FRACTAL DIMENSION AND THE PERSISTENT HOMOLOGY OF RANDOM GEOMETRIC COMPLEXES

BENJAMIN SCHWEINHART

ABSTRACT. We prove that the fractal dimension of a metric space equipped with an Ahlfors regular measure can be recovered from the persistent homology of random samples. Our main result is that if \(x_1, \ldots, x_n\) are i.i.d. samples from a \(d\)-Ahlfors regular measure on a metric space, and \(E^0_\alpha(x_1, \ldots, x_n)\) denotes the \(\alpha\)-weight of the minimum spanning tree on \(x_1, \ldots, x_n\):

\[
E^0_\alpha(x_1, \ldots, x_n) = \sum_{e \in T(x_1, \ldots, x_n)} |e|^{-\alpha},
\]

then there exist constants \(0 < C_1 \leq C_2\) so that

\[
C_1 \leq n^{-\frac{d-\alpha}{d}} E^0_\alpha(x_1, \ldots, x_n) \leq C_2
\]

with high probability as \(n \to \infty\). In particular,

\[
\log \left( \frac{E^0_\alpha(x_1, \ldots, x_n)}{\log(n)} \right) \to (d-\alpha)/d.
\]

This is a generalization of a result of Steele [63] from the non-singular case to the fractal setting. Our result is best possible, in the sense that there exist Ahlfors regular measures for which the limit \(\lim_{n \to \infty} n^{-\frac{d-\alpha}{d}} E^0_\alpha(x_1, \ldots, x_n)\) does not exist with high probability. We also prove analogous results for weighted sums defined in terms of higher dimensional persistent homology.

1. Introduction

The first precise notion of a fractal dimension was proposed by Hausdorff in 1918 [32, 38]. Since then, many other definitions have been put forward, including the box [13] and correlation [35] dimensions. These quantities do not agree in general, but coincide on a class of regular sets. Fractal dimension was popularized by Mandelbrot in the 1970s and 1980s [51, 50], and it has since found a wide range of applications in subjects including medicine [4, 47], ecology [36], materials science [24, 69], and the analysis of large data sets [5, 66]. It is also important in pure mathematics and mathematical physics, in disciplines ranging from dynamics [65] to probability [7].

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Recently, there has been a surge of interest in applications of topology, and of persistent homology in particular. Several authors have proposed estimators of fractal dimension defined in terms of minimum spanning trees and higher dimensional persistent homology [2, 49, 53, 56, 67], and provided empirical evidence that those quantities agreed with classical notions of fractal dimension. Here, we provide the first rigorous justification for the use of random minimum spanning trees and higher dimensional persistent homology to estimate fractal dimension.

We define a persistent homology dimension for measures (Definition 9) and prove that it equals the Hausdorff dimension for a wide class of “regular” measures (Corollary 10). In concurrent, separate work with J. Jaquette [42], we implement an algorithm to compute this persistent homology dimension and provide computational evidence that it performs as well or better than classical dimension estimation techniques.

Informally, a set of “fractal dimension” $d$ is self-similar in the sense that its “local properties” measured at scale $\epsilon$ scale as $\epsilon^d$ or $\epsilon^{-d}$ for some positive real number $d$ that may be non-integer. This is not well-defined in general, and there exist “multifractals” for which different local properties give different values of $d$. Here, we assume a standard regularity hypothesis that implies that the fractal dimension of a measure is well-defined in the sense that the various classical notions of fractal dimension — including the Hausdorff, box-counting, and correlation dimensions — coincide. This is done by taking the volumes of balls centered at points in our set as the defining “local property.”

**Definition 1 ([8, 22]).** A probability measure $\mu$ supported on a metric space $X$ is $d$-Ahlfors regular if there exist positive real numbers $c$ and $\delta_0$ so that

\[
\frac{1}{c} \delta^d \leq \mu(B_\delta(x)) \leq c \delta^d
\]

for all $x \in X$ and $\delta < \delta_0$, where $B_\delta(x)$ is the open ball of radius $\delta$ centered at $x$.

Ahlfors regularity is a common hypothesis used when studying geometry and analysis in the fractal setting [22, 23, 44, 48, 55]. If $\mu$ is $d$-Ahlfors regular on $X$ then it is comparable to the $d$-dimensional Hausdorff measure on $X$, and the Hausdorff measure is itself $d$-Ahlfors regular. Examples Ahlfors regular measures include the natural measures on the Sierpiński triangle and Cantor set, and, more generally, on any self-similar subset of Euclidean space defined by an iterated function system whose correct-dimensional Hausdorff measure is positive [1] (a weaker requirement than the usual open set condition); a certain well-studied measure on the fundamental group of a compact, negatively curved manifold [20, 23]; and bounded probability densities on a compact manifold, either with the intrinsic metric or one induced by
an embedding in Euclidean space (these are indeed “self-similar” sets). As such, our methods and results will be more general than previous papers on the absolutely continuous case. Standard arguments used in proofs for the non-singular case do not work here, and laws of large numbers that follow from them are false for some Ahlfors regular measures (as we will see in Theorem 4 below).

We study the asymptotic behavior of random variables of the form
\[ E^i(x_1, \ldots, x_n) = \sum_{I \in PH^i(x_1, \ldots, x_n)} |I|^\alpha, \]
where \( \{x_j\}_{j \in \mathbb{N}} \) are i.i.d. samples from a probability measure \( \mu \) on a metric space, \( PH^i(x_1, \ldots, x_n) \) denotes the \( i \)-dimensional reduced persistent homology of the Čech or Vietoris–Rips complex of \( \{x_1, \ldots, x_n\} \), and \( |I| \) is the length of a persistent homology interval. Unless otherwise specified, our results apply to the persistent homology of either the Čech or Vietoris–Rips complex, though the constants may differ. The case where \( i = 0 \) and \( \mu \) is absolutely continuous is already well-studied, under a different guise: if \( T(x_1, \ldots, x_n) \) denotes the minimum spanning tree on \( x_1, \ldots, x_n \) and
\[ E^0(x_1, \ldots, x_n) = \sum_{e \in T(x_1, \ldots, x_n)} |e|^{\alpha}, \]
then
\[ E_{\alpha}(x_1, \ldots, x_n) = E^0_{\alpha}(x_1, \ldots, x_n) \]
where persistent homology is taken with respect to the Vietoris–Rips complex.

In 1988, Steele [63] proved the following celebrated result.

**Theorem 2 (Steele).** Let \( \mu \) be a compactly supported probability measure on \( \mathbb{R}^m \), \( m \geq 2 \), and let \( \{x_n\}_{n \in \mathbb{N}} \) be i.i.d. samples from \( \mu \). If \( 0 < \alpha < m \),
\[ \lim_{n \to \infty} n^{-m/\alpha}E^0_{\alpha}(x_1, \ldots, x_n) \to c(\alpha, m) \int_{\mathbb{R}^m} f(x)^{(m-\alpha)/m} \, dx \]
with probability one, where \( f(x) \) is the probability density of the absolutely continuous part of \( \mu \), and \( c(\alpha, m) \) is a positive constant that depends only on \( \alpha \) and \( m \).

Steele wrote [63]:

One feature of Theorem 2 that should be noted is that if \( \mu \) has bounded support and \( \mu \) is singular with respect to Lebesgue measure, then we have with probability one that \( E^0_{\alpha}(x_1, \ldots, x_n) = o\left(n^{(d-\alpha)/d}\right) \). Part of the appeal of this observation is the indication that the length of the
minimum spanning tree is a measure of the *dimension* of the support of the distribution. This suggests that the asymptotic behavior of the minimum spanning tree might be a useful adjunct to the concept of dimension in the modeling applications and analysis of fractals; see, e.g., [50].

However, despite many subsequent sharper and more general results for non-singular measures on Euclidean space [3, 43, 70] and Riemannian manifolds [21], little is known about the asymptotic properties of random minimum spanning trees for singular measures. As far as we know, the only previous result toward that end is that of Kozma, Lotker and Stupp [44], who proved that if \( \mu \) is a \( d \)-Ahlfors regular measure with connected support, then the length of the longest edge of a minimum spanning tree on \( n \) i.i.d. points sampled from \( \mu \) is \( \approx \left( \log (n) / n \right)^{1/d} \), where the symbol \( \approx \) denotes that the ratio between the two quantities is bounded between two positive constants that do not depend on \( n \). They also raised the possibility that analogous asymptotics hold for the alpha-weight of a minimum spanning tree, which we prove here in Theorem 3.

More recently, as the field of stochastic topology has matured, several studies have examined the properties of the higher dimensional persistent homology of random geometric complexes defined by absolutely continuous measures on Euclidean space [6, 10, 9, 28, 68]. Most relevantly, Divol and Polonik [27] proved a strong law of large numbers akin to Steele’s theorem for the persistent homology of points sampled from bounded, absolutely continuous probability densities on \([0, 1]^m\). In the non-persistent setting, several authors have investigated the homology of random geometric complexes on manifolds [11, 12, 25, 54]. However, as far as we know, the current work is the first to study persistent homology of random geometric complexes beyond the world of absolutely continuous measures on \( \mathbb{R}^m \) with convex support (with the exception of our unpublished manuscript [60], which has largely been subsumed into the current work). With these broader hypotheses, we encounter difficult geometric issues related to non-locality and non-triviality of persistent homology, which we discuss below.

A relationship between persistent homology and fractal dimension has been observed in several experimental studies. In 1991, Weygaert, Jones, and Martinez [67] proposed using the asymptotics of \( E_0^0(x_1, \ldots, x_n) \) to estimate the generalized Hausdorff dimensions of chaotic attractors. The PhD thesis of Robins, which was arguably one of the first publications in the field of topological data analysis, studied the scaling of Betti numbers of fractals and proved results for the 0-dimensional persistent homology of disconnected sets [56]. In joint work with Robert MacPherson, we proposed a dimension for probability distributions of geometric objects based on persistent
Figure 1. Two sets of fractional dimension, and their $\epsilon$-neighborhoods: (a) a modified Sierpiński triangle and (b) a branched polymer. Their complex geometry is reflected by their persistent homology.

homology in 2012 [49]. Note that the quantities studied in that paper and in the thesis of Robins measure the complexity of a shape rather than the fractional dimension. Most recently, Adams et al. [2] defined a persistent homology dimension for measures in terms of the asymptotics of $E^1_i(x_1,\ldots,x_n)$. Their computational experiments helped to inspire this work. We study a modified version of their dimension here (Definition 9), and find hypotheses under which it agrees with the Ahlfors dimension (Corollary 10).

In the extremal setting, Kozma, Lotker and Stupp [45] defined a minimum spanning tree dimension for a metric space $M$ in terms of the behavior of $E^0_\alpha(Y)$ as $Y$ ranges over all subsets of $M$, and proved that it equals the upper box dimension. In 2018, we generalized this concept to higher dimensional persistent homology and established hypotheses under which it agrees with the upper box dimension [59]. In the course of this work, we investigated extremal questions about the number of persistent homology intervals of a set of $n$ points; these questions are also important in the probabilistic context, as we describe below.

1.1. Our Results. Our main result is:

**Theorem 3.** Let $\mu$ be a $d$-Ahlfors regular measure on a metric space, and let $\{x_n\}_{n \in \mathbb{N}}$ be i.i.d. samples from $\mu$. If $0 < \alpha < d$, then

$$E^0_\alpha(x_1,\ldots,x_n) \approx n^{\frac{d-\alpha}{d}}$$
with high probability as \( n \to \infty \), where the symbol \( \approx \) denotes that the ratio of the two quantities is bounded between positive constants that do not depend on \( n \).

We provide a proof of this result using the language of minimum spanning trees (rather than persistent homology) in Section 3. The special case where \( \mu \) is a measure on Euclidean space is also a consequence of either Theorems 5 or 8 below.

The hypotheses we require to prove Theorem 3 and our other results below are somewhat weaker than Ahlfors regularity. In particular, the proofs of our upper bounds only require that \( M_\delta (\mu) = O (\delta^{-d}) \), where \( M_\delta (\mu) \) is the maximal number of disjoint balls of radius \( \delta \) centered at points of supp \( \mu \). Also, the proofs of our lower bounds require that the uniform bounds in Equation 1 are satisfied on a set of positive measure, but not necessarily at every point in the support of \( \mu \). However, a regularity hypothesis on the underlying measure is necessary. Some definitions of fractals include the chaotic attractors studied in Section 4 of our computational paper [42]. Our computations suggest that for some such examples and each \( \alpha > 0 \) there is a different value of \( d_\alpha \) so that \( E_0^\alpha (y_1, \ldots, y_n) \approx n^{\frac{d_\alpha - \alpha}{d_\alpha}} \) (i.e. that the measure is “multifractal”). In particular, we could not replace \( d \) in the previous theorem with, say, the upper box or Hausdorff dimension of the support.

Our next theorem shows that bounding \( E_0^\alpha (x_1, \ldots, x_n) \) \( n^{-\frac{d_\alpha}{d}} \) between two constants is the best possible result. That is, laws of large numbers as in Theorem 2 are false for some Ahlfors regular measures.

**Theorem 4.** Let \( d = \log (2) / \log (3) \) and \( 0 < \alpha < d \). There exists a \( d \)-Ahlfors regular measure \( \mu \) on \( [0,1] \) so that \( \lim_{n \to \infty} n^{-\frac{d_\alpha}{d}} E_0^\alpha (x_1, \ldots, x_n) \) does not converge in probability.

In particular, we construct an example of a \( d \)-Ahlfors regular measure where \( n^{-\frac{d_\alpha}{d}} E_0^\alpha (x_1, \ldots, x_n) \) oscillates between two positive real numbers with high probability. Our proof is in Section 3.3.

As we noted in our earlier paper [59], proving asymptotic results for higher dimensional persistent homology is challenging due to extremal questions about the number of persistent homology intervals of a finite point set. While a minimum spanning tree on \( n \) points always has \( n - 1 \) edges, a set of \( n \) points may have trivial \( PH_i \) for all \( i > 0 \), and there exist families of finite metric spaces for which the number of persistent homology intervals grows faster than linearly in the number of points.

To prove upper bounds for the asymptotics of \( E_i^\alpha \) for \( i > 0 \), we require either extremal or probabilistic control of the number of persistent homology intervals of a
set of \( n \) points. Families of point sets in Euclidean space with more than a linear number of persistent homology intervals exist [34, 59], but are considered somewhat pathological. As far as we know, the Upper Bound Theorem [62] on the number of faces of a neighborly polytope provides the best extremal upper bound for the number of persistent homology intervals of the Čech complex of a finite subset of \( \mathbb{R}^m \):

\[
|PH_i(x_1, \ldots, x_n)| = \begin{cases} 
O \left( n^{i+1} \right) & i < \left\lceil \frac{m}{2} \right\rceil \\
O \left( n^{\left\lfloor \frac{m+1}{2} \right\rfloor} \right) & i \geq \left\lceil \frac{m}{2} \right\rceil
\end{cases}
\]

For the Vietoris–Rips complex of points in Euclidean space, we [59] showed that

\[
|PH_1(x_1, \ldots, x_n)| = O(n)
\]

by modifying an argument of Goff [34]. A different extremal question arises in the process of proving lower bounds for \( E_i^\alpha \). In particular, a subset \( \mathbb{R}^m \) must have dimension above a certain non-triviality constant \( \gamma^m_i \) (defined in Section 6.2) to guarantee the existence of subsets with non-trivial \( i \)-dimensional persistent homology. Note that \( \gamma^m_i \) may depend on whether persistent homology is taken with respect to the Čech complex or Vietoris–Rips complex. Unless otherwise noted, that dependence is left implicit. We showed that \( \gamma^1_1 < m - 1/2 \) for the Čech complex in our previous paper [59].

The proofs of the upper bounds in the next two theorems work for Ahlfors regular measures on arbitrary metric spaces, but the lower bound requires that the measure is defined on a subset of Euclidean space.

**Theorem 5.** Let \( \mu \) be a \( d \)-Ahlfors regular measure on \( \mathbb{R}^m \) with \( d > \gamma^m_i \), and let \( \{x_n\}_{n \in \mathbb{N}} \) be i.i.d. samples from \( \mu \). If there are positive real numbers \( D \) and \( a \) so that

\[
|PH_i(x_1, \ldots, x_n)| < Dn^a
\]

for all finite subsets of \( \text{supp} \, \mu \), and \( 0 < \alpha < ad \), then there are real numbers \( 0 < \zeta < Z \) so that

\[
\zeta n^{\frac{d-\alpha}{d}} \leq E^i_\alpha(x_1, \ldots, x_n) \leq Z n^{\frac{ad-\alpha}{d}}
\]

with high probability, as \( n \to \infty \). In fact, the upper bound holds with probability one.

The upper bound is shown in Proposition 30, and the lower bound in Proposition 43. The following is a corollary, using our previous results on \( \gamma^2_1 \) and the fact that the Alpha complex of a set of \( n \) points in \( \mathbb{R}^2 \) has \( O(n) \) faces.
Corollary 6. Let $\mu$ be a $d$-Ahlfors regular measure on $\mathbb{R}^2$, and let $\{x_n\}_{n \in \mathbb{N}}$ be i.i.d. samples from $\mu$. If $0 < \alpha < d$. If $d > 1.5$, $0 < \alpha < d$, and persistent homology is taken of the Čech complex, then

$$E_1^\alpha (x_1, \ldots, x_n) \approx n^{d-\alpha}$$

with high probability as $n \to \infty$. In fact, the upper bound holds with probability one.

Another corollary, based on our results on the Rips complex [59], is

Corollary 7. Let $\mu$ be a $d$-Ahlfors regular measure on $\mathbb{R}^m$, and let $\{x_n\}_{n \in \mathbb{N}}$ be i.i.d. samples from $\mu$. If persistent homology is taken of the Rips complex, $d > \gamma_1^m$, and $0 < \alpha < d$, then

$$E_1^\alpha (x_1, \ldots, x_n) \approx n^{d-\alpha}$$

with high probability as $n \to \infty$. In fact, the upper bound holds with probability one.

For $i > 0$ and $m > 2$, we show better upper bounds for $d$-Ahlfors regular measures for which the expectation and variance of $|PH_i(x_1, \ldots, x_n)|$ scale linearly and sub-quadratically, respectively. These quantities can be measured in practice, allowing one to determine whether higher dimensional persistent homology would be suitable for dimension estimation in applications.

Theorem 8. Let $\mu$ be a $d$-Ahlfors regular measure on $\mathbb{R}^m$ so that $d > \gamma_1^m$, and let $\{x_n\}_{n \in \mathbb{N}}$ be i.i.d. samples from $\mu$. If

$$E \left(|PH_i(x_1, \ldots, x_n)|\right) = O(n) \quad \text{and} \quad \text{Var} \left(|PH_i(x_1, \ldots, x_n)|\right) / n^2 \to 0$$

and $0 < \alpha < d$, then there are real numbers $0 < \lambda < \Lambda$ so that

$$\lambda n^{d-\alpha} \leq E_i^\alpha (x_1, \ldots, x_n) \leq \Lambda n^{d-\alpha} \log (n)^{\frac{\alpha}{d}}$$

with high probability, as $n \to \infty$.

The upper and lower bounds are shown in Propositions 35 and 43, respectively. Many of our other results can be viewed as special cases of this theorem, including Corollaries 6 and 7 and the particular case of Theorem 3 where the measure is supported on Euclidean space. More generally, although there are few rigorous results on the scaling of the number of persistent homology intervals in higher dimensions, computational results indicate that these hypotheses hold broadly — see the Appendix. Also, Stemeseder [64] showed that any positive, continuous probability density on the $m$-dimensional Euclidean sphere satisfies the hypothesis on the expected number of intervals, and the uniform measure on the sphere satisfies the hypothesis on the variance. However, we think this is interesting hypotheses not because they are easy to prove but because they can be estimated in data analysis.
1.2. Dimension Estimation. As we noted earlier, several authors have proposed to use persistent homology for dimension estimation. Here, we provide the first proof that these methods recover a classical fractal dimension, under certain hypotheses.

We define a family of $PH_i$ dimensions of a measure, one for each real number $\alpha > 0$ and $i \in \mathbb{N}$:

**Definition 9.**

$$\dim_{PH_\alpha^i}(\mu) = \frac{\alpha}{1 - \beta},$$

where

$$\beta = \limsup_{n \to \infty} \frac{\log \left( \mathbb{E} \left( E_{\alpha}^i(x_1, \ldots, x_n) \right) \right)}{\log(n)}.$$

That is, $\dim_{PH_\alpha^i}(\mu)$ is the unique real number $d$ so that

$$\limsup_{n \to \infty} \mathbb{E} \left( E_{\alpha}^i(x_1, \ldots, x_n) \right) n^{-\frac{k-\alpha}{k}}$$

equals $\infty$ for all $k < d$, and is bounded for $k > d$. The case $\alpha = 1$ is very closely related to the dimension studied by Adams et al. [2], and agrees with it if defined.

Theorem 2 [63] implies that if $\mu$ is a compactly supported, non-singular probability measure on $\mathbb{R}^m$, then $\dim_{PH_0}(\mu) = m$ for $0 < \alpha < m$. Similar, the results of Divol and Polonik [27] show that if $\mu$ is a bounded probability measure on the cube in $\mathbb{R}^m$, then $\dim_{PH_i}(\mu) = m$ for $0 < \alpha < m$ and $0 \leq i < m$.

The following is a corollary of our theorems on the asymptotic behavior of $E_{\alpha}^i$:

**Corollary 10.** If $\mu$ is a $d$-Ahlfors regular measure on a metric space and $0 < \alpha < d$ then

$$\dim_{PH_0} = d.$$

Furthermore, if $\mu$ is defined on $\mathbb{R}^m$, $d > \gamma_i^m$, and

$$\mathbb{E} \left( |PH_i(x_1, \ldots, x_n)| \right) = O(n) \quad \text{and} \quad \text{Var} \left( |PH_i(x_1, \ldots, x_n)| \right) / n^2 \to 0,$$

then

$$\dim_{PH_i} = d.$$

This result is weaker than our main theorems, and it can be proven with weaker hypotheses. For example, the upper bound $\dim_{PH_i}(\mu) \leq d$ holds if the hypothesis of $d$-Ahlfors regularity is replaced by the requirement that the upper box dimension of the support of $\mu$ is less than or equal to $d$. 
We also prove the following proposition, which does not require any regularity hypothesis on the measure:

**Proposition 11.** Let \( \mu \) be a measure on a bounded metric space \( X \), and let \( \dim_{\text{box}}(X) \) be the upper box dimension of \( X \) (defined below). If \( \alpha < \dim_{\text{box}}(X) \) then
\[
\dim_{PH}^\alpha(\mu) \leq \dim_{\text{box}}(X).
\]

In separate experimental work (joint with J. Jaquette), we implement an algorithm to compute the persistent homology dimensions and compare its practical performance below to classical techniques for estimating fractal dimension, such as box–counting and the estimation of the correlation dimension. The persistent homology dimension (for \( i = 0 \)) performs about as well as the correlation dimension, both in terms of the convergence rate and speed of computation, and significantly better than the box dimension. [42] Our results here imply that the computational estimates in the other paper will converge with high probability as the number of samples goes to infinity for several of the examples considered. These include the \( PH_0 \) dimension of the Cantor dust, Cantor set cross an interval, Sierpiński triangle, and Menger sponge, and the \( PH_1 \) dimensions of the Cantor set cross an interval and the Sierpiński triangle.

**1.3. A Conjecture.** We conjecture that if the persistent homology of the support of an Ahlfors regular measure is trivial, then the Lebesgue measure can be replaced with the \( d \)-dimensional Hausdorff measure \( \mathcal{H}^d \) in Theorem 2. Note that this would exclude the example in Theorem 4.

**Conjecture 12.** Let \( \mu \) be a \( d \)-Ahlfors regular measure on a metric space \( M \) and let \( \{x_n\}_{n \in \mathbb{N}} \) be i.i.d. samples from \( \mu \). If \( PH_0(\text{supp } \mu) \) is trivial and \( 0 < \alpha < d \), then
\[
\lim_{n \to \infty} n^{-\frac{d-\alpha}{d}} E_0^0(x_1, \ldots, x_n) \to c_0(\alpha, d) \int_M f(x)^{(d-\alpha)/d} \, dx
\]
with probability one, where \( f(x) \) is the probability density of the absolutely continuous part of \( \mu \) with respect to the \( d \)-dimensional Hausdorff measure \( \mathcal{H}^d \) and \( c_0(\alpha, d) \) is a continuous function of \( \alpha \) and \( d \).

Furthermore, if \( \mu \) is supported on \( \mathbb{R}^m \), \( d > \gamma_m \), and \( PH_i(\text{supp } \mu) \) is trivial then
\[
\lim_{n \to \infty} n^{-\frac{d-\alpha}{d}} E_i^0(x_1, \ldots, x_n) \to c_i(\alpha, d) \int_M f(x)^{(d-\alpha)/d} \, dx
\]
with probability one, where \( f(x) \) is the probability density of the absolutely continuous part of \( \mu \) with respect to the \( d \)-dimensional Hausdorff measure \( \mathcal{H}^d \) and \( c_i(\alpha, d) \) is a continuous function of \( \alpha \) and \( d \).
2. Preliminaries

Let \( X \) be a metric space, and let \( M_\delta (X) \) be the maximal number of disjoint open balls of radius \( \delta \) centered at points of \( X \). The upper box dimension is defined in terms of the asymptotic properties of \( M_\delta (X) \).

**Definition 13.** Let \( X \) be a bounded metric space. The upper box dimension of \( X \) is

\[
dim_{\text{box}} (X) = \limsup_{\delta \to 0} \frac{\log (M_\delta (X))}{-\log (\delta)}.
\]

If \( X \) admits a \( d \)-Ahlfors regular measure, we can control the behavior of \( M_\delta (X) \).

**Lemma 14** (Ball-counting Lemma). If \( \mu \) is a \( d \)-Ahlfors regular measure supported on a metric space \( X \) then

\[
\frac{1}{c} 2^{-d} \delta^{-d} \leq M_\delta (X) \leq c \delta^{-d}
\]

for all \( \delta < \delta_0 \), where \( c \) and \( \delta_0 \) are the constants given in the definition of Ahlfors regularity.

**Proof.** Let \( \{x_j\}_{j=1}^{M_\delta (X)} \) be the centers of a maximal set of disjoint balls of radius \( \delta \) centered at points of \( X \).

\[
1 = \mu (X) = \mu \left( \bigcup_{j=1}^{M_\delta (\mu)} B_\delta (x_j) \right) \geq \sum_{j=1}^{M_\delta (\mu)} \mu (B_\delta (x_j)) \geq \frac{1}{c} \delta^d M_\delta (\mu) \quad \text{by disjointness}
\]

\[
\implies M_\delta (\mu) \leq c \delta^{-d}.
\]

The maximality of \( \{B_\delta (x_i)\}_{i=1}^{M_\delta (\mu)} \) implies that the balls of radius \( 2\delta \) centered at the points \( \{x_i\} \) cover \( X \). It follows that

\[
1 = \mu (X) = \mu \left( \bigcup_{j=1}^{M_\delta (X)} B_{2\delta} (x_j) \right) \leq \sum_{j=1}^{M_\delta (X)} \mu (B_{2\delta} (x_j)) \leq c 2^d \delta^d M_\delta (X) \quad \text{by Ahlfors regularity}
\]

\[
\implies M_\delta (X) \geq \frac{1}{c} 2^{-d} \delta^{-d},
\]
We also require the following lemma of Cohen-Steiner et al. [19].

**Lemma 15.** Let $J \subset \mathbb{R}^+$ be a bounded set of positive real numbers and let

$$J_\epsilon = \{ j \in J : j > \epsilon \}.$$

If

$$|J_\epsilon| \leq f(\epsilon) < \infty$$

for all $\epsilon > 0$ then

$$\sum_{j \in J_\epsilon} j^\alpha \leq \epsilon^\alpha f(\epsilon) + \alpha \int_{\delta=\epsilon}^{\max J} f(\delta) \delta^{\alpha-1} \, d\delta$$

for all $\alpha > 0$. Furthermore, if $|J| \leq f(0) < \infty$ then

$$\sum_{j \in J} j^\alpha \leq \epsilon^\alpha \int_{\delta=0}^{\max J} f(\delta) \delta^{\alpha-1} \, d\delta.$$

For completeness, we reproduce the proof in [19]. $\sum_{j \in J_\epsilon} j^\alpha$ can be expressed as an integral involving the distributional derivative of $|J_\epsilon|$. Applying integration by parts yields:

$$\sum_{j \in J_\epsilon} j^\alpha = \int_{\delta=\epsilon}^{\infty} \frac{\partial |J_\delta|}{\partial \delta} \delta^\alpha \, d\delta$$

$$= \left[ -|J_\delta| \delta^\alpha \right]_{\delta=\epsilon}^{\infty} + \alpha \int_{\delta=\epsilon}^{\infty} |J_\delta| \delta^{\alpha-1} \, d\delta$$

$$= \epsilon^\alpha |J_\epsilon| + \alpha \int_{\delta=\epsilon}^{\sup J} |J_\delta| \delta^{\alpha-1} \, d\delta$$

$$\leq \epsilon^\alpha f(\epsilon) + \alpha \int_{\delta=\epsilon}^{\sup J} f(\delta) \delta^{\alpha-1} \, d\delta.$$

2.1. **Notation.** In the following, $X$ will denote a metric space and $x$ will denote a finite point set with an unspecified number of elements. Furthermore, $x_n$ will be shorthand for a finite point set $\{x_1, \ldots, x_n\} \subset X$ containing $n$ points. If the measure $\mu$ is obvious from the context, $\{x_j\}_{j \in \mathbb{N}}$ will be a collection of independent random variables with common distribution $\mu$. Finally, we will use symbols with the “mathcal” font (i.e. $\mathcal{A}, \mathcal{B}, \ldots$) for collections of sets.
2.2. Occupancy Events. Our strategy for proving lower bounds for the asymptotic behavior of $E_i^\alpha(x_1,\ldots,x_n)$ will be to define certain occupancy events that imply the existence of a persistent homology interval (or minimum spanning tree edge) whose length is bounded away from zero.

If $A$ and $B$ are sets define
\[
\delta(A,B) = \begin{cases} 
0 & A \cap B = \emptyset \\
1 & A \cap B \neq \emptyset
\end{cases}.
\]

Also, if $A$ is a set and $\mathcal{B}$ is a collection of sets define the occupancy event
\[
\Xi(x, A, \mathcal{B}) = \begin{cases} 
1 & \delta(A, x) = 0 \quad \text{and} \quad \delta(B, x) = 1 \quad \forall B \in \mathcal{B} \\
0 & \text{otherwise}
\end{cases}
\]

All occupancy events considered in this paper will satisfy $A \cap B = \emptyset$ for all $B \in \mathcal{B}$, and $B_1 \cap B_2 = \emptyset$ for all $B_1, B_2 \in \mathcal{B}$ so that $B_1 \neq B_2$. We say that two occupancy events $\Xi(x, A_1, B)$ and $\Xi(x, A_2, C)$ (where $x$ is the same sample for each) are disjoint if
\[
\left( A_1 \cup \bigcup_{B \in \mathcal{B}} B \right) \cap \left( A_2 \cup \bigcup_{C \in \mathcal{C}} C \right) = \emptyset.
\]

An $n,p,q,r$-bounded occupancy event is a random variable of the form
\[
\Xi(x_n, A, \mathcal{B}),
\]
where $\mathcal{B}$ is a collection of at least $r$ sets, and $x_n$ is a collection of $n$ independent random variables with common distribution $\nu$ satisfying
\[
\nu(A) \leq q/n \quad \text{and} \quad \nu(B) \geq p/n \quad \forall B \in \mathcal{B}.
\]

If the above conditions on $\nu$ and the number of sets in $\mathcal{B}$ hold with equality, we say that $\Xi(x_n, A, \mathcal{B})$ is a $n,p,q,r$-uniform occupancy event.

Disjoint $n,p,q,r$-uniform occupancy events satisfy something akin to a weak law of large numbers as $n \to \infty$.

**Lemma 16.** Let $r, a > 0$, and $0 < p, q < 1$. Also, for each $n \in \mathbb{N}$ let $X_1^n, \ldots, X_{[an]}^n$ be disjoint $n,p,q,r$-uniform occupancy events. If $Y_n = \frac{1}{n} \sum_{j=1}^{[an]} X_j^n$, then
\[
\lim_{n \to \infty} Y_n = \gamma
\]
in probability, where $\gamma = ae^{-q} \left(1 - e^{-p}\right)^r$. 
Proof. First, we compute the limiting expectation of the events $X_j^n$ as $n \to \infty$:

$$
\mathbb{E}(X_j^n) = \mathbb{P}(X_j^n = 1) = \left(1 - \frac{q}{n}\right)^n \sum_{j=0}^r (-1)^j \binom{r}{j} \left(1 - j \frac{p/n}{1 - q/n}\right)^n
$$

by inclusion-exclusion. Therefore

$$
\lim_{n \to \infty} \mathbb{E}(X_j^n) = e^{-q} \left(\sum_{k=0}^r (-1)^k \binom{r}{k} e^{-kp}\right) = e^{-q} (1 - e^{-p})^r
$$

where we factored the second term in the middle equation using the binomial theorem. Thus $\lim_{n \to \infty} \mathbb{E}(Y_n) = \gamma$ by linearity of expectation.

A similar computation shows that if $j \neq k$,

$$
\lim_{n \to \infty} \mathbb{E}(X_j^n X_k^n) = e^{-2q} (1 - e^{-p})^{2r}.
$$

It follows that

$$
\lim_{n \to \infty} \text{Cov}(X_j^n, X_k^n) = \lim_{n \to \infty} \left(\mathbb{E}(X_j^n X_k^n) - \mathbb{E}(X_j^n) \mathbb{E}(X_k^n)\right) = 0.
$$

Therefore

$$
\text{Var}(Y_n) = \frac{1}{n^2} \left(\sum_{j=1}^{\lfloor an \rfloor} \text{Var}(X_j^n) + 2 \sum_{j=1}^{\lfloor an \rfloor} \sum_{i=1}^{j-1} \text{Cov}(X_j^n, X_k^n)\right)
$$

$$
\sim \frac{a}{n} \text{Var}(X_1^n) + a \frac{n^2 - n}{n^2} \text{Cov}(X_1^n, X_2^n)
$$

$$
\leq \frac{a}{n} + a \left(1 - \frac{1}{n}\right) \text{Cov}(X_1^n, X_2^n)
$$

also converges to 0 as $n$ goes to $\infty$.

Let $\epsilon > 0$ and $0 < \rho < 1$. Choose $N$ sufficiently large so that

$$
|\mathbb{E}(Y_n) - \gamma| < \epsilon/2 \quad \text{and} \quad \text{Var}(Y_n) < \frac{\epsilon^2 \rho}{4}
$$
for all $n > N$. If $n > N$,
\[
P \left( |Y_n - \gamma| > \epsilon \right) \leq P \left( |Y_n - \mathbb{E}(Y_n)| > \epsilon/2 \right) \\
\leq P \left( |Y_n - \mathbb{E}(Y_n)| > \frac{1}{\sqrt{\rho}} \sqrt{\text{Var}(Y_n)} \right) \\
\leq \rho
\]
by Chebyshev’s Inequality.

The occupancy events we define below will not be uniform, but we can use the previous lemma to bound them. We will use a standard lemma on non-atomic measures [41, 61].

**Lemma 17.** If $\mu$ is a non-atomic measure on a metric space $Y$, and $0 < a < \mu(Y)$ then there exists $Y_0 \subset Y$ so that $\mu(Y_0) = a$.

**Lemma 18.** Let $r, a > 0, 0 < p, q < 1$, and $s_n \geq \lfloor an \rfloor$ for all $n \in \mathbb{N}$. Also, for each $n \in \mathbb{N}$ let $X_1^n, \ldots, X_{s_n}^n$ be disjoint $n, p, q, r$-bounded occupancy events. Under these hypotheses, there is a $\gamma > 0$ so that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{s_n} X_j^n \geq \gamma
\]
with high probability.

**Proof.** Let $a_0 = \min \left( a, 1/(p+q) \right), 1 \leq j \leq \lfloor a_0n \rfloor$, and
\[
X_j^n = \Xi(x_n, A_j^n, B_j^n).
\]
$\nu$ is non-atomic, so by the previous lemma we can find a subset $\hat{B}$ of each set $B \in B_j^n$ so that $\nu(\hat{B}) = p/n$. Let
\[
\hat{B}_j^n = \left\{ \hat{B} : B \in B_j^n \right\} \quad \text{and} \quad D_n = \bigcup_{j=1}^{[a_0n]} \bigcup_{\hat{B} \in \hat{B}_j^n} \hat{B}.
\]
We will show that there are disjoint sets $\hat{A}_1^n, \ldots, \hat{A}_{[a_0n]}^n$ so that $A_j^n \subseteq \hat{A}_j^n \subseteq D_j^n$ and $\nu(\hat{A}_j^n) = q/n$ for $j = 1, \ldots, [a_0n]$. Let $\hat{D}_n = D_j^n \setminus \bigcup_j A_j^n$. The maximum index of $j$
Fractal Dimension and Persistent Homology

is \([a_0n]\) and \(a_0 (p + q) \leq 1\) so

\[ \nu \left( \hat{D}_0 \right) \geq \sum_{j=1}^{[a_0n]} \left( \frac{q}{n} - \nu \left( A^n_j \right) \right). \]

Applying the previous lemma to \(\nu \mid_{\hat{D}_n}\) gives a \(C_1 \subset \hat{D}_n\) with \(\nu (C_1) = q/n - \nu (A^n_1)\), so if \(\hat{A}_1^n = C_1 \cup A_1^n\) then \(A_1^n \subseteq \hat{A}_1^n \subset \hat{D}_n\) and \(\nu \left( \hat{A}_1^n \right) = q/n\). Assuming we have found \(\hat{A}_1^n, \ldots, \hat{A}_k^n\) we can apply the same argument to \(\nu \mid_{\hat{D}_n \cup \bigcup_{j=1}^{k} \hat{A}_j^n}\) to find \(\hat{A}^{n}_{k+1}\).

Let

\[ \hat{X}_j^n = \Xi \left( x_n, \hat{A}_j^n, \hat{B}_j^n \right). \]

By construction, \(X_j^n = 1 \implies \hat{X}_j^n = 1\) so \(\frac{1}{n} \sum_{j=1}^{[a_0n]} X_j^n\) stochastically dominates \(\frac{1}{n} \sum_{j=1}^{[a_0n]} \hat{X}_j^n\). Applying the previous lemma to the latter sum implies the desired result. \(\square\)

3. The Proofs for Minimum Spanning Trees

If \(x\) is a finite metric space, \(T(x)\) denote the minimum spanning tree on \(x\), and let \(p(x, \epsilon)\) be the number of edges of \(T(x)\) of length greater than \(\epsilon\). Also, let \(G_{x,\epsilon}\) be the graph with vertex set \(x\) so that \(x_1\) and \(x_2\) are connected by an edge if and only if \(d(x_1, x_2) < \epsilon\) (this is the one-skeleton of the Vietoris-Rips complex on \(x\)). The following is a corollary of Kruskal’s algorithm.

**Lemma 19.**

\[ p(x, \epsilon) = \beta_0(G_{x,\epsilon}) - 1 \]

where \(\beta_0(G_{x,\epsilon})\) is the number of connected components of \(G_{x,\epsilon}\).

3.1. Proof of the Upper Bound in Theorem 3. Our strategy to prove an upper bound for the asymptotics of \(E_0^n(x_n)\) is to control the number of edges in \(T(x_n)\) of length greater than \(\delta\) in terms of the maximal number of disjoint balls of radius \(\delta/2\) centered at points of \(x_n\). The approach is similar to that in our earlier papers [59, 60].

**Lemma 20.** Let \(X\) be a metric space and suppose there are positive real numbers \(D\) and \(d\) so that

\[ M_\delta(X) \leq D \delta^{-d} \]

for all \(\delta > 0\), where \(M_\delta(X)\) was defined in the previous section. Then

\[ p(x, \delta) < D2^{-d} \delta^{-d} \]

for all finite subsets \(x\) of \(X\) and all \(\delta > 0\).
Proof. Let \( x \subseteq X \) and \( \delta > 0 \). Also, let \( y \) be set of centers of a maximal set of disjoint balls of radius \( \delta/2 \) centered at points of \( x \). The maximality of \( y \) implies that for every \( x \in x \) there exists a \( y \in y \) so that \( d(x, y) < \delta \). In particular, every connected component of \( G_{x, \delta} \) has a vertex that is an element of \( y \). Therefore,

\[
p(x, \delta) = \beta_0(G_{x, \delta}) - 1 \\
\leq |y| - 1 \\
\leq D(\delta/2)^{-d} \\
= 2^{-d}D\delta^{-d}.
\]

\( \square \)

We prove an extremal upper bound for \( E^0_\alpha(x_n) \) that, when combined with Lemma 14, implies the upper bound for Theorem 3.

**Proposition 21.** Let \( X \) be a metric space and suppose there are positive real numbers \( D \) and \( d \) so that

\[
M_\delta(X) \leq D \delta^{-d}
\]

for all \( \delta > 0 \). If \( 0 < \alpha < d \), then there exists a \( D_\alpha > 0 \) so that

\[
E^0_\alpha(x_n) \leq D_\alpha n^{\frac{d-\alpha}{d}}
\]

for all \( n \) and all collections \( x_n \) of \( n \) points in \( X \). Furthermore, there exists a \( D_d > 0 \) so that

\[
E^0_d(x_n) \leq D_d \log(n)
\]

for all \( n \) and all collections \( x_n \) of \( n \) points in \( X \).

**Proof.** Rescale \( X \) if necessary so that its diameter is less than 1, and let

\[
\kappa = \frac{1}{2} \left( \frac{D}{n-1} \right)^{1/d}.
\]

The previous lemma together with the fact that a minimum spanning tree on \( n \) points has \( n - 1 \) edges implies that \( p(\{x_n\}, \epsilon) \leq f(\epsilon) \) where

\[
f(\epsilon) = \min\left(n - 1, 2^{-d}D\epsilon^{-d}\right) = \begin{cases} 
  n - 1 & \epsilon \leq \kappa \\
  2^{-d}D\epsilon^{-d} & \epsilon \geq \kappa
\end{cases}.
\]
Applying Lemma 15 to the set of edge lengths of the minimum spanning tree on $x_n$ yields

$$E_\alpha^0(x_n) = \sum_{e \in T(x_n)} |e|^{\alpha}$$

$$\leq \alpha \int_{\delta=0}^{1} f(\delta) \delta^{\alpha-1} \, d\delta$$

$$= (n-1) \int_{\delta=0}^{\kappa} \alpha \delta^{\alpha-1} \, d\delta + \alpha 2^{-d} D \int_{\delta=\kappa}^{1} \delta^{\alpha-d-1} \, d\delta$$

$$= (n-1) \left[ \alpha \delta_0^{\alpha} - \frac{\alpha}{d-\alpha} \right] \frac{1}{D}$$

$$= 2^{\alpha} D_{\alpha}^{\frac{d}{d-\alpha}} \left( 1 + D \frac{\alpha}{d-\alpha} \right) (n-1) \frac{d^{\alpha}}{d^{\alpha-d}} - \frac{\alpha}{d-\alpha} 2^{-d} D$$

$$\leq D_{\alpha} n^{\frac{d^{\alpha}}{d-\alpha}},$$

where

$$D_{\alpha} = 2^{\alpha} D_{\alpha}^{\frac{d}{d-\alpha}} \left( 1 + D \frac{\alpha}{d-\alpha} \right).$$

The result for $\alpha = d$ follows from a similar computation. $\square$

Proposition 11 also follows from the previous result.

Proof of Proposition 11. Let $\mu$ be a measure on a metric space $X$, $d = \dim_{\text{box}}(X)$ be the upper box dimension of $X$, and $\alpha < d < d_0$. By Definition 13 there is a $D > 0$ so that

$$M_\delta(X) \leq D \delta^{-d_0}$$

for all sufficiently small $\delta$. Therefore, by the previous result, there is a $D_{\alpha} > 0$ so that

$$E_\alpha^0(x_n) \leq D_{\alpha} n^{\frac{d_0-\alpha}{d_0}}$$

for all sufficiently large $n$.

Then

$$\beta := \limsup_{n \to \infty} \frac{\log \left( \mathbb{E} \left( E_\alpha^i(x_n) \right) \right)}{\log (n)} \leq \frac{\log \left( n^{\frac{d_0-\alpha}{d_0}} \right)}{\log (n)} = \frac{d_0 - \alpha}{d_0}.$$
and
\[
\dim_{PH}^\alpha (\mu) = \frac{\alpha}{1 - \frac{\alpha}{d_0 - \alpha}} = d_0 ,
\]
where we have used that \(\frac{d_0 - \alpha}{d_0} > 0\). This inequality holds for any \(d_0 > d\), so
\[
\dim_{PH}^\alpha (\mu) \leq d = \dim_{box} (X) ,
\]
as desired. \(\Box\)

### 3.2. Proof of the Lower Bound in Theorem 3

Our strategy to prove a lower bound for the asymptotics of \(E_0^\alpha (x_n)\) is to define random variables in terms of occupancy patterns of disjoint balls of radius \(2r\). These random variables will imply the existence of minimum spanning tree edges of length at least \(r\).

Let \(M\) be a metric space and let \(\mu\) be a \(d\)-Ahlfors regular measure with support \(M\). If \(B\) is a ball of radius \(2r\) centered at a point \(y \in M\) and \(x\) is a finite subset of \(M\), define
\[
\omega (B, x) = \Xi \left( x, B \setminus B_r (y), \{ B_r (y) \} \right) .
\]
That is, \(\omega (B, x) = 1\) if \(x\) intersects \(B_r (y)\) but not the annulus centered at \(y\) with radii \(r\) and \(2r\).
Lemma 22. Let $B$ be a set of disjoint balls of radius $2r$ centered at points of $M$, and let $x$ be a finite subset of $M$. Then
\[ p(x, r) \geq \sum_{B \in B} \omega(B, x) - 1. \]

Proof. This is an immediate consequence of Lemma 19. See Figure 2. \qed

Fix $n \in \mathbb{N}$ and let $\epsilon = n^{-1/d}$. Let $B^n_1, \ldots, B^n_{s_n}$ be a maximal collection of disjoint balls of radius $2\epsilon$ centered at points of $X$, and let $y^n_j$ be the center of $B^n_j$ for $j = 1, \ldots, s_n$. We require one more lemma before proving the lower bound in Theorem 3.

Lemma 23. There is a positive real number $\gamma > 0$ so that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{s_n} \omega(B^n_j, x^n) \geq \gamma \]
with high probability as $n \to \infty$.

Proof. Let
\[ p = \frac{1}{c} \quad \text{and} \quad q = 2^d c - \frac{1}{c}. \]
where $c$ is the constant appearing in the definition of Ahlfors regularity. By that definition,
\[ \mu\left( B_\epsilon(y^n_j) \right) \geq p \epsilon^d = \frac{p}{n} \]
and
\[ \mu\left( B^n_j \setminus B_\epsilon(y^n_j) \right) \leq c(2\epsilon)^d - \frac{1}{c} \epsilon^d = \frac{q}{n}. \]
Also, Lemma 14 implies that
\[ s_n \geq \frac{1}{c} 2^{-d} \epsilon^{-d} = \frac{1}{c} 2^{-2d} n. \]

Therefore, the occupancy events $\omega\left( B^n_1, x^n \right), \ldots, \omega\left( B^n_{s_n}, x^n \right)$ satisfy the hypotheses of Lemma 18, which immediately implies the desired result. \qed

The lower bound in Theorem 3 follows quickly.
Proposition 24. Let $\mu$ be a $d$-Ahlfors regular measure on a metric space $M$. If $\{x_j\}_{j \in \mathbb{N}}$ are i.i.d. samples from $\mu$, and $\gamma$ is as given in the previous lemma, then

$$\lim_{n \to \infty} n^{-\frac{d}{d} - \frac{\alpha}{d}} E_\alpha^0 (x_n) \geq \gamma$$

with high probability.

Proof. We have that

$$\lim_{n \to \infty} n^{-\frac{d}{d} - \frac{\alpha}{d}} E_\alpha^0 (x_n) \geq \lim_{n \to \infty} n^{-\frac{d}{d} - \frac{\alpha}{d}} n^{-\alpha/d} p \left( x_n, n^{-1/d} \right)$$

$$\geq \lim_{n \to \infty} \frac{1}{n} \left( \sum_{j=1}^{s_n} \omega \left( B_{n,j}^n, x_n \right) - 1 \right)$$

by Lemma 22

$$\geq \gamma$$

by Lemma 23

with high probability as $n \to \infty$. \qed

3.3. Proof of Theorem 4. We will construct a $d$-Ahlfors regular measure $\sigma$ with $d = \frac{\log(2)}{\log(3)}$ so that if $\{z_j\}_{j \in \mathbb{N}}$ are i.i.d. samples from $\mu$ and $0 < \alpha < d$ then the quantity

$$n^{-\frac{d}{d} - \frac{\alpha}{d}} E_\alpha^0 (z_1, \ldots, z_n)$$

oscillates with high probability as $n \to \infty$. Our example will be constructed as the intersection of a nested sequence of closed subsets $Y_1 \supset Y_2 \supset Y_3 \ldots$ of $[0, 1]$, where each $Y_j$ is the union of finitely many congruent, disjoint intervals. At some scales the set will resemble the Cantor set, while at others it will resemble the Cantor set scaled by a factor $\frac{5}{7}$. 

Figure 3. The nested sequences of interval collections defining $C$ and $\frac{5}{7} C$. 

\begin{center}
\begin{tabular}{ccc}
$C$ & $\frac{5}{7} C$ \\
$T_1$ & \hline & \hline \\
$T_2$ & --- & --- \\
$T_3$ & -- & -- \\
$T_4$ & -- & -- \\
$T_5$ & -- & -- \\
$T_6$ & -- & -- \\
\end{tabular}
\end{center}
The Cantor set can be defined in terms of a “middle thirds” operation on sets of intervals. If \( I \) is an interval, let \( K (I) \) be the set of two intervals of length \( 1/3 |I| \)

\[
K (I) = \left\{ 1/3 I, 1/3 I + 2/3 |I| \right\},
\]
and if \( \mathcal{I} \) is a set of intervals let

\[
K (\mathcal{I}) = \left\{ K (I) : I \in \mathcal{I} \right\}.
\]

Define \( T_m \) to be the set of intervals obtained by applying \( K \) to \( \{ [0, 1] \} \) \( m - 1 \) times. \( T_m \) consists of \( 2^{m-1} \) intervals of length \( (1/3)^{m-1} \). Then the Cantor set is

\[
C = \cap_{m \in \mathbb{N}} T_m.
\]

See Figure 3.

We introduce notation and shorthand related to sets of intervals. Call a finite set of disjoint intervals \( \mathcal{I} \) an “interval collection.” We will abuse notation, and use \( \mathcal{I} \) to refer to both the collection \( \mathcal{I} \) and the union \( \bigcup_{I \in \mathcal{I}} I \). Let \( |\mathcal{I}| \) be the number of intervals in the collection, and \( \|\mathcal{I}\| \) be the minimum length of an interval. A natural map between two interval collections \( \mathcal{I} \) and \( \mathcal{J} \) is an order-preserving homeomorphism \( f : \mathcal{I} \to \mathcal{J} \) so that, for any \( I \in \mathcal{I}, \phi |_I \) is a translation and \( f(I) \) is an interval in \( \mathcal{J} \). Note that if \( \mathcal{I} \) and \( \mathcal{J} \) are sets of disjoint, congruent intervals so that \( |\mathcal{I}| = |\mathcal{J}| \) and \( \|\mathcal{I}\| = \|\mathcal{J}\| \), then there is a unique natural map between them.

If \( S_1 \supset S_2 \supset S_3 \ldots \) is a nested sequence of interval collections, there is natural probability measure \( \mu_S \) on \( \cap_n S_n \) so that

\[
\mu_S (I) = \frac{1}{|S_j|} \text{ for } I \in S_j.
\]

The natural measure on the Cantor set is defined this way.

Before proving Theorem 4, we prove three technical lemmas.

**Lemma 25.** Let \( \mathcal{I} \) and \( \mathcal{J} \) be interval collections contained in \([0, 1]\), and suppose that there is a natural map \( f : \mathcal{I} \to \mathcal{J} \). Let \( 0 < \alpha < d \), and let \( \mu \) be a probability measure supported on \( \mathcal{I} \) so that

\[
n^{- \frac{d-\alpha}{d}} E^0_\alpha (x_n) \to c
\]

in probability as \( n \to \infty \), for some real number \( c \). Then

\[
n^{- \frac{d-\alpha}{d}} E^0_\alpha (f(x_1), \ldots, f(x_n)) \to c
\]

in probability as \( n \to \infty \).
Proof. First, note that if \( \{y_1, \ldots, y_n\} \) is an ordered set of points in \( \mathbb{R} \), the edges of \( T(y_1, \ldots, y_n) \) are the intervals \([y_1, y_2], \ldots, [y_{n-1}, y_n] \). For a finite point set \( x \subseteq I \) let

\[
T_0(x) = \{ e \in T(x) : e \subseteq I \text{ for some } I \in \mathcal{I} \}
\]

and

\[
T_1(x) = T(x) \setminus T_0(x).
\]

See Figure 4. Note that \( |T_1(x)| < k := |\mathcal{I}| \).

Recall that \( x_n \) is shorthand for \( \{x_1, \ldots, x_n\} \). If the edge \([x_j, x_{j+1}]\) is contained in \( T_0(x_n) \), then \([x_j, x_{j+1}] \subseteq I \) for some \( I \in \mathcal{I} \), and by the definition of a natural map there is a \( J \in \mathcal{J} \) so that \( f(I) = J \) and \( f|_I \) is a translation. Therefore \([f(x_j), f(x_{j+1})]\) is an interval in \( T_0(f(x_n)) \) of the same length as \([x_j, x_{j+1}]\). It follows that there is a length-preserving bijection between the edges of \( T_0(x_n) \) and \( T_0(f(x_n)) \), and

\[
\sum_{e \in T_0(x_n)} |e|^\alpha = \sum_{e \in T_0(f(x_n))} |e|^\alpha.
\]

For \( \epsilon > 0 \), choose \( N \) sufficiently large so that for all \( n > N \)

\[
\left| n^{-\frac{2-\alpha}{d}} E^0_\alpha(x_n) - c \right| < \epsilon/2
\]

with probability greater than \( 1 - \epsilon \), and also \( kn^{-\frac{2-\alpha}{d}} < \epsilon/4 \).
Then, if $n > N$,

$$
\left| n^{\frac{d-\alpha}{d}} E^{0}_\alpha (f(x_n)) - c \right| \leq \left| n^{\frac{d-\alpha}{d}} E^{0}_\alpha (f(x_n)) - n^{\frac{d-\alpha}{d}} E^{0}_\alpha (x_n) \right| + \left| n^{\frac{d-\alpha}{d}} E^{0}_\alpha (x_n) - c \right|
$$

$$
< n^{\frac{d-\alpha}{d}} \left| \sum_{e \in T_0(f(x_n))} |e|^\alpha - \sum_{e \in T_0(x_n)} |e|^\alpha \right|
+ n^{\frac{d-\alpha}{d}} \left| \sum_{e \in T_1(f(x_n))} |e|^\alpha + n^{\frac{d-\alpha}{d}} \sum_{e \in T_1(x_n)} |e|^\alpha + \epsilon \right| / 2
$$

$$
< 0 + 2kn^{\frac{d-\alpha}{d}} 1^\alpha + \epsilon \left/ 2 \right.< \epsilon,
$$

with probability greater than $1 - \epsilon$, where the $1^\alpha$ in the penultimate line comes from the fact that the edge lengths of $T(x_n)$ and $T(f(x_n))$ are at most 1. □

If $I = \{I_j\}_{j=1}^k$ and $\{x_1, \ldots, x_n\} \subset I$ let $\phi_I(x_1, \ldots, x_n)$ record the interval membership of the points $x_1, \ldots, x_n$:

$$
\phi_I(x_1, \ldots, x_n) = (l_1, \ldots, l_n) \text{ if } x_1 \in I_{l_1}, \ldots, x_n \in I_{l_n}.
$$

Also, if $\mu$ is supported on $I$, let $\mu^n_I$ be the discrete random variable $\phi_I(x_1, \ldots, x_n)$.

**Lemma 26.** Let $n \in \mathbb{N}, \epsilon_0 > 0$, and $\alpha > 0$. There exists a $\delta > 0$ so that if $I$ is an interval collection with $\|I\| < \delta$ and $\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\} \subset I$ satisfy

$$
\phi_I(x_1, \ldots, x_n) = \phi_I(y_1, \ldots, y_n)
$$

then

$$
\left| E^{0}_\alpha(x_1, \ldots, x_n) - E^{0}_\alpha(y_1, \ldots, y_n) \right| < \epsilon_0.
$$

**Proof.** The function $x \rightarrow x^\alpha$ is $\alpha$-Holder continuous on $[0, 1]$ so there exists a $C > 0$ so that $|x^\alpha - y^\alpha| < C|x - y|^\alpha$ for all $x, y \in [0, 1]$. Let $\delta = \frac{1}{2} \left( \frac{\epsilon_0}{C \alpha} \right)^{1/\alpha}$.

If $I, \{x_1, \ldots, x_n\}$, and $\{y_1, \ldots, y_n\}$ satisfy the hypotheses then $|x_i - y_i| < \delta$ for $i = 1, \ldots, n$, because $x_i$ and $y_i$ are contained in an interval whose length is less than $\delta$. 
It follows that
\[ |E_0^\alpha (x_1, \ldots, x_n) - E_0^\alpha (y_1, \ldots, y_n)| = \left| \sum_{i=1}^{n-1} (x_{i+1} - x_i)^\alpha - (y_{i+1} - y_i)^\alpha \right| \]
\[ \leq \sum_{i=1}^{n-1} \left| (x_{i+1} - x_i)^\alpha - (y_{i+1} - y_i)^\alpha \right| \]
\[ \leq \sum_{i=1}^{n} C \left| (x_{i+1} - y_{i+1}) + (x_i - y_i) \right|^\alpha \]
\[ < nC2^\alpha \delta^\alpha \]
\[ = \epsilon_0. \]

\[ \square \]

**Lemma 27.** Let \( \mu \) be a probability measure supported on a nested sequence of interval collections \( \{S_1, S_2, \ldots\} \) with \( \|S_j\| \to 0 \). Let \( 0 < \alpha < d \), and assume that
\[ n^{-\frac{d-\alpha}{d}} E_0^\alpha (x_1, \ldots, x_n) \to c \]
in probability as \( n \to \infty \). For all \( \epsilon > 0 \) and all sufficiently large \( n \) there exists an \( M(\epsilon, n) \) so that if \( j > M(\epsilon, n) \) and \( \nu \) is a probability measure supported on \( S_j \) satisfying \( \nu_{S_j} = \mu_{S_j}^n \), then
\[ |c - n^{-\frac{d-\alpha}{d}} E_0^\alpha (y_1, \ldots, y_n)| < \epsilon \]
with probability greater than \( 1 - \epsilon \), where \( \{y_k\}_{k \in \mathbb{N}} \) are i.i.d. points sampled from \( \nu \).

**Proof.** For convenience, define
\[ F(x_1, \ldots, x_n) = |c - n^{-\frac{d-\alpha}{d}} E_0^\alpha (x_1, \ldots, x_n)|. \]
Let \( \epsilon > 0 \) and choose \( n \) sufficiently large so that \( F(x_1, \ldots, x_n) < \epsilon/2 \) with probability greater than \( 1 - \epsilon \). Also, let \( \delta > 0 \) be as given in by the previous lemma for \( \epsilon_0 = n^{-\frac{d-\alpha}{d}} \epsilon/2 \). Choose \( M(\epsilon, n) \) sufficiently large so that \( \|S_j\| < \delta(n) \) for \( j > M(\epsilon, n) \).

Let \( j > M(\epsilon, n) \) and define
\[ V = \phi_{S_j} \left( F^{-1} \left( \epsilon/2 \right) \right). \]
That is, \( (l_1, \ldots, l_n) \in V \) if there exists points \( x_1 \in I_{l_1}, \ldots, x_n \in I_{l_n} \) so that \( F(x_1, \ldots, x_n) < \epsilon/2 \) (where \( S_j = \{I_s\} \)).
The discrete random variables \( \mu^\alpha_S \) and \( \nu^\alpha_S \) coincide by hypothesis, so
\[
\nu \left( \{ y_1, \ldots, y_n : \phi_I(y_1, \ldots, y_n) \in V \} \right) > 1 - \epsilon.
\]
Therefore, with probability greater than \( 1 - \epsilon \), \( \phi_I(y_1, \ldots, y_n) \in M \) and there exists \( x_1, \ldots, x_n \) so that \( F(x_1, \ldots, x_n) < \epsilon/2 \) and \( \phi_I(y_1, \ldots, y_n) = \phi_I(x_1, \ldots, x_n) \).

\( x_1 \ldots, x_n \) and \( y_1 \ldots, y_n \) satisfy the hypotheses of the previous lemma, so
\[
F \left( E^0_\alpha (y_1, \ldots, y_n) \right) \leq \left| n^{-d-\alpha} \left( E^0_\alpha (x_1, \ldots, x_n) - E^0_\alpha (y_1, \ldots, y_n) \right) \right| + F (x_1, \ldots, x_n)
\leq n^{-d-\alpha} \epsilon_0 + \epsilon/2
= \epsilon.
\]

We define two more operations on sets of intervals, related to the middle thirds operation \( K \). One produces slightly thicker intervals than \( K \) does, and the other produces slightly thinner intervals. If \( I \) is an interval, let
\[
L (I) = \left\{ \left( \frac{5}{7} \right) \left( \frac{1}{3} \right) I, \left( \frac{5}{7} \right) \left( \frac{1}{3} \right) |I| + \left( 1 - \left( \frac{5}{7} \right) \left( \frac{1}{3} \right) \right) I \right\}
\]
and
\[
\Gamma (I) = \left\{ \left( \frac{7}{5} \right) \left( \frac{1}{3} \right) I, \left( \frac{7}{5} \right) \left( \frac{1}{3} \right) |I| + \left( 1 - \left( \frac{7}{5} \right) \left( \frac{1}{3} \right) \right) I \right\}.
\]
If \( \mathcal{I} \) is a collection of intervals, define \( L (\mathcal{I}) \) and \( \Gamma (\mathcal{I}) \) by performing the operation on each interval in the collection. See Figure 5.
Proof or Theorem 4. Let \(0 < \alpha < d = \frac{\log(2)}{\log(3)}\), and let \(\mu\) and \(\nu\) be the natural probability measures on \(C\) and \(\frac{2}{3}C\), where \(C\) is the Cantor set. Let \(\{x_j\}_{j \in \mathbb{N}}\) and \(\{y_j\}_{j \in \mathbb{N}}\) be i.i.d. samples from \(\mu\) and \(\nu\), respectively. Assume that there is a real number \(c\) so that

\[ n^{-d-\alpha} E_0^\alpha(x_1, \ldots, x_n) \to c \]

in probability as \(n \to \infty\). If this were false, \(C\) would be our desired example. Theorem 3 implies that \(c > 0\). Also,

\[ n^{-d-\alpha} E_0^\alpha(y_1, \ldots, y_n) \to \left(\frac{5}{7}\right)^\alpha c \]

in probability as \(n \to \infty\).

We will construct a nested sequence of interval collections \(S_1 \supset S_2 \supset S_3 \ldots\) mostly by applying the middle thirds operation \(K\), but infrequently and alternately applying the operations \(L\) and \(\Gamma\).

Let \(\{\epsilon_i\}\) be a sequence of real numbers converging to zero. Let \(n_1\) be large enough so

\[ \left| c - n_1^{-d-\alpha} E_0^\alpha(x_1, \ldots, x_{n_1}) \right| < \epsilon_1/2 \]

with probability greater than \(1 - \epsilon_1\), choose \(m_1 > M(\epsilon_1, n_1)\), where \(M(\epsilon_1, n_1)\) is as given in Lemma 27. Let \(S_1 = [0, 1] = T_1, S_j = T_j = K(S_{j-1})\) for \(j = 2, \ldots, m_1\), and \(S_{m_1+1} = L(S_{m_1})\). \(S_{m_1+1}\) consists of \(2^{m_1}\) disjoint intervals of width \(\frac{5}{7} \left(\frac{1}{3}\right)^{m_1}\), so there is a natural map of interval collections \(f_1: \frac{5}{7} T_{m_1+1} \to S_{m_1+1}\). Let \(\nu_1\) be the pushforward of \(\nu\) by \(\phi_1\), and suppose \(\{z_j\}_{j \in \mathbb{N}}\) are i.i.d. samples from \(\nu_1\). Lemma 27 implies that

\[ \left| c - n_1^{-d-\alpha} E_0^\alpha(z_1, \ldots, z_{n_1}) \right| < \epsilon_1, \]

but by Lemma 25,

\[ E_0^\alpha(z_1, \ldots, z_n) \to \left(\frac{5}{7}\right)^\alpha c, \]

in probability as \(n \to \infty\). Repeat the previous argument to find an \(n_2\) so that

\[ \left| \left(\frac{5}{7}\right)^\alpha c - n_2^{-d-\alpha} E_0^\alpha(z_1, \ldots, z_{n_2}) \right| < \epsilon_2/2 \]

with probability greater than \(1 - \epsilon_2\), and \(m_2 > M(\epsilon_2, n_2)\) as given by Lemma 27. Let \(S_j = f_1(\frac{5}{7} T_j) = K(S_{j-1})\) for \(j = m_1 + 2, \ldots, m_2\), and \(S_{m_2+1} = \Gamma(S_{m_2})\). \(S_{m_2+1}\) consists of \(2^{m_2}\) disjoint intervals of width \(\left(\frac{1}{3}\right)^{m_2}\), so there is a natural map of interval
collections $f_2: T_{m_2+1} \to S_{m_2+1}$. Let $\nu_2$ be the pushforward of $\mu$ by $f_2$, and apply the same arguments as before to find an $n_3$ and $m_3$ so that the weighted sum will be within $\epsilon_3/2$ of $c$ with probability greater than $1 - \epsilon_3$.

Continue this process inductively to find a nested sequence of interval collections $S_1 \supset S_2 \supset S_3 \ldots$. Let $\sigma$ be the natural probability measure on $S = \cap_j S_j$ (the one that assigns equal probability to each interval of $S_j$), and let $\{w_j\}_{j \in \mathbb{N}}$ be i.i.d. samples from $\sigma$. By construction

$$n_{2k}^{d-\alpha} E_0^\alpha (w_1, \ldots, w_{n_{2k}}) \to c$$

but

$$n_{2k+1}^{d-\alpha} E_0^\alpha (w_1, \ldots, w_{n_{2k+1}}) \to \left( \frac{5}{7} \right)^\alpha c$$

in probability as $k \to \infty$.

To complete the proof, we will show that $\sigma$ is $d$-Ahlfors regular. Let $x \in S$. As a first case, let $m \in \mathbb{N}$ and consider the ball of radius $\frac{1}{3^m}$ centered at $x$, an interval of width $\frac{2}{3^m}$. $S_{m+1}$ contains $2^m$ intervals whose widths are either $\frac{1}{3^m}$ or $\frac{5}{7} \frac{1}{3^m}$. $B_{3^{-m}} (x)$ contains at least 1 interval of $S_{m+1}$ (the one that has $x$ as an element) and intersects at most 4 such intervals. Therefore,

$$\sigma (B_{3^{-m}} (x)) \geq \frac{1}{|S_m|} = 2^{-m} = \left( 3^{-m} \right)^d$$

and

$$\sigma (B_{3^{-m}} (x)) \leq \frac{4}{|S_m|} = 4 \left( 2^{-m} \right) = 4 \left( 3^{-m} \right)^d.$$ 

Let $0 < \delta < 1$, $\epsilon_0 = 3^{[\log_3 (\delta)]}$, and $\epsilon_1 = 3^{[\log_3 (\delta)]}$, so

$$\epsilon_0 \leq \delta \leq 3\epsilon_0 \text{ and } \epsilon_1 / 3 < \delta \leq \epsilon_1.$$ 

By our previous computations,

$$\sigma (B_{\delta} (x)) \geq \sigma (B_{\epsilon_0} (x)) \geq \epsilon_0^d \geq 3^{-d} \delta^d$$

and

$$\sigma (B_{\delta} (x)) \leq \sigma (B_{\epsilon_1} (x)) \leq 4 \epsilon_1^d \leq 4 \left( 3^d \delta^d \right).$$

Therefore, $\sigma$ is $d$-Ahlfors regular with $\delta_0 = 1$ and $c = 4 \left( 3^d \right)$. 

$\square$
4. Persistent Homology

We provide a brief introduction to the persistent homology [29] of a filtration, loosely following [59]. For a more in-depth survey refer to, e.g., [14, 15, 30, 31, 33]. A filtration of topological spaces is a family \( \{ X_\epsilon \}_{\epsilon \in \mathcal{E}} \) of topological spaces indexed by an ordered set \( \mathcal{E} \), with inclusion maps \( X_\epsilon_1 \to X_\epsilon_2 \) for all pairs of indices \( \epsilon_1 < \epsilon_2 \). For example, if \( X \) is a subset of a metric space \( M \), the \( \epsilon \)-neighborhood filtration of \( X \), \( X_\epsilon \in \mathbb{R}_+ \), is the family of \( \epsilon \)-neighborhoods of \( X \), together with inclusion maps \( X_\epsilon_1 \hookrightarrow X_\epsilon_2 \) for \( \epsilon_1 < \epsilon_2 \). See Figure 1. If \( X \) is a subset of Euclidean space, this construction is homotopy equivalent to the \( \check{C}ech \) complex of \( X \). The \( \check{C}ech \) complex of \( X \) (defined for any metric space) is the simplicial complex defined by

\[
(x_1, \ldots, x_n) \in C(X, \epsilon) \quad \text{if} \quad \cap_{j=1}^n B_\epsilon (x_j) \neq \emptyset.
\]

Note that the \( \check{C}ech \) complex depends on the ambient metric space. For example if \( p_1, p_2, p_3 \) are the vertices of an acute triangle \( \mathbb{R}^m \) and the ambient space is \( \mathbb{R}^m \) then the 2-simplex \( (p_1, p_2, p_3) \) will enter \( C(X, \epsilon) \) when \( \epsilon \) equals the circumradius of the triangle. If the ambient space \( \{ p_1, p_2, p_3 \} \), the simplex \( (p_1, p_2, p_3) \) will enter the complex when \( \epsilon \) equals the maximum pairwise distance between the three points.

In Euclidean space, the Alpha complex of a finite set \( x \) is filtration on the Delaunay triangulation on \( x \). We do not define the Alpha complex here; see [29] for a definition.

Another common construction is the Vietoris–Rips complex: if \( Y \) is a metric space, let \( V(Y, \epsilon) \) be the simplicial complex defined by

\[
(y_1, \ldots, y_n) \in V(Y, \epsilon) \quad \text{if} \quad d(y_j, y_k) < \epsilon \quad \text{for} \quad j, k = 1, \ldots, n.
\]

The family \( \{ V(Y, \epsilon) \}_{\epsilon > 0} \) together with inclusion maps for \( \epsilon_1 < \epsilon_2 \) is a filtration indexed by the positive real numbers. As noted earlier, all of our results apply to both the \( \check{C}ech \) and Vietoris–Rips complexes except for Corollaries 6 and 7, though the constants may differ. We will suppress the dependence of persistent homology on the underlying filtration, unless otherwise noted.

The **persistent homology module** of a filtration is the product \( \prod_{\epsilon \in \mathcal{E}} H_i(X_\epsilon) \), together with the homomorphisms \( j_{\epsilon_0, \epsilon_1} : H_i(X_{\epsilon_0}) \to H_i(X_{\epsilon_1}) \) for \( \epsilon_0 < \epsilon_1 \), where \( H_i(X_\epsilon) \) denotes the reduced homology of \( X_\epsilon \) with coefficients in a field. If the rank of \( i_{\epsilon_0, \epsilon_1} \) is finite for all \( \epsilon_0 < \epsilon_1 \), — a hypothesis satisfied by all filtrations considered in this paper [17, 15] — the persistent homology module decomposes canonically into a set of interval modules [16, 71]. We denote the collection of these intervals as
$PH_i(X)$; each interval $(b,d) \in PH_i(X)$ corresponds to a homology generator that is “born” at $\epsilon = b$ and “dies” at $\epsilon = d$.

If $x$ is a finite metric space and persistent homology is taken with respect to the Vietoris–Rips complex, Kruskal’s algorithm implies that there is a length-preserving bijection between intervals of $PH_0(x)$ and the edges of the minimum spanning tree on $x$. The same is true if persistent homology is taken with respect to the Čech complex if the ambient space is $\mathbb{R}^m$, except that an interval is matched with an edge of twice its length.

4.1. **Properties of Persistent Homology.** Let $X$ be a metric space. For each $\epsilon > 0$, let $PH^\epsilon_i(X)$ denote the set of intervals of $PH_i(X)$ of length greater than $\epsilon$:

$$PH^\epsilon_i(X) = \{ I \in PH_i(X) : |I| > \epsilon \}.$$

Also, define

$$p_i(X, \epsilon) = |PH^\epsilon_i(X)|.$$

If $X, Y \subset X$, let $d_H(X, Y)$ denote the Hausdorff distance between $X$ and $Y$:

$$d_H(X, Y) = \inf \{ \epsilon \geq 0 : Y \subseteq X_\epsilon \text{ and } X \subset Y_\epsilon \}.$$

Also, let $d(X, Y)$ be the infimal distance between pairs of points, one in each set:

$$d(X, Y) = \inf_{x \in X, y \in Y} d(x, y).$$

We use the following properties of persistent homology in our proofs:

1. **Stability:** If $d_H(X, Y) < \epsilon$, there is an injection

$$\eta : PH^{2\epsilon}_i(X) \rightarrow PH_i(Y)$$

so that if $\eta((b_0, d_0)) = (b_1, d_1)$ then

$$\max(|b_0 - b_1|, |d_0 - d_1|) < \epsilon$$

In particular,

$$p_i(X, 2\epsilon + \delta) \leq p_i(Y, \delta)$$

for all $\delta \geq 0$. [17, 18]

2. **Additivity for well-separated sets:** If $X_1, \ldots, X_n \subset M$ and

$$d(X_j, X_k) > \max(\text{diam } X_j, \text{diam } X_k) (1 - \delta_{j,k}) \quad \forall j, k$$

then

$$p_i(\cup_j X_j, \epsilon) \geq \sum_j p_i(X_j, \epsilon).$$
(3) Translation invariance: \( PH_i(X) = PH_i(X + t) \) for all \( t \in \mathbb{R}^n \).

(4) Scaling: For all \( \rho > 0 \),
\[
PH_i(\rho X) = \{ (\rho b, \rho d) : (b, d) \in PH_i(X) \} .
\]

We use property (1) in our proofs of both the upper and lower bounds of Theorems 5 and 8, and property (2) for our proof of the lower bound. For these results, we also require a non-triviality property (as in Definition 38) and an upper bound for the number of \( i \)-dimensional persistent homology intervals of a set of \( n \) points.

4.2. A Lemma. If \( X \) is a metric space, let \( F^i_\alpha(X, \epsilon) \) denote the \( \alpha \)-weighted sum of the persistent homology intervals of \( X \) of length greater than \( \epsilon \):
\[
F^i_\alpha(X, \epsilon) = \sum_{I \in PH^i_\epsilon(X)} |I|^\alpha .
\]

We will use the following lemma in the next section.

Lemma 28. If \( d_H(X, Y) < \epsilon/4 \) then
\[
F^i_\alpha(X, \epsilon) < 2^\alpha F^i_\alpha(Y, \epsilon/2) .
\]

Proof. By stability, there is an injection
\[
\eta : PH^\epsilon_i(X) \rightarrow PH^{\epsilon/2}_i(Y)
\]
so that
\[
|I| < |\eta(I)| + \epsilon/2 \leq 2|\eta(I)|
\]
for all \( I \in PH^\epsilon_i(X) \).

It follows that
\[
F^i_\alpha(X, \epsilon) = \sum_{I \in PH^\epsilon_i(X)} |I|^\alpha
< \sum_{I \in PH^\epsilon_i(X)} 2^\alpha |\eta(I)|^\alpha
\leq 2^\alpha \sum_{J \in PH^{\epsilon/2}_i(Y)} |J|^\alpha
= 2^\alpha F^i_\alpha(Y, \epsilon/2) .
\]

\[\Box\]
5. Upper Bounds

In this section, we prove the upper bounds in Theorems 5 and 8. Our strategy is similar to that in Section 3.1: we control the number of persistence intervals of length greater than $\epsilon$ by approximating $x_n$ by a set consisting of the centers of disjoint balls of radius $\epsilon/2$ centered at points of $x_n$.

5.1. Extremal Hypotheses. First, we prove the upper bound in Theorem 5, which implies the upper bound in our result for measures supported on a subset of $\mathbb{R}^2$.

**Lemma 29** (Interval Counting Lemma). If $X$ is a bounded metric space so that

$$|\text{PH}_i (x_1, \ldots, x_n)| < Dn^a.$$  

for some positive real numbers $a$ and $D$ and all finite subsets $\{x_1, \ldots, x_n\}$ of $X$, then

$$p_i (Y, \epsilon) < D' \epsilon^{-ad}$$  

for some $D' > 0$, all $Y \subseteq X$, and all $\epsilon > 0$.

**Proof.** Let $Y \subseteq X$, $\epsilon > 0$, and $\{y_j\}$ be the centers of a maximal set of disjoint balls of radius $\epsilon/4$ centered at points of $Y$. The balls of radius $\epsilon/2$ centered at the points $\{y_j\}$ cover $Y$ so

$$d_H (\{y_i\}, Y) < \epsilon/2$$

It follows that

$$p_i (Y, \epsilon) \leq p_i (\{y_i\}, 0)$$  

by stability

$$\leq D |y_i|^a$$  

by hypothesis

$$\leq DM_{\epsilon/4} (X)^a$$

$$\leq Dc^{a-4} \epsilon^{-ad}$$  

by Lemma 40

as desired. □

**Proposition 30.** If $X$ satisfies the hypotheses of the previous lemma and $\alpha < ad$, then there exists a $D_\alpha > 0$ so that

$$E_{\alpha}^i (x_1, \ldots, x_n) \leq D_\alpha n^{\frac{ad-\alpha}{d}}$$

for all finite subsets $\{x_1, \ldots, x_n\} \subseteq X$ and all $n \in \mathbb{N}$. Furthermore there exists a $D_d > 0$ so that

$$E_{ad}^i (x_1, \ldots, x_n) \leq D_d \log (n)$$

for all finite subsets $\{x_1, \ldots, x_n\} \subseteq X$ and all $n \in \mathbb{N}$.

**Proof.** The proof is nearly identical to that of Proposition 21, and we omit it here. □
5.2. Probabilistic Hypotheses. While the extremal hypotheses of the previous section allow us to prove the desired upper bound in Corollary 6, they are inadequate to show a similar upper bound for subsets of higher dimensional Euclidean space. Here, we show that hypotheses on the the expectation and variance of the number of $\text{PH}_i$ intervals of a set of $n$ points imply better asymptotic upper bounds (the upper bound in Theorem 8). The idea of the proof is to control the behavior of $\text{PH}_i(X)$ in terms of the persistent homology of point samples from $X$. With that, we write $\text{PH}_i(x_n)$ a sum of two terms, one which approximates $\text{PH}_i(X)$ and one which corresponds to “$d$-dimensional noise” at a certain scale.

First, we require the following lemma, which follows from a standard argument using the union bound; see [54] for a proof.

**Lemma 31.** Let $\mu$ be a probability measure on $X$, and $\{B_j\}_{j=1}^l \subset X$ be a collection of balls so that so that $\mu(B_j) \geq a$ for all $j$. Then

$$\mathbb{P}(x_n \cap B_j \neq \emptyset \text{ for } j = 1, \ldots, l) \geq 1 - le^{-an}.$$ 

Next, we apply the previous lemma to control the Hausdorff distance between $X$ and finite samples from an Ahlfors regular measure on $X$.

**Lemma 32.** If $\mu$ is a $d$-Ahlfors regular measure with support $X$ then there exists a positive real number $A_0$ that depends only on the constants $c$ and $d$ appearing in the definition of Ahlfors regularity so that

$$\mathbb{P}
d_H \left( \{x_n\}, X \right) < \epsilon
\geq 1 - ce^{-d}e^{-A_0\epsilon n}$$

for all $\epsilon > 0$.

**Proof.** Let $\left\{y_1, \ldots, y_{M_{\epsilon/3}(X)}\right\}$ be the centers of a maximal set of disjoint balls of radius $\epsilon/3$ centered at points of $X$. By the definition of Ahlfors regularity,

$$\mu \left( B_{\epsilon/3}(y_j) \right) \geq A_0\epsilon^d$$

for all $j$, where $A_0 = 3^{-d}/c$.

The balls of radius $2\epsilon/3$ centered at the points $\{y_j\}$ cover $X$ so

$$d_H \left( \{y_1\}, X \right) < 2\epsilon/3.$$ 

Therefore, if $x_n \cap B_{\epsilon/3}(y_j) \neq \emptyset$ for $j = 1, \ldots, M_{\epsilon/3}(X)$

$$d_H \left( \{x_n\}, X \right) < \epsilon/3 + 2\epsilon/3 = \epsilon.$$
It follows that

\[
P\left(d_H \left( \{x_n\} , X \right) < \epsilon \right) \geq \frac{1}{2} \]

\[
P\left( \{x_n\} \cap B_{\epsilon/3} (y_j) \neq \emptyset \text{ for } j = 1, \ldots, M_{\epsilon/3} (X) \right)
\]

\[
\geq 1 - M_{\epsilon/3} (X) e^{-A_0 \epsilon d n}
\]

\[
\geq 1 - c \epsilon^{-d} e^{-A_0 \epsilon d n}
\]

by Lemma 31.

Similarly,

\[
P\left( d_H \left( \{x_n\} \cap B_{\epsilon/3} (y_j) \neq \emptyset \text{ for } j = 1, \ldots, M_{\epsilon/3} (X) \right) \right)
\]

\[
\geq 1 - c \epsilon^{-d} e^{-A_0 \epsilon d n}
\]

by Lemma 14.

\[
\square
\]

In the next lemma, we show that if the expected number of persistent homology intervals of \(x_n, X\) is \(O(n)\), then we can control the number of “long” persistent homology intervals of \(X\) itself.

**Lemma 33.** Let \(X\) be a bounded metric space that admits a \(d\)-Ahlfors regular measure \(\mu\) satisfying

\[
E \left( \left| PH_i (x_n) \right| \right) = O(n).
\]

Then there are positive real numbers \(A_1\) and \(\epsilon_0\) so that

\[
p_i (X, \epsilon) \leq A_1 \epsilon^{-d} \log \left( \frac{1}{\epsilon} \right)
\]

for all \(\epsilon < \epsilon_0\).

**Proof.** There are positive real numbers \(D_1\) and \(N_1\) so that

\[
E \left( \left| PH_i (x_n) \right| \right) \leq nD_1/2
\]

for all \(n > N_1\). By Markov’s inequality,

\[
P \left( \left| PH_i (x_n) \right| > nD_1 \right) < 1/2.
\]

Manipulating the inequality in the previous lemma gives that

\[
P \left( d_H \left( \left\{ x_1, \ldots, x_{m(\epsilon)} \right\} , X \right) < \epsilon/2 \right) \geq 1/2
\]

where

\[
m(\epsilon) = \left\lceil \frac{2d}{A_0} \epsilon^{-d} \log \left( 2^{d+1} c \epsilon^{-d} \right) \right\rceil.
\]

Note that \(m(\epsilon)\) is chosen to give distances of less than \(\epsilon/2\), rather than less than \(\epsilon\). Let \(\epsilon\) be sufficiently small so that \(m(\epsilon) > N_1\). We have that
\[ |PH_i(x_1, \ldots, x_{m(\epsilon)})| \leq D_1 m(\epsilon) \quad \text{and} \quad d_H\left(\{x_1, \ldots, x_{m(\epsilon)}\}, X\right) < \epsilon \]

for some finite subset \(x_1, \ldots, x_{m(\epsilon)}\) of \(X\). Therefore, by stability

\[
p_i(X, \epsilon) \leq p_i\left(\{x_1, \ldots, x_{m(\epsilon)}\}, 0\right) \\
\leq D_1 m(\epsilon) \\
= D_1 \left[ \frac{2^d}{A_0} \epsilon^{-d} \log \left( \frac{2^{d+1}}{A_0 \epsilon^{-d}} \right) \right] \\
= O\left( \epsilon^{-d} \log \left( \frac{1}{\epsilon} \right) \right)
\]

as \(\epsilon \to 0\). \qed

Next, we use the previous lemma to control \(F^i_\alpha(X, \epsilon)\), the truncated \(\alpha\)-weighted sum defined in Section 4.2.

**Proposition 34.** If \(X\) satisfies the hypotheses of the previous lemma and \(0 < \alpha < d\), then there exist positive real numbers \(A_2\) and \(\epsilon_1\) so that

\[ F^i_\alpha(X, \epsilon) \leq A_2\epsilon^{-d} \log \left( \frac{1}{\epsilon} \right) \]

for all \(\epsilon < \epsilon_1\).

**Proof.** Without loss of generality, we may rescale \(X\) so its diameter is less than one. By the previous lemma

\[ p_i(X, \epsilon) \leq f(\epsilon) := A_1 (\epsilon)^{-d} \log \left( \frac{1}{\epsilon} \right) \]

for all \(\epsilon < \epsilon_0\). Applying Lemma 15 yields

\[ F^i_\alpha(Y, \epsilon) \leq \epsilon^\alpha f(\epsilon) + \alpha \int_{t=\epsilon}^{\epsilon_0} f(t) t^{\alpha-1} dt + F^i_\alpha(Y, \epsilon_0) . \]

The first term equals

\[ A_1 \epsilon^{\alpha-d} \left( \log \left( \frac{1}{\epsilon} \right) \right) , \]
which has the desired asymptotics as $\epsilon \to 0$. The second term equals
\[
\alpha \int_{t=\epsilon}^{\epsilon_0} A_1 t^{\alpha-d-1} \log \left( \frac{1}{t} \right) \, dt = \\
A_1 \left[ -\frac{1}{d-\alpha} t^{\alpha-d} \log \left( \frac{1}{t} \right) - \frac{1}{(d-\alpha)^2} t^{\alpha-d} \log \left( \frac{1}{t} \right) \right]_{\epsilon}^{\epsilon_0} \\
= A_1 \left( \frac{1}{d-\alpha} e^{\alpha-d} \log \left( \frac{1}{\epsilon} \right) + \frac{1}{(d-\alpha)^2} e^{\alpha-d} \right) \\
- A_1 \left( \frac{1}{d-\alpha} e_0^{\alpha-d} \log \left( \frac{1}{\epsilon_0} \right) + \frac{1}{(d-\alpha)^2} e_0^{\alpha-d} \right) \\
= O \left( e^{\alpha-d} \log \left( \frac{1}{\epsilon} \right) \right).
\]
Therefore, $p_i(X, \epsilon) = O \left( e^{\alpha-d} \log \left( \frac{1}{\epsilon} \right) \right)$, as desired.

Finally, we can bootstrap the previous result to control $E^i_\alpha (x_n)$ and prove the upper bound in Theorem 8.

**Proposition 35.** Let $\mu$ be a $d$-Ahlfors regular measure on a bounded metric space. If
\[
E \left( \left| PH_i (x_n) \right| \right) = O \left( n \right)
\]
and
\[
Var \left( \left| PH_i (x_n) \right| \right) / n^2 \to 0,
\]
then there is a $\Lambda > 0$ so that
\[
E^i_\alpha (x_n) \leq \Lambda n^{\frac{d-\alpha}{d}} \log \left( n \right)^{\frac{\alpha}{d}}
\]
with high probability as $n \to \infty$.

**Proof.** Let
\[
G^i_\alpha (x, \epsilon) = \sum_{I \in PH_i (x) \setminus PH^i_\epsilon (x)} |I|^\alpha.
\]
Our strategy is to write
\[
E^i_\alpha (x_n) = G^i_\alpha (x_n, \epsilon) + F^i_\alpha (x_n, \epsilon)
\]
for a well-chosen $\epsilon$. The former term can be interpreted as “noise,” and the latter approximates the persistent homology of the support of $\mu$. 

Let $0 < p < 1$, and let $D$ be a positive real number so that
\[
\mathbb{E} \left( \left| PH_i (x_n) \right| \right) \leq (D/2) n
\]
for all sufficiently large $n$. By Chebyshev’s inequality,
\[
\mathbb{P} \left( \left| PH_i (x_n) \right| > Dn \right) \leq
\mathbb{P} \left( \left| PH_i (x_n) \right| - \mathbb{E} \left( \left| PH_i (x_n) \right| \right) > Dn/2 \right)
\leq \frac{\text{Var} \left( \left| PH_i (x_n) \right| \right)}{D^2 n^2}
\]
which converges to 0 as $n \to \infty$, by hypothesis. Therefore, there is a $M$ so that
\[
\mathbb{P} \left( \left| PH_i (x_n) \right| > Dn \right) < p/2
\]
for all $n > M$.

Solving for $\epsilon$ in the expression in Lemma 32 gives that
\[
\mathbb{P} \left( d_H (\{x_n\}, X) > \epsilon (n) / 4 \right) < p/2
\]
if
\[
\epsilon (n) = 4A_0^{-1/d} n^{-1/d} W \left( \frac{2cA_0 n}{p} \right)^{1/d},
\]
where $W$ is the Lambert $W$ function. $W (m) \sim \log (m)$ as $m \to \infty$, and $W (m) \leq \log (m)$ for $m \geq e [40]$. Therefore, there is an $A_3 > 0$ that does not depend on $p$ and an $N_1 (p)$ that does depend on $p$ so that
\[
(2) \quad \frac{A_3}{2} n^{-1/d} \log (n)^{1/d} \leq \epsilon (n) \leq A_3 n^{-1/d} \log (n)^{1/d}
\]
for all $n > N_1 (p)$.

Choose $N_2 (p) > N_1 (p)$ to be sufficiently large so that $\epsilon (n) < \epsilon_1$ for all $n > N_2 (p)$, where $\epsilon_1$ is given in Proposition 34. Let $n > N_2 (p)$ and suppose that $x_n$ satisfies $|PH_i (x_n)| < Dn$ and $d_H (x_n, X) < \epsilon (n) / 4$ — an event which occurs with probability greater than $1 - p$. Write
\[
E_{\alpha}^i (x_n) = F_{\alpha}^i (x_n, \epsilon (n)) + G_{\alpha}^i (x_n, \epsilon (n)) .
\]

We consider the two terms separately.
\[ G_{\alpha}^{i}(x_n, \epsilon(n)) \leq D |x_n| \epsilon(n)^\alpha \]
\[ \leq 2^\alpha DA_3^\alpha n^{\frac{d-\alpha}{\alpha}} \log(n)^{\alpha/d} \]
\[ = A_4 n^{\frac{d-\alpha}{\alpha}} \log(n)^{\frac{\alpha}{d}}, \]
where \( A_4 = 2^\alpha DA_3^\alpha \) is a positive constant that does not depend on \( n \) or \( p \).

To bound the second term, we apply Lemma 28 to find
\[ F_{\alpha}^{i}(X, \epsilon(n)/2) \leq A_2 \left( \frac{\epsilon(n)}{\epsilon(n)} \right)^{\alpha-d} \log \left( \frac{1}{\epsilon(n)} \right) \]
by Prop. 34
\[ \leq A_2 A_3^{\alpha-d} n^{\frac{d-\alpha}{\alpha}} \log(n)^{\frac{d-\alpha}{\alpha}} \left( \frac{1}{2} \log(n) - \log \left( 2A_3 \log(n)^{1/d} \right) \right) \]
by Eqn. 2
\[ = A_2 A_3^{\alpha-d} n^{\frac{d-\alpha}{\alpha}} \log(n)^{\frac{\alpha}{d}} \]
\[ = A_5 n^{\frac{d-\alpha}{\alpha}} \log(n)^{\frac{\alpha}{d}}, \]
where \( A_5 = \frac{1}{d} A_2 A_3^{\alpha-d} \) is a positive constant that does not depend on \( n \) or \( p \).

In summary, if \( \Lambda = A_4 + A_5 \) and \( 0 < p < 1 \), then there exists an \( N_2(p) > 0 \) so that
\[ \mathbb{P} \left( E_{\alpha}^{i}(x_n) \leq \Lambda n^{\frac{d-\alpha}{\alpha}} \log(n)^{\frac{\alpha}{d}} \right) > 1 - p \]
for all \( n > N_2(p) \).

\[ \square \]

6. The Lower Bound

In this section, we prove the lower bounds in Theorems 5 and 8. While our proofs of the upper bounds work for Ahlfors regular measures on arbitrary bounded metric spaces, here we restrict our attention Ahlfors regular measures on Euclidean space. This will allow us to use the structure of the cubical grid on \( \mathbb{R}^m \). First, we consider the special case of an \( m \)-Ahlfors regular measure on \( \mathbb{R}^m \). The argument will be more straightforward than in the general case, but will contain some of the same elements. These arguments also appear in our unpublished manuscript [60], which has largely been subsumed into the current work.
6.1. The Absolutely Continuous Case. Note that if $\mu$ is an $m$-Ahlfors regular measure on $\mathbb{R}^m$, then $\mu$ is comparable to the Lebesgue measure on its support and is thus absolutely continuous with respect to it.

**Proposition 36.** Let $\mu$ be an $m$-Ahlfors regular measure on $\mathbb{R}^m$. There exist a constant $\Psi > 0$ so that if
\[
\lim_{n \to \infty} n^{-\frac{d_{\alpha}}{\alpha}} E_{\alpha}^{(i)} (x_1, \ldots, x_n) \geq \Psi
\]
with high probability.

Before proving the previous proposition, we state and prove a preliminary lemma. Let
\[
J_{d,i} (X) = \{(b, d) \in PH_i (X) : d \leq \varepsilon\}.
\]

**Lemma 37.** Let $0 \leq i < m$, and $0 < b < d < 1/8$. There exists a $\lambda_0 > 0$ so that if $C \subset \mathbb{R}^m$ is an $m$-dimensional cube of width $R$ and $\lambda > \lambda_0$, there exists a collection $\mathcal{B}$ of disjoint cubes disjoint, congruent cubes of width $R \lambda^{-\frac{1}{m}}$ so that if $A = C \setminus \bigcup_{B \in \mathcal{B}} B$ and $\Xi (x, A, \mathcal{B}) = 1$ then $PH_i (x \cap C)$ contains an interval $(\hat{b}, \hat{d})$ with
\[
0 < \hat{b} < Rb < Rd < \hat{d}.
\]

Furthermore,
\[
J_{d,i} (x) = J_{d,i} (x \cap C) \cup J_{d,i} (x \setminus (x \cap C)).
\]

**Proof.** We may assume without loss of generality that $R = 1$ and $C$ is centered at the origin. Let $S^i \subset \mathbb{R}^m$ be an $i$–dimensional sphere of diameter $1/4$ centered at the origin; note that $PH_i (S^i)$ consists of a single interval $(0, 1/8)$ for the Čech complex (a slightly different argument is required for the Rips complex).
Let
\[
\kappa = \min \left( b, \frac{1}{8} - d, \frac{1}{24} \right)
\]
and
\[
\lambda_0 = \frac{m^{m/2}}{\kappa^m}.
\]
Choose \( \lambda > \lambda_0 \) and set \( \delta = \lambda^{-\frac{1}{m}} \). Let \( \mathcal{C} \) be the cubes in the standard tessellation of \( \mathbb{R}^m \) by cubes of width \( \delta \) and let
\[
\mathcal{B} = \left\{ c \in \mathcal{C} : S^i \cap c \neq \emptyset \right\}.
\]
See Figure 6.

If \( \Xi (x, A, \mathcal{B}) = 1 \), then
\[
d_H \left( x \cap C \cup S^i \right) < \kappa
\]
where we used the fact that the length of the diagonal of an \( m \)-dimensional cube of width \( \delta \) is \( \delta \sqrt{m} \). Stability and Equation 4 imply that \( PH_i (x \cap C) \) includes an interval \( (\hat{b}, \hat{d}) \) so that
\[
\hat{b} < \kappa \leq b < d < \frac{1}{8} - \kappa < \hat{d} < \frac{1}{8} + \kappa \leq 1 / 6.
\]
By construction,
\[
\frac{1}{2} d(x \cap C, C^c) > \frac{1}{2} \left( d \left( S^i, C^c \right) - \kappa \right) = \frac{3}{16} - \kappa/2 \geq 1 / 6 > \hat{d},
\]
where \( C^c \) is the complement of \( C \). Therefore, the the \( \epsilon \)-neighborhoods of \( x \cap C' \) and \( C^c \) are disjoint for all \( \epsilon \leq \hat{d} \) and Equation 3 holds.

Proof of Proposition 36. \( \mu \) is absolutely continuous so its support contains a cube \( C \). Without loss of generality, we may assume that \( C \) is a unit cube. Let \( b_0 = 1/16, d_0 = 1/8 \). Let \( \lambda > \lambda_0 \), where \( \lambda_0 \) is as in the previous lemma. Let \( \delta = n^{-1/m} \) and let \( \{ D_1, \ldots, D_s \} \) be the cubes in the cubical tessellation of width \( \delta \) which are fully contained within \( C \). Assume \( \delta \) is sufficiently small so that \( k > \delta^m/2 \). Let \( l \in \{1, \ldots, s\} \), and let \( A_l \) and \( B_l \) be the set and collection of disjoint cubes of width \( \delta \lambda^{-\frac{1}{m}} \) contained in \( C_l \) given by the previous lemma. It follows from the statement of that lemma that
\[
p_i \left( x_n, \frac{1}{16} n^{-1/m} \right) \geq \frac{1}{n} \sum_{j=1}^{s} \Xi (x_n, A_l, B_l).
\]
Let $c$ be the constant appearing in the definition of Ahlfors regularity, $v_0$ be the volume of a unit ball in $\mathbb{R}^m$, and

$$ q = \frac{cm^{m/2}v_0}{2^m} \quad \text{and} \quad p = \frac{v_0}{\lambda^{m/2}}. $$

If $E_0$ is a ball of radius $\delta \sqrt{m}/2$ containing $C$,

$$ \mu(A_l) \leq \mu(E_0) \leq cv_0 \left( \delta \sqrt{m}/2 \right)^m = q\delta^m = \frac{q}{n}. $$

Similarly, if $B$ is a cube of $\mathcal{B}_l$, and $E_1$ is a ball of radius $\delta \lambda^{-\frac{1}{m}}/2$ contained in $B$

$$ \mu(B) \geq \mu(E_1) \geq \frac{1}{c} \left( \delta \lambda^{-\frac{1}{m}} \right)^m v_0 = \frac{p}{n}. $$

Let $r = |\mathcal{B}_l|$ and note that $r$ depends only on $\lambda_0, b_0$, and $d_0$. $\Xi(x_n, A_l, \mathcal{B}_l)$ is a $n, p, q, r$-bounded occupancy event for each $l$, so by Lemma 18 there exists a $\gamma_0 > 0$ so that

$$ p_i \left( x_n, \frac{1}{16} n^{-1/m} \right) \geq \sum_{j=1}^{\delta} \Xi(x_n, A_l, \mathcal{B}_l) > \gamma_0 $$

with high probability as $n \to \infty$.

Then

$$ \lim_{n \to \infty} n^{-d-\alpha} E^s_\alpha(x_n) \geq \lim_{n \to \infty} n^{-d-\alpha} p_i \left( x_n, \frac{1}{16} n^{-1/d} \right) 16^{-\alpha} \left( n^{-1/d} \right)^\alpha $$

$$ = 16^{-\alpha} \lim_{n \to \infty} \frac{1}{n} p_i \left( x_n, \frac{1}{16} n^{-1/d} \right) $$

$$ \geq 16^{-\alpha} \Psi $$

$$ := \Omega_0 $$

with high probability. \qed

Note that it is straightforward to modify the previous argument to work for any metric space $X$ with a subset $Y$ that is the bi-Lipschitz image of a cube in Euclidean space. In particular, if the bi-Lipschitz constant is $L$, it would suffice to take $b_0$ to be $\frac{1}{16L^2}$ and $d_0$ to be $\frac{1}{8}$ and argue that an interval $(b, d) \in PH_1(C)$ with $b < \frac{\delta}{16L^2} < \frac{\delta}{8}$ corresponds to an interval $(b_1, d_1) \in PH_1(X)$ with $b_1 < \frac{\delta}{16L} < \frac{\delta}{8L} < d_1$. 
Figure 7. The Čech \( PH_1 \) class of the lattice points corresponding to the gray cubes in (a) and (b) is stable — any choice of one point in each cube will give the vertices of an acute triangle, and therefore a set with non-trivial \( PH_1 \). The one in (c) and (d) is not, because the points in (d) form an obtuse triangle so the persistent homology of that set is trivial. [59]

6.2. Non-triviality Constants. To prove the lower bound, we modify the approach in our paper on extremal \( PH \)-dimension [59] to work in a probabilistic context. If \( \mu \) is a \( d \)-Ahlfors regular measure on \( \mathbb{R}^m \) and \( \delta > 0 \), let \( C_\delta (\mu) \) be the cubes in the grid of mesh \( \delta \) that intersect the support of \( \mu \). The basic idea is to sub-divide the grid of mesh \( \delta \) so each cube contains \( k^m \) sub-cubes. If \( k \) is chosen carefully, we can find a positive fraction of cubes in \( C_\delta (\mu) \) that contain enough cubes of \( C_{\delta/k} (\mu) \) to guarantee a stable \( PH_i \) class. In fact, we can require that the sub-cubes have probability exceeding a certain threshold. We then control the number of stable \( PH_i \) classes realized by a random sample \( x_n \) with Lemma 18.

In previous work [59], we raised the question of how large a subset of the integer lattice can be without having a subset with “stable” \( i \)-dimensional persistent homology.

Definition 38. For \( x \in \mathbb{Z}^m \), let the cube corresponding to \( x - C(x) \) be the cube of width 1 centered at \( x \). A subset \( X \) of \( \mathbb{Z}^m \) has a stable \( i \)-dimensional persistent homology class if there is a \( c > 0 \) so that if \( Y \) is any subset of \( \bigcup_{x \in X} C(x) \) satisfying

\[
Y \cap C(x) \neq \emptyset \quad \forall \ x \in X,
\]

then there is an \( I \in PH_i (Y) \) so that \( |I| > c \) (see Figure 7). The supremal such \( c \) is called the size of the stable persistence class.

Note that this notion depends on whether persistent homology is taken with respect to the Rips complex or the Čech complex, but is defined for both.
Definition 39. Let $\xi_m^m(N)$ be the size of the largest subset $X$ of $\{1, \ldots, N\}^m \subset \mathbb{Z}^m$ so that no subset $Y$ of $X$ has a stable $PH_i$-class. Define

$$\gamma_i^m = \liminf_{N \to \infty} \frac{\log(\xi_i^m(N))}{\log(N)}.$$ 

Note that $\gamma_i^m$ may depend on whether persistent homology is taken with respect to the Rips complex or the Čech complex. $\gamma_i^m = 0$ for all $m \in \mathbb{N}$: any subset of $\mathbb{Z}^m$ with more than $3^m$ points has a stable $PH_0$ class (at least $3^m$ points are necessary to rule out point sets with points from neighboring cubes). In [59], we proved that $\gamma_1^m \leq m - \frac{1}{2}$ if persistent homology is taken with respect to the Čech complex. Note that the previous definition does not include the same restriction on the size as in our previous paper.

6.3. Ahlfors Regular Measures and Box Counting. Before proceeding to the proof of the lower bound, we prove a lemma about the asymptotics of the number of cubes that intersect the support of a $d$-Ahlfors regular measure. Let $\mathcal{C}_{\delta,a}(\mu)$ be the set of closed cubes $C$ in the cubic grid of mesh $\delta$ in $\mathbb{R}^m$ centered at the origin that satisfy

$$\mu(C) \geq a\delta^d,$$

and let $N_{\delta,a}(\mu) = |\mathcal{C}_{\delta,a}(\mu)|$. (The upper and lower box dimensions of a subset of Euclidean space can be defined in terms of the asymptotic properties of $N_{\delta,0}(X)$).

Lemma 40. If $\mu$ is a $d$-Ahlfors regular measure with support $X \subset \mathbb{R}^m$, then there exist real numbers $0 < c_0 \leq c_1 < \infty$ depending on $m$ and the constants $c$ and $d$ appearing in the definitions of Ahlfors regularity so that

$$c_0\delta^{-d} \leq N_{\delta,\hat{c}}(\mu) \leq c_1\delta^{-d}$$

for all $\delta < \delta_0$, where $\hat{c} = \frac{c}{c^{2m}}$. Similarly, there exist real numbers $0 < c_0' \leq c_1' < \infty$ depending on $c, d, \mu$, and $m$ so that

$$c_0'\delta^{-d} \leq N_{\delta,0}(\mu) \leq c_1'\delta^{-d}$$

for all $\delta < \delta_0$.

Proof. Let $C$ be a cube in the grid of mesh $\delta$ that intersects $X$, and $x \in C \cap X$. $\mu(B_\delta(x)) > 1/c\delta^d$ and $B_\delta(x)$ intersects at most $2^m$ cubes in the grid of mesh $\delta$, so at least one cube adjacent to $C$ has measure exceeding $\hat{c}\delta^d$ (where two cubes are adjacent if they share at least one point). Also, each cube of $\mathcal{C}_{\delta,\hat{c}}(\mu)$ is adjacent to at most $3^m$ cubes of $\mathcal{C}_\delta(\mu)$. It follows that

$$\frac{1}{3^m}N_{\delta,0}(\mu) \leq N_{\delta,\hat{c}}(\mu) \leq N_{\delta,0}(\mu)$$
where the upper bound is trivial. Thus, bounds for $N_{\delta,0}(\mu)$ imply bounds for $N_{\delta,\hat{c}}(\mu)$, and visa versa.

We have that

$$1 = \mu(X) \leq \sum_{C \in C_{\delta,0}(\mu)} \mu(C) \leq c \delta^d m^{d/2} N_{\delta,0}(\mu) \leq 3^m c \delta^d m^{d/2} N_{\delta,\hat{c}}(\mu) \implies N_{\delta,\hat{c}}(\mu) \geq 3^{-m} m^{-d/2} \delta^{-d}.$$  

For the upper bound, note that the intersection of two cubes may have positive measure, but a cube can share measure with only $3^m - 1$ adjacent cubes. It follows that

$$1 = \mu(X) \geq \frac{1}{3^m} \hat{c} \delta^d N_{\delta,\hat{c}}(\mu) \implies N_{\delta,\hat{c}}(\mu) \leq c 6^m \delta^{-d}.$$  

\[\square\]

For each $k \in \mathbb{N}$, $\delta > 0$, and $C \in C_{\delta}(\mu)$, let $D_k(C)$ be the set of cubes in $C_{\delta,k,\hat{c}}(\mu)$ that are contained in $C$, and let $D_k(C) = |D_k(C)|$. See Figure 8.

**Lemma 41.** Let $0 < \beta < d$ and let

$$C^{k,\beta}_{\delta} = \left\{ C \in C_{\delta}(\mu) : D_k(C) > k^\beta \right\}$$
and

\[ M(\delta, k, \beta) = |C_{\delta}^{k, \beta}|. \]

Then there exists a \( K > 0 \) so that for all \( k > K \) there exist \( \delta_1, c_2 > 0 \) so that

\[ M(\delta, k, \beta) > c_2 \delta^{-d} \]

for all \( \delta < \delta_1 \).

**Proof.** Let \( c_0, c'_1, \) and \( \delta_0 \) be the constants from the previous lemma so \( N_{\delta, 0}(\mu) \leq c'_1 \delta^{-d} \) and \( N_{\delta, \hat{c}}(\mu) \geq c_0 \delta^{-d} \) for all \( \delta < \delta_1 \).

There are at least \( c_0 k^d \delta^{-d} \) cubes in \( C_{\delta/k, \hat{c}}(\mu) \), each of which is either a sub-cube of \( C_{\delta}^{k, \beta} \) or \( C_{\delta, 0}(\mu) \setminus C_{\delta}^{k, \beta} \). A cube in \( C_{\delta}^{k, \beta} \) can contain at most \( k^m \) sub-cubes of \( C_{\delta/k, \hat{c}}(\mu) \), and a cube in \( C_{\delta, 0}(\mu) \setminus C_{\delta}^{k, \beta} \) can contain at most \( k^\beta \) sub-cubes of \( C_{\delta/k, \hat{c}}(\mu) \). Therefore, \( M(\delta, k, \beta) \) is bounded below by the smallest integer \( a_{k, \delta} \) satisfying

\[ a_{k, \delta} k^m + (c'_1 \delta^{-d} - a_{k, \delta}) k^\beta \geq c_0 k^d \delta^{-d}. \]

Rearranging terms, we have that

\[ a_{k, \delta} = \left\lfloor \frac{\delta^{-d} \left( c_0 k^{d-\beta} - c'_1 \right)}{k^{m-\beta} - 1} \right\rfloor. \]

Let

\[ K = \left( \frac{c'_1}{c_0} \right)^{\frac{1}{d-\beta}}, \]

so both the numerator and the denominator of the previous expression are positive for \( k > K \). Let \( k > K \) and set

\[ c_2 = \frac{1}{2} \left( c_0 k^{d-\beta} - c'_1 \right), \]

so

\[ a_{k, \delta} \sim 2c_2 \delta^{-d} \]

as \( \delta \to 0 \). It follows that

\[ M(\delta, k, \beta) \geq a_{k, \delta} > c_2 \delta^{-d} \]

for all sufficiently small \( \delta \), as desired. \( \square \)
6.4. Proof of the Lower Bound in Theorems 5 and 8. We require one more lemma before proving the lower bound. The idea is similar to that of Lemma 23.

**Lemma 42.** If \( \mu \) be a \( d \)-Ahlfors regular measure on \( \mathbb{R}^m \) with \( d > \gamma_i^m \), then there exist positive real numbers \( \epsilon_0 \) and \( \Omega_0 \) so that

\[
\lim_{n \to \infty} \frac{1}{n} p_i \left( x_n, \epsilon_0 n^{-1/d} \right) \geq \Omega_0
\]

with high probability.

**Proof.** Let \( \gamma_i^m < \beta < d \). By the definition of \( \gamma_i^m \) we can find a \( K_0 \) so that \( k^{\beta} > \xi_i^m (k) \) for all \( k > K_0 \). Let \( k_0 > \min(K,K_0) \), where \( K \) is given in the previous lemma, and let \( \delta_1 \) and \( c_2 \) also be as in the previous lemma. There are only finitely many collections of sub-cubes of \( [k_0]^m \), so there are only finitely many possible stable \( PH_i \) classes of subsets of \( [k_0]^m \). Let \( \epsilon_0 \) be the minimum of the sizes of these stable classes.

Let \( \delta = n^{-1/d} \) and choose \( n \) large enough so that \( \delta < \delta_1 \). Also, let \( \{D_1, \ldots, D_s\} \) be a maximal collection of cubes in \( C_{k_0,\beta} \) so that \( d(D_j, D_k) > \delta \sqrt{m} \) for all \( j, k \in \{1, \ldots, s\} \) so that \( j \neq k \). See Figure 9. There is a constant \( 0 < \kappa < 1 \) that depends only on \( d \) so that

\[
s \geq \kappa N (\delta, k, \beta) > \kappa c_2 \delta^{-d} = \kappa c_2 n
\]

Let \( l \in \{1, \ldots, s\} \). By the definition of \( \gamma_i^m \), there is a collection of sub-cubes \( \mathcal{B}_l \subset \mathcal{D}_{k_0} (D_l) \) with a stable \( PH_i \) class. Let \( A_l = \hat{B}_{\delta \sqrt{m}} (C) \setminus \bigcup_{B \in \mathcal{B}_l} B \).
where \( \hat{B}_{\delta \sqrt{m}}(D_j) \) is the union of all cubes in the grid of mesh \( \delta/k \) within distance \( \delta \sqrt{m} \) of \( D_j \). Also, let \( B'_n \) be collection of the interiors of the sets \( B_l \). It follows from property (2) in Section 4.1 that

\[
p_i \left( x_n, \epsilon_0 n^{-1/d} \right) \geq \frac{1}{n} \sum_{j=1}^{s} \Xi \left( x_n, A_l, B'_l \right).
\]

There is a \( q > 0 \) depending only on \( k_0, c, d, \) and \( m \) so that

\[
\mu \left( A_l \right) \leq q \delta^d = \frac{q}{n}.
\]

for all \( l \in \{1, \ldots, s\} \). Also, each \( B \in B_l \) is a cube of width \( \delta/k_0 \) in \( \mathbb{R}^m \) so

\[
\mu \left( B \right) \geq \frac{1}{c} \left( \frac{\delta \sqrt{m}}{2k_0} \right)^d = \frac{p}{n},
\]

where \( p = 2^{-d} k_0^{-d} m^{d/2} / c \). Therefore, \( \Xi \left( x_n, A_l, B'_l \right) \) is a \( n, p, q, k^m \)-bounded occupancy event for each \( l \), and the desired result follows from Lemma 18. \( \square \)

The proof of the lower bound in Theorems 5 and 8 and is now straightforward.

**Proposition 43.** Let \( \mu \) be a \( d \)-Ahlfors regular measure on \( \mathbb{R}^m \) with \( d > \gamma_i^m \). Then there is an \( \Omega > 0 \) so that

\[
\lim_{n \to \infty} n^{-d-\alpha} E^\alpha_{\alpha} (x_1, \ldots, x_n) \geq \Omega
\]

with high probability.

**Proof.** It follows immediately from the previous lemma that

\[
\lim_{n \to \infty} n^{-d-\alpha} E^\alpha_{\alpha} (x_n) \geq \lim_{n \to \infty} n^{-d-\alpha} p_i \left( x_n, \epsilon_0 n^{-1/d} \right) \left( \epsilon_0 n^{-1/d} \right)^\alpha
\]

\[
= \epsilon_0^\alpha \lim_{n \to \infty} \frac{1}{n} p_i \left( x_n, \epsilon_0 n^{-1/d} \right)
\]

\[
\geq \epsilon_0^\alpha \Omega_0 \quad \text{by Lemma 42}
\]

:= \Omega

with high probability. \( \square \)
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References


Figure 10. The (a) Menger sponge, (b) Sierpiński triangle, and (c) two stacked tori. Figures were generated in Mathematica.


Appendix A. Scaling of Persistent Homology

We provide computational evidence that the hypotheses of Theorem 8 hold in many cases. We examine four examples in $\mathbb{R}^3$ — the natural measures on the Menger sponge and the Sierpiński triangle cross an interval, the uniform measure on two tori stacked one above the other, and empirical data from earthquake hypocenters. See Figure 10. The first three are Ahlfors regular measures, with dimensions of $\frac{\log(20)}{\log(3)} \approx 2.727$, $1 + \frac{\log(3)}{\log(2)} \approx 2.585$, and 2, respectively. Note that $\gamma_1^2 \leq 2.5$ [59], so the first two examples are known to meet all requirements of Theorem 8 for $i = 1$ except for perhaps the scaling of the expectation and variance of the number of intervals.
Figure 11. Scaling of $|PH_i(x_1, \ldots, x_n)|/n$ for four examples, and $i = 1, 2$.

Figure 12. Variance of $|PH_i(x_1, \ldots, x_n)|$ divided by $n^2$ for three examples, and $i = 1, 2$.

We sample points from the natural measures on the Menger sponge and the Sierpiński triangle using the procedures described in [42]. The rejection sampling algorithm developed in [26] was used to sample points from the uniform distribution of the torus $\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2$ with $R = 2$ and $r = 1$. The $z$-coordinate was translated by 3 with probability $\frac{1}{2}$. The earthquake hypocenter data comes from the Hauksson–Shearer Waveform Relocated Southern California earthquake catalog [37, 46, 58]; data was processed as in [42]. Persistent homology was computed using the implementation of the Alpha complex in GUDHI [52, 57].

Figure 11 shows the empirical expectation of $PH_i(x_1, \ldots, x_n)$ divided by $n$ for each of the four examples, and $i = 1, 2$. The expectation was averaged over 100 trials for
each example except for the earthquake data, which was averaged over 7. In each case, the quantity appears to limit to a constant with $n$, indicating linear scaling of the number of intervals. Figure 12 shows the empirical variance of $PH_i(x_1, \ldots, x_n)$ divided by $n^2$ for the three regular examples. This quantity decreases toward zero for all examples.