manifold of non-positive sectional curvature.*

These examples are indeed different from those in [7], since their examples have $\mathbb{Z}^2$-periodic flats that are wild knots in the boundary, where as in the examples that I give, all such flats are unknotted in the boundary. In particular, one can see that the obstruction comes from linking of the flats in the boundary.

2.2 Dimension 4

Here I briefly describe the Davis-Januszkiewicz-Lafont obstruction to non-positively curved Riemannian smoothing in dimension 4.

One starts with a non-trivial knot k in S^3, and constructs a triangulation of S^3 that is *flag* with *isolated squares* (i.e., no two squares in the triangulation intersect). In fact, one can have the triangulation to contain a single square which gives the knot k (Sec. 3, [7]). The isotopy class of the knot is said to be the *type* of the triangulation. Since we are in dimension 3, the triangulation is automatically smooth. Let Σ denote such a smooth triangulation of S^3.

Define $M := P_\Sigma$, where P_Σ is a 4-dimensional cubical complex such that $\tilde{P}_\Sigma$ is the *Davis complex* corresponding to the simplicial complex Σ. There is a natural piecewise Euclidean metric on P_Σ, obtained by making each k-dimensional face in the cubulation of P_Σ isometric to $[-1,1]^k \subset \mathbb{R}^k$. Using propertries of Davis complexes in the right angled case [5], we see that M has the following properties:

- Since Σ is a smooth triangulation, P_Σ is a smooth 4-manifold.
- Since Σ is a flag complex, the piecewise flat metric on P_Σ is locally CAT(0).
- The boundary at infinity, $\partial^\infty \tilde{P}_\Sigma$, is homeomorphic to S^3.
- By construction, links of vertices in P_Σ, $lk_{P_\Sigma}(v)$, are simplicially isomorphic to Σ. This shows that the link at each vertex of P_Σ contains a square, $\square$, given by the knot k. The cube complex $P_\square$ corresponding to this square is contained in $P_\Sigma = M$. It can be seen that $P_\square$ is isomorphic to a flat torus T^2 that is locally knotted in M.

The important property of M is that $\pi_1(M)$ is not isomorphic to the fundamental group of any non-positively curved Riemannian manifold. Suppose there is a Riemannian manifold M' with non-positive sectional curvature such that $\pi_1(M) \cong \pi_1(M')$. Let $\Gamma := \pi_1(M)$. The idea is to produce a Γ-equivariant homeomorphism $\tilde{\phi} : \partial^\infty \tilde{M} \to \partial^\infty \tilde{M}'$. The existence of such a homeomorphism may not be true in general (Croke-Kleiner example [4]). However, it is true if one of the spaces has *isolated flats* (Hruska-Kleiner [12]). Due to results by Hruska-Kleiner [12] and a criterion by Caprace [3], the isolated squares condition on Σ ensures isolated flats in $\tilde{M}$.

We know that M contains a locally knotted flat torus T^2. By lifting to the universal covers, there is a flat F in $\tilde{M}$ which is locally knotted at lifts of the vertices in T^2. This gives an embedding of $\partial^\infty F \cong S^1$ into $\partial^\infty \tilde{M} \cong S^3$. It turns out that the local knottedness propogates to the boundary at infinity, and this embedding defines a nontrivial knot, $\partial^\infty F$, in $\partial^\infty \tilde{M}$.

The Γ-equivariant homeomorphism $\tilde{\phi} : \partial^\infty \tilde{M} \to \partial^\infty \tilde{M}'$ maps the knot $\partial^\infty F$ to some $\partial^\infty F'$ in $\partial^\infty \tilde{M}'$ where F' is a flat in the Riemannian manifold $\tilde{M}'$, by the flat torus theorem.

Now, for any $p \in F'$, the geodesic retraction $\rho' : \partial^\infty \tilde{M}' \to T_p \tilde{M}'$ is a homeomorphism which takes the knotted subset $\partial^\infty F'$ to the unknotted subset $T_p F'$ lying inside $T_p \tilde{M}' \cong S^3$. This gives a contradiction.

2.3 New Obstruction in dimension 4

In the construction in the previous section, one makes use of a knot k in S^3 to produce a particular triangulation. Similarly, one can construct a triangulation of the type of a link L in S^3, in place of a knot. One would like the same argument to go through for links, which will help generate a lot of new examples of 4-manifolds with locally CAT(0) metrics but that do not support non-positive curvature Riemannian smoothing.

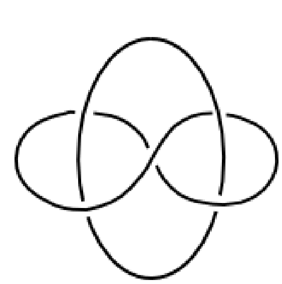
I prove in [17] that one can do this, as long as the link we start with does not have all the pairwise linking numbers = 1. The flats corresponding to the components of a non-trivial link do give a non-trivial link in $\partial^\infty \tilde{M}$. The restriction comes from the fact that the link in $T_p\tilde{M}'$ obtained under the homeomorphism $\rho' : \partial^\infty \tilde{M}' \to T_p\tilde{M}'$ can only be a *great circle link* (i.e., all its components are great circles in S^3), and hence does not give a contradiction.

We have the following picture. We start with an n-component link L in S^3 and construct a flag triangulation Σ of S^3 with isolated squares giving the n components of the link, and then let $M := P_\Sigma$ as described in the previous section. We get a collection of tori $\{T_i^2\}_{i=1}^n$ in M which lift to flats $\{F_i\}_{i=1}^n$ in $\tilde{M}$. Then each $\partial^\infty F_i$ is a copy of S^1 in $\partial^\infty \tilde{M} \cong S^3$, giving a link a link L_∞ in the boundary $\partial^\infty \tilde{M}$ with components $\{\partial^\infty F_1, \ldots, \partial^\infty F_n\}$.

Now if there is a non-positively curved Riemannian manifold M' with $\pi_1(M') \cong \pi_1(M')$, then using the $\pi_1(M)$-equivariant homeomorphism $\tilde{\phi} : \partial^\infty \tilde{M} \to \partial^\infty \tilde{M}'$ we get a link $L'_\infty \hookrightarrow \partial^\infty \tilde{M}'$ isotopic to L_∞ in $\partial^\infty \tilde{M}'$. By the flat torus theorem, there exist flats F'_i in $\tilde{M}'$ such that the collection $\{\partial^\infty F'_1, \ldots, \partial^\infty F'_n\}$ gives the components of the link L'_∞.

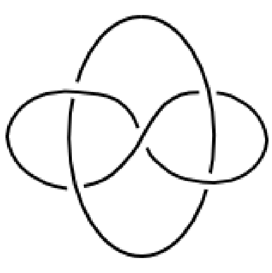
Let us first focus on links with two components. Let $L = (l_1, l_2)$. The linking number between l_1 and l_2, $Lk(l_1, L_2)$, is preserved under the near-homeomorphism $\rho : \partial^\infty \tilde{M} \to lk_{\tilde{M}}(v)$, where $lk_{\tilde{M}}(v)$ is the link of a vertex v in $\tilde{M}$, and is also preserved under the homeomorphism $\tilde{\phi}$.

If $Lk(l_1, l_2)$ is non-zero, then the flats must intersect at a point, say p, in $\tilde{M}'$. Let us further assume that the linking number is not ± 1. For example, consider the link in Figure 1. It has linking number 2.



Rysunek 1

The geodesic retraction map from $\partial^\infty \tilde{M}'$ to the unit tangent space $T_p\tilde{M}'$ maps L'_∞ homeomorphically to a link, L', in $T_p\tilde{M}'$. The components of this link are given by the unit tangent space of p in F'_i. However, $\cup T_p F'_i \hookrightarrow T_p\tilde{M}'$ is a great circle link. It is known that great circle links with two components have linking numbers $= \pm 1$ [19]. Thus, this gives a contradiction to the existence of such a Riemannian manifold M'.



Rysunek 2: Whitehead link

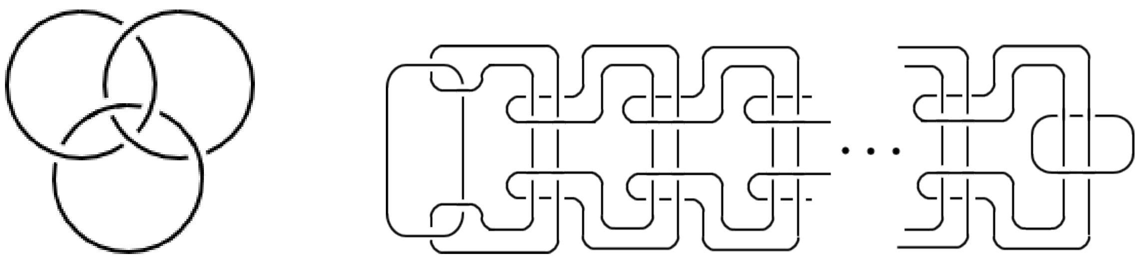
Now, if the linking number is zero, then the flats cannot intersect. In fact, if they intersected, then by the above argument we will obtain an isotopic link in $T_p\tilde{M}'$ which is a great circle link and hence must have linking number ± 1.

Let p be a point in $\tilde{M}'$ which does not lie on either of the flats and is at the same distance from both the flats. Then there is an $r > 0$ such that $S_p(r) \cap (\sqcup F'_i)$ is an unlink, and an $s > r$ such that $S_p(s) \cap (\sqcup F'_i)$ is a nontrivial link. The geodesic retraction map $\rho' : S_p(s) \to S_p(r)$ is a homeomorphism that maps the non-trivial link $S_p(s) \cap (\sqcup F'_i)$ to the trivial link $S_p(r) \cap (\sqcup F'_i)$. This is a contradiction.

An example is given by the Whitehead link as shown in Figure 2. It is a non-trivial link with two components with linking number zero.

Now suppose $L = (l_1, \dots, l_n)$ is a link with more than 2 components with at least one of the pairwise linking numbers, $Lk(l_i, l_j)$, not equal to ± 1, where $1 \leqslant i, j \leqslant n$, $i \neq j$. Then by a similar argument, there is a great circle link, L', in $T_p\tilde{M}'$ which is link-isotopic to L and hence has the same pairwise linking numbers. However, all the pairwise linking numbers of a great circle link have to be ± 1. This gives a contradiction.

A few examples of links that generate such manifolds are given by links given in Figure 3. In



(a) Borromean rings (b) Brunnian link with m components, B_m (Milnor [15])

Rysunek 3: Non-trivial links with pairwise linking numbers = 0

fact, the *Brunnian links* (Fig. 3b) give a family of links that generate locally CAT(0) 4-manifolds that do not support Riemannian metric with a non-positive sectional curvature. A Brunnian link is a non-trivial link such that each of its sublink is isotopic to an unlink.

I also show that this is indeed a new obstruction. This is done by showing that the obstruction does come from the linking and if each of the components are unknotted, they stay unknotted at the boundary. This can be seen from the following result proved in [17].

Let k be a knot in S^3, and Σ be the triangulation of S^3 of the type of the knot k. Let $M = P_\Sigma$ be the 4-manifold as described in Section 2.2. We know that there is a flat F in the universal cover $\tilde{M}$ such that if x is a vertex of the cubulation of $\tilde{M}$ in F, then the link of x in F, $lk_F(x)$, gives a copy of the knot k in $lk_{\tilde{M}}(x) \cong S^3$. Also the boundary of F, $\partial^\infty F$, is a copy of S^1 in $\partial^\infty \tilde{M} \cong S^3$.

Theorem 2.3.1. *Let M, F and v be as described above. If $lk_F(v)$ is an unknot in $lk_{\tilde{M}}(v)$, then so is $\partial^\infty F \subset \partial^\infty \tilde{M}$.*

3 Link homotopy vs isotopy

3.1 Overview

An n-component link L in S^3 is a collection of piecewise linear maps $(l_1, \dots, l_n) : S^1 \to S^3$, where the images $l_1(S^1), \dots, l_n(S^1)$ are pairwise disjoint. A link with one component is a knot. Two links in S^3, L_1 and L_2, are said to be *isotopic* if there is an orientation preserving homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$ and h is isotopic to the identity map. The notion of link homotopy was introduced by Milnor in [15]. Two links L and L' are said to be *link homotopic* if there exist homotopies $h_{i,t}$, between the maps l_i and the maps l'_i, so that the sets $h_{1,t}(S^1), \dots, h_{n,t}(S^1)$ are disjoint for each value of $t \in [0, 1]$. In particular, a link is said to be link homotopically trivial if it is link homotopic to the unlink. Notice that this equivalence allows self-crossings, that is, crossing changes involving two strands of the same component.

The question that arises now is: how different are these two notions of link equivalence? From the definition it is clear that a link isotopy is a link homotopy as well. Link homotopy allows self-crossings, which dramatically simplifies the equivalence between links. Under link homotopy, all knots become trivial knots, despite the fact that they are non-trivial under isotopy. Link homotopy in some sense measures the interactions between different components of a link. So the question reduces to whether two link homotopic links are also isotopic. We have the example of the Whitehead link (Fig. 2), which is a non-trivial link that becomes trivial under link homotopy. This gives us an example of a 2-component link that is link homotopic, but not isotopic, to the unlink, answering our question for links with two components.

I provide examples of links with more than 2 components that are link homotopic, but not link isotopic, to any given link. In particular, I prove the following in [16].

Theorem 3.1.1. *Let L be any m-component link in S^3 for $m \geqslant 2$. There exists a family of m-component links, $\mathcal{L}_m$, with the following properties:*

1. *Each $\mathcal{L}_m$ is link homotopic to L*
2. *Each $\mathcal{L}_m$ is not link isotopic to L*
3. *Every proper sublink of $\mathcal{L}_m$ is link isotopic to the corresponding sublink of L. In particular, if L is an unlink then every proper sublink of $\mathcal{L}_m$ is isotopic to the unlink.*

I also provide examples which have an additional property about Milnor invariants by proving the following [16].

Theorem 3.1.2. *There exist links in S^3 with the following properties.*

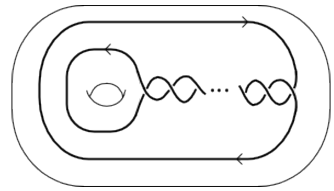
1. *It is link homotopic to the unlink.*
2. *It is not link isotopic to the unlink.*
3. *Every proper sublink is link isotopic to the unlink.*
4. *All Milnor's μ-invariants, $\overline{\mu}(i_1 \dots i_s)$, are 0.*

3.2 Examples of links homotopic but not isotopic to the unlink

Here I will briefly sketch the idea behind the proof of Theorem 3.1.1 when L is an unlink [16]. Notice that a good candidate for such a link is a *Brunnian link* (Fig. 3b). These links satisfy properties (2) and (3) of our theorem, but not property (1). To fix this we consider its Whitehead double.

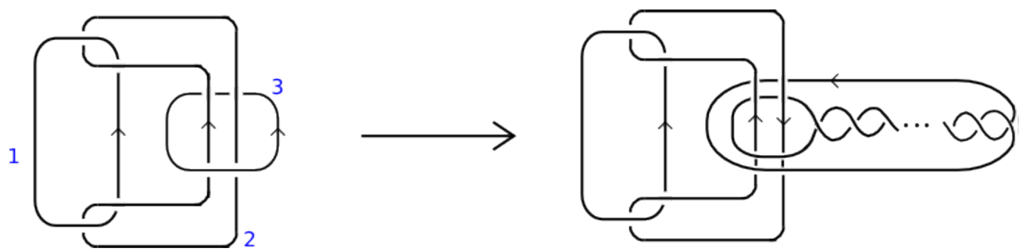
Definition 3.2.1 (Whitehead double)**.** Let l be a component of a link L in S^3, regarded as $h(\{0\} \times S^1)$ for some embedding $h : D^2 \times S^1 \to S^3 \setminus (L \setminus l)$, such that l and $h((0,1) \times S^1)$ have linking number zero. Let n be a non-zero integer. Consider in the solid torus $T = D^2 \times S^1$ the knot $\mathcal{W}_n$ with n crossings, as depicted in Figure 4. The knot $h(\mathcal{W}_n)$ is called the Whitehead n-double of l, and is denoted by $W_n(l)$.

If the components of the link L are numbered, $L = (l_1, \dots, l_m)$, then the link obtained by considering the Whitehead n-double on the i-th component of L will be denoted by $W_n^i(L)$.



Rysunek 4: $\mathcal{W}_n$

We consider here the case of 3-component links. Let B_3 be the Brunnian link with 3 components, with its components numbered as shown in Figure 5. Consider the Whitehead n-double of the third component of B_3 to obtain the new link, $W_n^3(B_3)$. Observe that when n is even, a single crossing change in $W_n^3(B_3)$ unwinds the link to make it link homotopic to an unlink, which shows that $W_n^3(B_3)$ satisfies property (1) of Theorem 3.1.1. It is also clear that is satisfies property (3).



Rysunek 5: Brunnian link, B_3, and Whitehead double of B_3, $W_n^3(B_3)$

I show that after some $N > 0$ for every even integer n, $W_n^3(B_3)$ is not link isotopic to the unlink, that is, it also satisfies property (2). Further, all the links $W_n(B_m)$ are non-isotopic, giving a family of links that satisfy all three properties. This is achieved by showing that the Jones polynomial of $W_n^3(B_3)$ for some n is not the same as that of the unlink. This involves making use of Skein relations of the Jones polynomial multiple times. Similar methods show that this result is true for $W_n^m(B_m)$, the Whitehead double of B_m for $m > 3$.

3.3 Examples of links homotopic but not isotopic to a given link

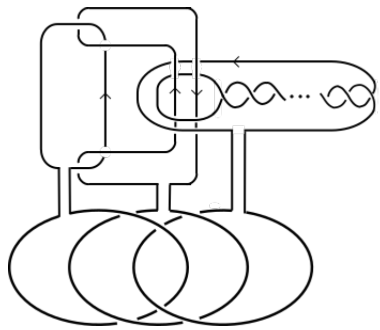
In this section we will prove Theorem 3.1.1 for a general link [16]. To construct the desired links we use what we call *connect sum of links*.

Definition 3.3.1 (Connect sum of links)**.** Let $L = (l_1, \dots, l_n)$ and $L' = (l'_1, \dots, l'_n)$ be two n-component links in S^3. Let B be a 3-ball in S^3 such that each component l_i intersects B in an unknotted arc. Further, we arrange that two distinct arcs do not link with each other. Similarly, let B' be such a ball in another copy of S^3. We define a new link by removing the two balls from the two copies of S^3 and attaching the punctured 3-spheres by a homeomorphism $h : \partial B \to \partial B'$ such that the $l_i \cap \partial B$ gets mapped to $l'_i \cap \partial B'$. We call this new link the connect sum of L and L' and denote it by $L\#L' = (l_1\#l'_1, \dots, l_n\#l'_n)$.

If the links L and L' are oriented links, we can define $L\#L'$ to be an oriented link by having h preserve the orientation of each component.

Note that the definition of connect sum of links is not well-defined in general. It depends on several factors like, the ordering of the components, the balls B and B' that are removed, the homeomorphism h, and the ways in which the link components might be extended to intersect the balls B and B'. However, our construction works for any such connect sum.

Now let L be any given link. Let us consider the example of a 3-component link in S^3 with all linking numbers $= 1$. Let $W_n(B_3)$ be the Whitehead double of the 3-component Brunnian link with n twists, where n is even and $n \geqslant 2$. Define $L_n(B_3) := L \# W_n(B_3)$ to be a connect sum of the two links.



Rysunek 6: Connect Sum $L_n = L \# W_n(B_3)$

One can easily see that these links satisfy properties (1) and (3) of Theorem 3.1.1. We show using Jones polynomials that each of the links $L_n(B_3)$ are non-isotopic for each even n. In particular, there exist infinitely many links that are not isotopic to L and hence satisfy all three properties of our theorem.

Our computations don't depend on the Jones polynomial of the link L, and hence this method works for any link m-component L with $m \geqslant 2$.

4 Future Research plans

4.1 Locally CAT(0) manifolds with no Riemannian smoothing

4.1.1 Branched coverings

The construction for the 4-manifold [7] described in Section 2.2 relies also on a basic rigidity phenomenon in non-positive curvature, namely that a free abelian subgroup in the fundamental group of a closed non-positively curved manifold stabilizes a flat in the universal cover of the manifold. The obstruction in the construction is that the flats in the universal cover $\tilde{M}$ corresponding to copies of $\mathbb{Z}^2$ in the fundamental group of M are knotted at infinity.

Stadler uses this rigidity to provide different examples of closed smooth 4-manifolds which support singular metrics of non-positive curvature but no smooth ones. In fact, the example was first pointed out by Gromov in 1985 and it first appeared as Exercise 1 in [1]. This famous exercise remained open for a long time, and was finally solved by Stadler in 2013 [18].

The manifold is constructed by considering branched coverings of $\Sigma \times \Sigma$, with the branching locus being the diagonal, Δ_Σ, where Σ is a higher genus surface. One notices that in this example the diagonal $\Delta_\Sigma \hookrightarrow \Sigma \times \Sigma$ is a codimension 2 submanifold locally modelled on the inclusion $\mathbb{H}^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{H}^2$. This is in contrast with the result by Gromov and Thurston [11], who proved that any finite ramified covering of a compact hyperbolic manifold, along a codimension 2 totally geodesic submanifold, can be endowed with a Riemannian metric of negative sectional curvature, provided the ramification locus has large normal neighborhood. The branching locus in the Gromov-Thurston examples are locally modelled on $\mathbb{H}^{n-2} \hookrightarrow \mathbb{H}^n$.

One could generalize Stadler's method to contruct examples in dimension 5. To do this one can look at branched coverings over a 3-dimension submanifold in a non-positively curved 5-manifold. One candidate is a pair of spaces locally modelled on the inclusion $\mathbb{H}^2 \times \mathbb{R} \hookrightarrow SL(3,\mathbb{R})/SO(3,\mathbb{R})$. I wish to construct co-compact lattices in $SL(3,\mathbb{R})/SO(3,\mathbb{R})$ to get pairs of spaces as described above, and then check whether these examples of a 5-manifolds support Riemannian metric of non-positive curvature or not.

4.1.2 In higher dimensions

I am also interested in exploring whether one can extend the methods described in Section 2 to produce examples of manifolds in dimensions $\geqslant 5$. For higher dimensions one might have to consider knotting and linking spheres of co-dimension 2. We still have a flat torus but one might have to develop new tools to construct the appropriate triangulations of higher dimensional spheres. However, due to the work by Januszkiewicz and Świątkowski [13], one cannot get flag-no-squares triangulations for higher dimensions. Nonetheless, one might be able to use relative hyperbolization relative to flats to produce such manifolds.

4.2 Link homotopy vs isotopy

I am interested in exploring other ways to find examples of links that are link homotopic but not isotopic to a given link in S^3. One can consider the map from the pure braid group, PB_n, to the group of link homotopy classes of string links, $\mathcal{H}_n$. This map is surjective by work of Goldsmith [10]. The kernel of this map is generated by commutators of a generator and one of its conjugates. One can use this map to find non-trivial pure braids that will be link homotopic to the trivial (closed) link. If one can prove that the closure of these links is also non-trivial, then one has another way to find examples of links that are homotopic but not isotopic to the unlink.

4.3 Pure braid group

It is known that the pure braid groups are residually torsion-free nilpotent by work of Falk and Randell [8, 9]. This implies a number of things, including that the Atiyah conjecture about integrality of L^2-Betti numbers is true for the pure braid groups. Peter Linnell and Thomas Schick [14] have been able to push almost the same statement to the full braid group. In particular, they show that the quotients of the full braid group by deep enough lower central series subgroups are torsion-free and virtually nilpotent. Their work can be used to show that the Atiyah conjecture holds true for the full braid group as well.

A significant question is whether this result carries over to naturally given quotients of the braid groups. The homotopy braid groups are intersting candidates for this. I am interested in working on this question.

Another question along these lines is whether the central quotients of the full braid groups have the algebraic properties listed above. Some related work has been done by Barré and Pichot in [2], where they study some interesting quotients of the braid group and their properties. I am interested in exploring it further.

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