LIMITING DISTRIBUTIONS OF CURVES UNDER GEODESIC FLOW ON HYPERBOLIC MANIFOLDS

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Abstract

We consider the evolution of a compact segment of an analytic curve on the unit tangent bundle of a hyperbolic n-manifold of finite volume under the geodesic flow. Suppose that the curve is not contained in a stable leaf of the flow. It is shown that under the geodesic flow, the normalized parameter measure on the curve gets asymptotically equidistributed with respect to the normalized natural Riemannian measure on the unit tangent bundle of a closed totally geodesically immersed submanifold.

Moreover, if this immersed submanifold is a proper subset, then a lift of the curve to the universal covering space $T^1(\mathbb{H}^n)$ *is mapped into a proper subsphere of the ideal boundary sphere* $\partial\mathbb{H}^n$ *under the visual map. This proper subsphere can be realized as the ideal boundary of an isometrically embedded hyperbolic subspace in* $\mathbb{H}^n$ *covering the closed immersed submanifold.*

In particular, if the visual map does not send a lift of the curve into a proper subsphere of $\partial\mathbb{H}^n$*, then under the geodesic flow the curve gets asymptotically equidistributed on the unit tangent bundle of the manifold with respect to the normalized natural Riemannian measure.*

The proof uses dynamical properties of unipotent flows on homogeneous spaces of $\mathrm{SO}(n,1)$ *of finite volume.*

1. Introduction

It is instructive to note the following dynamical property. Let $\psi : I = [0,1] \to \mathbb{R}^n$ be a C^2-curve such that for any proper rational hyperplane (say, H) in $\mathbb{R}^n$, the set $\{s \in I : \psi(s) \in H\}$ has null measure. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ denote the quotient map. Then for any continuous function f on $\mathbb{T}^n$,

$$\lim_{\alpha\to\infty} \int_0^1 f\big(\pi(\alpha\psi(s))\big)\,\mathrm{d}s = \int_{\mathbb{T}^n} f(x)\,\mathrm{d}x, \tag{1}$$

DUKE MATHEMATICAL JOURNAL
Vol. 148, No. 2, © 2009 DOI 10.1215/00127094-2009-026
Received 29 August 2007. Revision received 1 December 2008.
2000 *Mathematics Subject Classification.* Primary 37A17; Secondary 37D40, 22E40.

where $\mathrm{d}x$ denotes the normalized Haar integral on $\mathbb{T}^n$. Using Fourier transforms, we can verify (1) for the characters

$$f_m(x) := \exp\big(2\pi(m\cdot x)\big), \quad \forall x \in \mathbb{T}^n, \text{ where } m \in \mathbb{Z}^n.$$

The above observation was used in [10] for $\psi(t) = (\cos(2\pi t), \sin(2\pi t))$, the unit circle in $\mathbb{R}^2$. Later we learned that a general result in this direction was obtained earlier by Randol [13] in response to a question raised by Sullivan.

Now we ask a similar question for the hyperbolic spaces. Consider the unit ball model B^n for the hyperbolic n-space $\mathbb{H}^n$ of constant curvature (-1). Let $\Gamma \subset \mathrm{SO}(n,1)$ be a discrete subgroup such that $M := \mathbb{H}^n/\Gamma$ is a hyperbolic manifold of finite Riemannian volume. Let $\pi : \mathbb{H}^n \to M$ be the quotient map. As a special case of a more general result proved in [3] and [4], we have that, if we project the invariant probability measure on the sphere $\alpha\mathbb{S}^{n-1} \subset B^n$ for $0 < \alpha < 1$, under π to M, then asymptotically as $\alpha \to 1^-$, the measure gets equidistributed with respect to the normalized measure associated to the Riemannian volume form on M. The case of $n = 3$ was proved earlier in [13].

In this article, we address the following much more refined problem: instead of the invariant measure on the sphere, we take a smooth measure on a one-dimensional curve on $\mathbb{S}^{n-1}$ and describe the limiting distribution of the projection of its dilations on $\alpha\mathbb{S}^{n-1}$ as $\alpha \to 1^-$.

THEOREM 1.1
Let $\bar\psi : I = [0,1] \to \mathbb{S}^{n-1}$ be an analytic map. If $\bar\psi(I)$ is not contained in a proper subsphere in $\mathbb{S}^{n-1}$, then for any $f \in \mathrm{C_c}(M)$,

$$\lim_{\alpha\to 1^-} \int_I f\big(\pi(\alpha\bar\psi(s))\big)\,\mathrm{d}s = \int_M f(x)\,\mathrm{d}x, \tag{2}$$

where $\mathrm{d}x$ denotes the normalized integral associated to the Riemannian volume form on M.

By a *proper subsphere* of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ we mean the intersection of $\mathbb{S}^{n-1}$ with a proper affine subspace of $\mathbb{R}^n$.

Now we describe a generalization of the phenomenon observed in Theorem 1.1 in a suitable geometric framework. Let $\partial\mathbb{H}^n$ denote the ideal boundary of $\mathbb{H}^n$. Let $T^1(\mathbb{H}^n)$ denote the unit tangent bundle on $\mathbb{H}^n$. We identify $\partial\mathbb{H}^n$ with $\mathbb{S}^{n-1}$. Let

$$\mathrm{Vis} : T^1(\mathbb{H}^n) \to \partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$$

denote the visual map sending a unit tangent vector to the equivalence class of the directed geodesics tangent to it. Thus, any fiber of the visual map is a (weakly) stable leaf of the geodesic flow. Now, let M be any n-dimensional hyperbolic manifold (with constant curvature -1) and of finite Riemannian volume, let $T^1(M)$ denote the unit tangent bundle on M, and let $\{g_t\}$ denote the geodesic flow on $T^1(M)$. Let $\pi : \mathbb{H}^n \to M$ be a universal covering map, and let $D\pi : T^1(\mathbb{H}^n) \to T^1(M)$ denote its derivative.

THEOREM 1.2
Let $\psi : I = [a,b] \to T^1(M)$ be an analytic curve such that $\mathrm{Vis}(\tilde{\psi}(I))$ is not a singleton set, where $\tilde{\psi} : I \to T^1(\mathbb{H}^n)$ denotes a lift of ψ to the covering space; that is, $D\pi \circ \tilde{\psi} = \psi$. Then there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic manifold M_1 of finite volume such that the following holds: $\forall f \in \mathrm{C_c}(T^1(M))$, we have

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f\big(g_t\psi(s)\big)\,\mathrm{d}s = \int_{T^1(M_1)} f\big((D\,\Phi)(v)\big)\,\mathrm{d}v, \tag{3}$$

where $|\cdot|$ denotes the Lebesgue measure, and $\mathrm{d}v$ denotes the normalized integral on $T^1(M_1)$ associated to the Riemannian volume form on M_1.

Moreover, if $\pi' : \mathbb{H}^m \to M_1$ denotes a locally isometric covering map, then there exists an isometric embedding $\tilde{\Phi} : \mathbb{H}^m \hookrightarrow \mathbb{H}^n$ such that

$$\pi \circ \tilde{\Phi} = \Phi \circ \pi' \qquad \text{and} \qquad \mathrm{Vis}\big(\tilde{\psi}(I)\big) \subset \partial\big(\tilde{\Phi}(\mathbb{H}^m)\big).$$

In order to describe the relation between $\mathrm{Vis}(\tilde{\psi}(I))$ and the totally geodesic immersion Φ, we recall the following.

THEOREM 1.3 (see [15], [16])
Let M be a hyperbolic manifold of finite Riemannian volume. For $k \geq 2$, let $\Psi : \mathbb{H}^k \to M$ be a totally geodesic immersion. Then there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic manifold M_1 of finite Riemannian volume such that

$$\overline{\Psi(\mathbb{H}^k)} = \Phi(M_1) \qquad \text{and} \qquad \overline{D\,\Psi\big(T^1(\mathbb{H}^k)\big)} = D\,\Phi\big(T^1(M_1)\big).$$

This result can be obtained as a direct consequence of the orbit closure theorem for unipotent flows (Raghunathan's conjecture) proved by Ratner [15] and, more specifically, as a result of the fact that the closure of any $\mathrm{SO}(k,1)$-orbit in $\mathrm{SO}(n,1)/\Gamma$ is a closed orbit of a subgroup of the form $Z \cdot \mathrm{SO}(m,1)$, where Z is a compact subgroup of the centralizer of $\mathrm{SO}(m,1)$ in $\mathrm{SO}(n,1)$.

Remark 1.1
Let notation be as in Theorem 1.2. Let $\mathbb{S}^{k-1}$ be the smallest dimensional subsphere of $\partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$ such that $\mathrm{Vis}(\tilde{\psi}(I)) \subset \mathbb{S}^{k-1}$. Since $\mathrm{Vis}(\tilde{\psi}(I))$ is not a singleton set,

we have $2 \leq k \leq n$. Therefore, there exists an isometric embedding $\mathbb{H}^k \hookrightarrow \mathbb{H}^n$ such that $\partial \mathbb{H}^k = \mathbb{S}^{k-1}$. If $\{\tilde{g}_t\}$ denotes the geodesic flow and $\tilde{d}(\cdot, \cdot)$ denotes the distance function on $T^1(\mathbb{H}^n)$, then

$$\lim_{t\to\infty} \sup_{s\in I} \tilde{d}\big(\tilde{g}_t \tilde{\psi}(s), T^1(\mathbb{H}^k)\big) = 0. \tag{4}$$

Since $\pi : \mathbb{H}^k \to M$ is a totally geodesic immersion, by Theorem 1.3 there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic manifold of finite Riemannian volume such that

$$\Phi(M_1) = \overline{\pi(\mathbb{H}^k)}. \tag{5}$$

This describes the map Φ as involved in the statement of Theorem 1.2. Also, by (4) and (5), if $d(\cdot, \cdot)$ denotes the distance function on M, then

$$\lim_{t\to\infty} \sup_{s\in I} d\big(g_t \psi(s), D\,\Phi(T^1(M_1))\big) = 0. \tag{6}$$

We have the following consequences.

THEOREM 1.4
Let M be a hyperbolic Riemannian manifold of finite volume. Let $\psi : I \to M$ be an analytic map such that $\mathrm{Vis}(\tilde{\psi}(I))$ *is not contained in a proper subsphere in $\partial\mathbb{H}^n$, where $\tilde{\psi} : I \to T^1(\mathbb{H}^n)$ is a lift of ψ such that $D\pi \circ \tilde{\psi} = \psi$. Then, given any $f \in \mathrm{C_c}(T^1(M))$, we have*

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f\big(g_t \psi(s)\big)\, \mathrm{d}s = \int_{T^1(M)} f \,\mathrm{d}v,$$

where dv *is the normalized integral on $T^1(M)$ associated to the Riemannian volume form on M.*

COROLLARY 1.5
Let M be a hyperbolic manifold of finite volume. Let $x \in M$, and let $\psi : I = [a, b] \to {T^1}_x(M)$ be an analytic map such that $\psi(I)$ is not contained in any proper subsphere in ${T^1}_x(M)$. Then

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f\big(g_t \psi(s)\big)\, \mathrm{d}t = \int_{T^1(M)} f(v)\, \mathrm{d}v, \quad \forall f \in \mathrm{C_c}\big(T^1(M)\big),$$

where dv *is the normalized Riemannian volume integral on $T^1(M)$.*

It may be interesting to compare Theorem 1.2 with [20], where any rectifiable invariant set for the geodesic flow is shown to be a conull subset of the unit tangent bundle of a closed totally geodesic submanifold of finite volume.

1.1. Reformulation in terms of flows on homogeneous spaces

Let $G = \mathrm{SO}(n, 1)$, let P^- be a minimal parabolic subgroup of G, and let $K \cong \mathrm{SO}(n)$ be a maximal compact subgroup of G. Then $M := P^- \cap K \cong \mathrm{SO}(n-1)$. Since $G = P^- K$, then

$$P^-\backslash G \cong M\backslash K \cong \mathrm{SO}(n-1)\backslash\mathrm{SO}(n) \cong \mathbb{S}^{n-1}. \tag{7}$$

We let $\mathcal{I} : G \to \mathbb{S}^{n-1}$ denote the quotient map corresponding to (7). Let A be a maximal connected $\mathbb{R}$-diagonalizable subgroup of G contained in $Z_G(M) \cap P^-$. Since G is of $\mathbb{R}$-rank 1, A is a one-parameter group, and the centralizer of A in G is $Z_G(A) := MA$. Let N^- denote the unipotent radical of P^-. Then $P^- = MAN^-$. Define

$$A^+ = \{a \in A : a^k g a^{-k} \to e \text{ as } k \to \infty \text{ for any } g \in N^-\}, \tag{8}$$

and

$$N = \{g \in G : a^{-k} g a^k \to e \text{ as } k \to \infty \text{ for any } a \in A^+\}. \tag{9}$$

Let $\mathfrak{n}$ denote the Lie algebra on N. Then $\mathfrak{n}$ is abelian, and we identify it with $\mathbb{R}^{n-1}$. Let $u : \mathbb{R}^{n-1} \to N$ be the map $u(v) = \exp(v)$ for any $v \in \mathbb{R}^{n-1} \cong \mathfrak{n}$.

Let $\alpha : A \to \mathbb{R}^*$ be the character such that $au(v)a^{-1} = u(\alpha(a)v)$ for all $v \in \mathbb{R}^{n-1}$. Then $A^+ = \{a \in A : \alpha(a) > 1\}$.

Let Γ be a lattice in G, and let μ_G be the G-invariant probability measure on G/Γ.

THEOREM 1.6

Let $\theta : I = [a, b] \to G$ be an analytic map such that $\mathcal{I}(\theta(I))$ is not contained in a subsphere of $\mathbb{S}^{n-1}$. Then, given any $f \in \mathrm{C_c}(G/\Gamma)$, any compact set $\mathcal{K} \subset G/\Gamma$, and any $\epsilon > 0$, there exists $R > 0$ such that for any $a \in A^+$ with $\alpha(a) > R$, we have

$$\Big|\frac{1}{|I|}\int_I f\big(a\theta(t)x\big)\,\mathrm{d}t - \int_{G/\Gamma} f\,\mathrm{d}\mu_G\Big| < \epsilon, \quad \forall x \in \mathcal{K}. \tag{10}$$

First, we consider the following crucial case of Theorem 1.6.

THEOREM 1.7

Let $\varphi : I = [a, b] \to \mathbb{R}^{n-1}$ be an analytic curve such that $\varphi(I)$ is not contained in any sphere or an affine hyperplane. Let $x_i \xrightarrow{i\to\infty} x$ be a convergent sequence in G/Γ, *and let* $\{a_i\}_{i\in\mathbb{N}}$ *be a sequence in* A^+ *such that* $\alpha(a_i) \xrightarrow{i\to\infty} \infty$. *Then*

$$\lim_{i\to\infty} \frac{1}{|I|}\int_I f\big(a_i u(\varphi(t)x_i)\big)\,\mathrm{d}t = \int_{G/\Gamma} f\,\mathrm{d}\mu_G, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{11}$$

We will deduce Theorem 1.7 from the following general statement, which is the main result of this article.

THEOREM 1.8
Let $\varphi : I \to \mathbb{R}^{n-1}$ *be a nonconstant analytic map, and let* $x \in G/\Gamma$. *Then there exist a closed subgroup* H *of* G, *an analytic map* $\zeta : I \to M(= Z_G(A)\cap K)$, *and* $h_1 \in G$ *such that* $\pi(H)$ *is closed and admits a finite* H*-invariant measure (say,* μ_H*), and the following holds: for any sequence* $\{a_i\}_{i\in\mathbb{N}} \subset A^+$, *if* $\alpha(a_i) \xrightarrow{i\to\infty} \infty$, *then*

$$\lim_{i\to\infty} \int_I f\big(a_i u(\varphi(t))x\big)\,\mathrm{d}t = \int_{t\in I}\Big(\int_{y\in G/\Gamma} f(\zeta(t)h_1 y)\,\mathrm{d}\mu_H\Big)\mathrm{d}t. \tag{12}$$

Moreover, $A \subset h_1 H h_1^{-1}$, $N \cap h_1 H h_1^{-1} \neq \{e\}$, *and there exists* $g \in G$ *such that* $x = \pi(g)$ *and*

$$u\big(\varphi(t)\big)g \in N^-\zeta(t)h_1 H, \quad \forall t \in I. \tag{13}$$

Remark 1.2
Suppose that we are given a convergent sequence $x_i \to x$ in G/Γ. We consider (12) for x_i in place of x in the statement of Theorem 1.8. Then the limiting distribution depends on the choice of the sequence $\{a_i\}$. We can still conclude that the analogue of (12) holds after passing to a subsequence.

Relation with unipotent flows
We now indicate how we involve unipotent flows to resolve our problem. For simplicity, we assume that for any $s \in I$, the derivative $\dot\varphi(s) \neq 0$. Note that $\mathrm{Ad}(Z_G(A))$ acts transitively on $\mathrm{Lie}(N) \setminus \{0\}$. Therefore, there exists a continuous function $z : I \to Z_G(A)$ such that $z(s)u(\dot\varphi(s))z(s)^{-1} = u(e_1)$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$. By uniform continuity of f, given that $\epsilon > 0$, there exists $\delta > 0$ such that for any subinterval $J \subset I$ with $|J| < \delta$ and $y \in G/\Gamma$, we have

$$\big|f\big(z(s)a_i u(\varphi(s))y\big) - f\big(a_i u(\varphi(s))y\big)\big| < \epsilon.$$

Note that it is enough to prove Theorem 1.7 for all intervals I of length less than δ. Therefore, in view of the above observation, it is enough to show that if λ_i is the pushforward of the normalized Lebesgue measure on I under the map $s \mapsto z(s)a_i u(\varphi(s))x_i$, then $\lambda_i \to \mu_G$ as $i \to \infty$.

Now, suppose that we have shown that, after passing to a subsequence, $\lambda_i \to \lambda$ weakly in the space of probability measures on G/Γ. Take any $r \in \mathbb{R}$; we want to show that $u(re_1)\lambda = \lambda$. To see this, let $s \in I$, and let $f \in \mathrm{C_c}(G/\Gamma)$. Then

$$\begin{aligned}
&f\big(u(re_1)z(s)a_i u(\varphi(s))x_i\big)\\
&= f\big(z(s)a_i u(re^{-t_i}\dot{\varphi}(s) + \varphi(s))x_i\big)\\
&= f\big(z(s)a_i u(\varphi(s + re^{-t_i}) + O((re^{-t_i})^2))x_i\big) \quad \text{(because } \varphi \text{ is } C^2\text{)}\\
&= f\big(u(O(r^2e^{-t_i}))z(s)a_i u(\varphi(s + re^{-t_i}))x_i\big)\\
&\approx f\big(z(s)a_i u(\varphi(s + re^{-t_i}))x_i\big), \quad \forall i \gg 0.
\end{aligned}$$

For $i \gg 0$, $s + re^{-t_i}$ is a very small translate in the parameter s, and hence, $\lambda_i \to u(re_1)\lambda$ as $i \to \infty$. Therefore, λ is invariant under the one-parameter subgroup $\{u(re_1) : r \in \mathbb{R}\}$ (see Theorem 3.1). We then apply Ratner's theorem and linearization technique in order to further analyze the measure λ. Using an observation on dynamics of linear actions of $\mathrm{SL}(2, \mathbb{R})$ on vector spaces, we show that λ is indeed G-invariant.

2. Nondivergence of translated measures

Let $\varphi : I \to \mathbb{R}^{n-1}$ be a nonconstant analytic map. Let $\{a_i\} \subset A^+$ be a sequence such that $\alpha(a_i) \overset{i\to\infty}{\longrightarrow} \infty$. Let $x_i \overset{i\to\infty}{\longrightarrow} x$ be a convergent sequence in G/Γ. For each $i \in \mathbb{N}$, let μ_i be the measure on G/Γ defined by

$$\int_{G/\Gamma} f \,\mathrm{d}\mu_i := \frac{1}{|I|}\int_I f\big(a_i u(\varphi(t))x_i\big)\,\mathrm{d}t, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{14}$$

This section is devoted to the proof of the following theorem.

THEOREM 2.1
Given $\epsilon > 0$, there exists compact set $\mathcal{K} \subset G/\Gamma$ such that $\mu_i(\mathcal{K}) \geq 1 - \epsilon$ for all $i \in \mathbb{N}$.

2.1. (C, α)-good family

Let $V = \bigoplus_{d=1}^{\dim \mathfrak{g}} \bigwedge^d \mathfrak{g}$, and consider the linear action of any $g \in G$ on V via $\bigoplus_{d=1}^{\dim \mathfrak{g}} \bigwedge^d \mathrm{Ad}\, g$.

Let $\Upsilon : I \to \mathrm{End}(V)$ be the map given by $\Upsilon(s) = \bigwedge \mathrm{Ad}\,\big(u(\varphi(s))\big)$ for all $s \in I$. Fix $s_0 \in I$, and let $\mathcal{E}$ be the smallest subspace of $\mathrm{End}(V)$ such that $\Upsilon(I) \subset \mathcal{E} + \Upsilon(s_0)$.

Then $\Upsilon(I) \subset \mathcal{E} + \Upsilon(s)$ for all $s \in I$. For any $s \in I$, we have $\mathcal{E}_s := \operatorname{span}\{\Upsilon^{(k)}(s) : k \geq 1\} \subset \mathcal{E}$, where $\Upsilon^{(k)}(s)$ denotes the kth derivative at s. Since Υ is an analytic function, we have $\Upsilon(I) \subset \Upsilon(s) + \mathcal{E}_s$. Therefore, $\mathcal{E} \subset \mathcal{E}_s$. Hence, $\mathcal{E}_s = \mathcal{E}$ for all $s \in I$.

Let $\mathscr{F}$ denote the linear span of coordinate functions of Υ.

By [8, Proposition 3.4], applied to the function $s \mapsto \Upsilon(s) - \Upsilon(s_0)$ from I to $\mathcal{E}$, there exist constants $C > 0$ and $\alpha > 0$ such that $\mathscr{F}$ consists of (C, α)-*good functions*; that is, for any subinterval $J \subset I$, $\xi \in \mathscr{F}$, and $r > 0$, we have

$$\left|\left\{s \in J : |\xi(s)| < r\right\}\right| \leq C\Big(\frac{r}{\sup_{s\in J} |\xi(s)|}\Big)^{\alpha} |J|. \tag{15}$$

It may be noted that (since I is compact) by the result quoted above, a priori (15) holds only for subintervals J with $|J|$ smaller than a fixed constant depending on Υ and I. Then by a straightforward argument using a finite covering of I by short intervals of fixed length with half-length overlapping between two successive intervals, and by applying equation (15) successively, we can choose a much larger C such that (15) holds for all subintervals $J \subset I$.

Let $\mathscr{F}(G)$ be the collection of all functions $\psi : I \to G$ such that for any $p \in V$ and for any linear functional f on V, if we define $\xi(s) = f(\psi(s)p)$ for all $s \in I$, then $\xi \in \mathscr{F}$.

PROPOSITION 2.2 (see [1])
Fix any norm on V. Let $d = \dim N$, and let $\boldsymbol{p} \in \bigwedge^d \mathfrak{n} \smallsetminus \{0\}$. There exists a finite set $\Sigma \subset G$ such that $\Gamma\Sigma\boldsymbol{p}$ is a discrete subset of V and the following holds: given $\epsilon > 0$ and $R > 0$, there exists a compact set $\mathcal{K} \subset G/\Gamma$ such that for any $\psi \in \mathscr{F}(G)$ and a subinterval $J \subset I$, one of the following holds:

(I) *there exist $\gamma \in \Gamma$ and $\sigma \in \Sigma$ such that $\sup_{s\in J} \|\psi(s)\gamma\sigma\boldsymbol{p}\| < R$;*

(II) $|\{s \in J : \pi(\psi(s)) \in \mathcal{K}\}| \geq (1 - \epsilon)|J|$.

In Proposition 2.2, $(\sigma N\sigma^{-1}) \cap \Gamma$ is a cocompact lattice in $\sigma N\sigma^{-1}$ for each $\sigma \in \Sigma$.

Although the proof in [1] considers only the case of $\psi(s) = u(s)h$ for some $h \in G$, the proof only uses the property that, for any $\gamma \in \Gamma$, $\sigma \in \Sigma$, and a linear functional f, if we define $\xi(s) := f(\psi(s)\gamma\sigma\boldsymbol{p})$ for all $s \in I$, then $\xi \in \mathscr{F}$; hence, (15) holds.

Now we make an observation that allows us to exclude possibility (I) of Proposition 2.2 in the situation of our interest, namely, when $\psi(s) = a_i u(\varphi(s))g$ for some fixed $g \in G$ and large $i \in \mathbb{N}$.

2.2. Basic lemma

Consider a linear representation of SL(2, R) on a finite-dimensional vector space V. Let $a = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$ for some $\alpha > 1$, and define

$$\begin{aligned} V^+ &= \{v \in V : a^{-k}v \to 0 \text{ as } k \to \infty\}, \\ V^0 &= \{v \in V : av = v\}, \\ V^- &= \{v \in V : a^k v \to 0 \text{ as } k \to \infty\}. \end{aligned} \tag{16}$$

Then any $v \in V$ can be uniquely expressed as $v = v^+ + v^0 + v^-$, where $v^\pm \in V^\pm$ and $v^0 \in V^0$. We also write $V^{+0} = V^+ + V^0$ and $V^{0-} = V^0 + V^-$. Let $q^+ : V \to V^+$, $q^0 : V \to V^0$, $q^{+0} : V \to V^{+0}$, and $q^{0-} : V \to V^{0-}$ denote the projections $q^+(v) = v^+$, $q^0(v) = v^0$, $q^{+0}(v) = v^{+0} := v^+ + v^0$, and $q^{0-}(v) = v^{0-} := v^0 + v^-$ for all $v \in V$. We consider the Euclidean norm on V such that V^+, V^0, and V^- are orthogonal.

LEMMA 2.3

Let $\boldsymbol{u} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some $t \neq 0$. Then there exists a constant $\kappa = \kappa(t) > 0$ such that

$$\max\left\{\|v^+\|, \|(\boldsymbol{u}v)^{+0}\|\right\} \geq \kappa\|v\|, \quad \forall v \in V. \tag{17}$$

Proof

It is enough to prove the result for each of the irreducible SL(2, $\mathbb{R}$)-submodules of V. Therefore, we may assume that SL(2, $\mathbb{R}$) acts irreducibly on V.

Let $m = \dim V - 1$. Then $m = 2r - 1$ or $m = 2r$ for some $r \in \mathbb{N}$. Consider the associated representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ on V. Let $\boldsymbol{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\boldsymbol{h} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, and $\boldsymbol{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ denote the standard $\mathfrak{sl}_2$-triple. Then there exists a basis of V consisting of elements $v_0, v_1, \ldots, v_m$ such that

$$\boldsymbol{h}v_k = (m - 2k)v_k \qquad \text{and} \qquad \boldsymbol{e}v_k = kv_{k-1}, \quad \forall 0 \leq k \leq m,$$

where $v_{-1} = 0$. Then

$$V^{+0} = \operatorname{span}\{v_0, \ldots, v_{m-r}\} \qquad \text{and} \qquad V^{0-} = \operatorname{span}\{v_r, \ldots, v_m\}. \tag{18}$$

Since $\boldsymbol{u} = \exp(t\boldsymbol{e})$, we have

$$\boldsymbol{u}v_k = \sum_{l=0}^{k} \binom{k}{l} t^{k-l} v_l, \quad 0 \leq k \leq m.$$

Let A denote the restriction of the map $\boldsymbol{u}$ from V^+ to V^+ with respect to the basis $\{v_0, \ldots, v_{r-1}\}$. Let B denote the matrix of the map $q^{+0} \circ \boldsymbol{u} : V^{0-} \to V^{+0}$ with respect to the basis given by (18).

Next, we want to show that B is invertible. We write $b_{l,k} = t^{k-l}\binom{k}{l}$ for $r \le k \le m$ and $0 \le l \le m - r$. And for any $r \le m_1 \le m$, we consider the $(m_1 - r + 1) \times (m_1 - r + 1)$-matrix

$$B(m_1, r) = (b_{l,k})_{0 \le l \le m_1 - r,\, r \le k \le m_1}.$$

Then $B = B(m, r)$. In view of the binomial relations

$$\binom{k+1}{l+1} - \binom{k}{l+1} = \binom{k}{l} \qquad \text{and} \qquad b_{l+1,k+1} - t b_{l+1,k} = b_{l,k},$$

we apply the column operations $C_{k+1} - tC_k$, successively, in the order $k = (m_1 - 1), \dots, r$. We obtain

$$\det B(m_1, r) = t^r \det B(m_1 - 1, r).$$

Since $\det B(r, r) = t^r$, we get

$$\det B = \det B(m, r) = t^{r(m-r+1)}.$$

Since $t \neq 0$, B is invertible.

Now

$$\|(\boldsymbol{u}v)^{+0}\| = \|Av^+ + Bv^{0-}\|. \tag{19}$$

Since A is a unipotent matrix, $\|A\| \ge 1$. We put

$$\kappa = \frac{1}{3}\min\{1, \|B^{-1}\|^{-1}\|A\|^{-1}\} \le \frac{1}{3}\min\{1, \|B^{-1}\|^{-1}\}. \tag{20}$$

Now to prove (17), it is enough to consider the case when

$$\|v^+\| \le \kappa\|v\| \le \frac{1}{3}\|v\|. \tag{21}$$

In particular,

$$\|v^{0-}\| \ge \|v\| - \|v^+\| \ge \|v\| - \frac{1}{3}\|v\| = \frac{2}{3}\|v\|. \tag{22}$$

Then by (19), (20), (22), and (21),

$$\begin{aligned}
\|(\boldsymbol{u}v)^{+0}\| &\ge \|Bv^{0-}\| - \|Av^+\| \\
&\ge \|B^{-1}\|^{-1}\|v^{0-}\| - \|A\|\|v^+\| \\
&\ge \|B^{-1}\|^{-1}\|v^{0-}\| - \kappa\|A\|\|v\| \\
&\ge \left(\|B^{-1}\|^{-1} - \frac{3}{2}\kappa\|A\|\right)\|v^{0-}\| \\
&\ge \frac{1}{2}\|B^{-1}\|^{-1}\|v^{0-}\| \\
&\ge \frac{1}{2}\|B^{-1}\|^{-1}\frac{2}{3}\|v\| \\
&\ge \kappa\|v\|.
\end{aligned}$$

□

COROLLARY 2.4
Let V be a finite-dimensional normed linear space. Consider a linear representation of $G = \mathrm{SO}(n, 1)$ on V, where $n \geq 2$. Let

$$\begin{aligned} V^+ &= \{v \in V : a^{-k} v \overset{k\to\infty}{\longrightarrow} 0,\ \forall a \in A^+\}, \\ V^- &= \{v \in V : a^{k} v \overset{k\to\infty}{\longrightarrow} 0,\ \forall a \in A^+\}, \\ V^0 &= \{v \in V : Av = v\}. \end{aligned} \tag{23}$$

Then given a compact set $F \subset N \setminus \{e\}$, there exists a constant $\kappa > 0$ such that for any $\boldsymbol{u} \in F$,

$$\max\big\{\|v^+\|, \|(\boldsymbol{u}v)^{+0}\|\big\} \geq \kappa\|v\|, \quad \forall v \in V. \tag{24}$$

In particular, for any $a \in A^+$, and any $\boldsymbol{u} \in F$,

$$\max\big\{\|av\|, \|a\boldsymbol{u}v\|\big\} \geq \kappa\|v\|, \quad \forall v \in V.$$

Proof
Given any $a \in A^+$ and $\boldsymbol{u} \in F$, there exists a continuous homomorphism of $\mathrm{SL}(2, \mathbb{R})$ into G such that a is the image of $\begin{pmatrix} \alpha & \\ & \alpha^{-1}\end{pmatrix}$ for some $\alpha > 1$, and $\boldsymbol{u}$ is the image of $\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$ for some $t \neq 0$. We apply Lemma 2.3 to obtain a constant $\kappa_1 > 0$ such that (24) holds for $\boldsymbol{u}$.

Now there exists a compact set $Z_1 \subset Z_G(A)$ such that any $\boldsymbol{u}_1 \in F$ is of the form $z\boldsymbol{u}z^{-1}$ for some $z \in Z_1$. Also, there exists a constant $\kappa_2 > 0$ such that

$$\kappa_2\|v\| \leq \|zv\| \leq \kappa_2^{-1}\|v\|, \quad \forall z \in Z_1,\ \forall v \in V.$$

Therefore, (24) holds for any $\boldsymbol{u}_1 \in F$ in place of $\boldsymbol{u}$ and $\kappa := \kappa_2^2\kappa_1$. □

2.3. Proof of Theorem 2.1
Let $t_1, t_2 \in I$ be such that $\boldsymbol{u} := u(\varphi(t_2) - \varphi(t_1))^{-1} \neq e$. By Corollary 2.4, there exists $\kappa > 0$ such that

$$\sup\big\{\|a_i v\|, \|a_i\boldsymbol{u}v\|\big\} \geq \kappa\|v\|, \quad \forall v \in V, \forall i \in \mathbb{N}. \tag{25}$$

Let a sequence $g_i \to g \in G$ be such that $\pi(g_i) = x_i$. By Proposition 2.2, $\Gamma\Sigma\boldsymbol{p}$ is discrete in V. Therefore,

$$R_1 := \inf\big\{\|u(\varphi(t_1))g_i\gamma\sigma\boldsymbol{p}\| : \gamma \in \Gamma,\ \sigma \in \Sigma\big\} > 0.$$

For any $\gamma \in \Gamma$, $\sigma \in \Sigma$ and $i \in \mathbb{N}$, if we put $v = u(\varphi(t_1))g_i\gamma\sigma\boldsymbol{p}$ in (25), then have

$$\sup_{t\in\{t_1,t_2\}}\big\{\big\|a_i u(\varphi(t))g_i\gamma\sigma\boldsymbol{p}\big\|\big\} \geq \kappa\,\big\|u(\varphi(t_1))g_i\gamma\sigma\boldsymbol{p}\big\| \geq \kappa R_1. \tag{26}$$

Given that $\epsilon > 0$, we obtain a compact set $\mathcal{K} \subset G/\Gamma$ such that the conclusion of Proposition 2.2 holds for $R = (1/2)\kappa R_1$. Then by (26), for any $i \in \mathbb{N}$, possibility (I) of Proposition 2.2 does not hold for $h_1 = a_i$ and $h_2 = g_i$. Therefore, possibility (II) of Proposition 2.2 must hold for all i. Thus, Theorem 2.1 follows. □

We obtain the following immediate consequence of Theorem 2.1.

COROLLARY 2.5
After passing to a subsequence, $\mu_i \to \mu$ in the space of probability measures on G/Γ with respect to the weak-topology; that is,*

$$\lim_{i\to\infty} \int_{G/\Gamma} f \,\mathrm{d}\mu_i = \int_{G/\Gamma} f \,\mathrm{d}\mu, \quad \forall f \in \mathrm{C_c}(G/\Gamma).$$

3. Invariance under a unipotent flow

Let $I = [a, b] \subset \mathbb{R}$ with $a < b$. Let $\varphi : I \to \mathbb{R}^{n-1}$ be a C^2-curve such that $\dot\varphi(t) \neq 0$ for all $t \in I$, where $\dot\varphi(t)$ denotes the tangent to the curve φ at t. Fix $w_0 \in \mathbb{R}^{n-1} \smallsetminus \{0\}$, and define

$$W = \big\{u(tw_0) : t \in \mathbb{R}\big\}.$$

Consider the $Z_G(A)$-action on $\mathbb{R}^{n-1}$ via the correspondence $u(zv) = zu(v)z^{-1}$ for all $v \in \mathbb{R}^{n-1}$ and $z \in Z_G(A)$. Then $Z_G(A) = MA$ acts transitively on $\mathbb{R}^{n-1} \smallsetminus \{0\}$. Therefore, there exists a continuous function $z : I \to Z_G(A)$ such that

$$z(t)\dot\varphi(t) = w_0, \quad \forall t \in I. \tag{27}$$

Now, assume that φ is analytic. Let a sequence $\{a_i\}_{i\in\mathbb{N}} \subset A^+$ be such that $\alpha(a_i) \to \infty$ as $i \to \infty$. Let $x_i \to x$ a convergent sequence in G/Γ. For each $i \in \mathbb{N}$, let λ_i be the probability measure on G/Γ such that

$$\int_{G/\Gamma} f \,\mathrm{d}\lambda_i = \frac{1}{|I|}\int_{t\in I} f\big(z(t)a_i u(\varphi(t))x_i\big)\,\mathrm{d}t, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{28}$$

Since $z(I)$ is compact, by Theorem 2.1 there exists a probability measure λ on G/Γ such that, after passing to a subsequence, $\lambda_i \to \lambda$ as $i \to \infty$ in the space of finite measures on G/Γ with respect to the weak*-topology.

THEOREM 3.1
The measure λ is W-invariant.

Proof (cf. [18])
We use the notation $\eta_1 \overset{\epsilon}{\approx} \eta_2$ to say $|\eta_1 - \eta_2| \leq \epsilon$.

Let both $f \in C_c(G/\Gamma)$ and $\epsilon > 0$ be given. Let Ω be a neighborhood of e in G such that

$$f(\omega y) \overset{\epsilon}{\approx} f(y), \quad \forall \omega \in \Omega^2 \text{ and } \forall y \in G/\Gamma. \tag{29}$$

Let $t_0 \in \mathbb{R}$. Let $t \in I = [a, b]$, and let $i \in \mathbb{N}$. By (27),

$$u(t_0 w_0)z(t)a_i = z(t)a_i u\big(\alpha(a_i)^{-1} t_0 z(t)^{-1} \cdot w_0\big) = z(t)a_i u\big(\xi_i \dot\varphi(t)\big), \tag{30}$$

where $\xi_i := \alpha(a_i)^{-1} t_0$. Since φ is a C^2-map,

$$\varphi(t + \xi_i) = \varphi(t) + \xi_i \dot\varphi(t) + \epsilon_i(t), \tag{31}$$

where, by Taylor's formula, there exists a constant $M > 0$ such that

$$|\epsilon_i(t)| \leq M|\xi_i|^2 \leq (M|t_0|^2)\alpha(a_i)^{-2}, \quad \forall t \in [a, b]. \tag{32}$$

As $i \to \infty$, we have $\alpha(a_i) \to \infty$, and hence $\xi_i \to 0$ and $\alpha(a_i)\epsilon_i(t) \to 0$. Since $t \mapsto z(t)$ is continuous, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$z(t + \xi_i)z(t)^{-1} \in \Omega \qquad \text{and} \qquad u\big(z(t) \cdot (\alpha(a_i)\epsilon_i(t))\big) \in \Omega. \tag{33}$$

Therefore,

$$\begin{aligned} &z(t + \xi_i)\, a_i u\, \big(\varphi(t + \xi_i)\big) \\ &\quad = \big(z(t + \xi_i)z(t)^{-1}\big) z(t)a_i u\big(\varphi(t) + \xi_i \dot\varphi(t) + \epsilon_i(t)\big) \quad \text{(by (31))} \\ &\quad \in \Omega u\big(z(t) \cdot (\alpha(a_i)\epsilon_i(t))\big) z(t)a_i \big(u(\varphi(t) + \xi_i \dot\varphi(t))\big) \\ &\quad \subset \Omega^2 z(t)a_i u\big(\xi_i \dot\varphi(t)\big) u\big(\varphi(t)\big) \quad \text{(by (33))} \\ &\quad \subset \Omega^2 u(t_0 w_0) z(t) a_i u\big(\varphi(t)\big) \quad \text{(by (30))}. \end{aligned} \tag{34}$$

Therefore, by (29),

$$f\big(z(t + \xi_i)a_i u(\varphi(t + \xi_i))x_i\big) \overset{\epsilon}{\approx} f\big(u(t_0 w_0)z(t)a_i u(\varphi(t))x_i\big). \tag{35}$$

Hence,

$$
\begin{aligned}
&\int_a^b f\bigl(z(t)\ a_i u(\varphi(t))x_i\bigr)\,\mathrm{d}t\\
&\qquad \overset{\xi_i \sup|f|}{\approx} \int_a^{b-\xi_i} f\bigl(z(t+\xi_i)a_i u(\varphi(t+\xi_i))x_i\bigr)\,\mathrm{d}t\\
&\qquad \overset{\epsilon|I|}{\approx} \int_a^{b-\xi_i} f\bigl(u(t_0w_0)z(t)a_i u(\varphi(t))x_i\bigr)\,\mathrm{d}t \quad \text{(by (35))}\\
&\qquad \overset{\xi_i \sup|f|}{\approx} \int_a^{b} f\bigl(u(s_0w_0)z(t)a_i u(\varphi(t))x_i\bigr)\,\mathrm{d}t.
\end{aligned} \tag{36}
$$

Therefore, since $\epsilon > 0$ is chosen arbitrarily and since $\xi_i \to 0$ as $i \to 0$, we have

$$\int_{G/\Gamma} f\bigl(u(s_0w_0)y\bigr)\,\mathrm{d}\lambda(y) = \int_{G/\Gamma} f(y)\,\mathrm{d}\lambda(y). \tag{37}$$

□

4. Dynamical behavior of translated trajectories near singular sets

Let notation be as in Section 3. We further assume that $\varphi : I \to \mathbb{R}^{n-1}$ is an analytic function. In this case, we also observe that the function $z : I \to Z_G(A)$ such that $z(t)\dot\varphi(t) = w_0$ for all $t \in I$ is also an analytic function. Given a convergent sequence $x_i \to x$ in G/Γ, we obtain a sequence of measures $\{\lambda_i : i \in \mathbb{N}\}$ on G/Γ as defined by (28). Due to Theorem 2.1, by passing to a subsequence we assume that $\lambda_i \to \lambda$ as $i \to \infty$, where λ is a probability measure on G/Γ. By Theorem 3.1, λ is invariant under the action of the one-parameter subgroup $W = \{u(sw_0) : s \in \mathbb{R}\}$. We would like to describe the measure λ using the description of ergodic invariant measures for unipotent flows on homogeneous spaces due to Ratner [14].

4.1. Ratner's theorem, singular sets, and linearization

Let $\mathcal{H}$ denote the collection of closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H, and suppose that a unipotent one-parameter subgroup of H acts ergodically with respect to the H-invariant probability measure on $H/H \cap \Gamma$. Then $\mathcal{H}$ is a countable collection (see [17], [14]).

For a closed connected subgroup H of G, define

$$N(H, W) = \{g \in G : g^{-1}Wg \subset H\}.$$

Now, suppose that $H \in \mathcal{H}$. We define the associated singular set

$$S(H, W) = \bigcup_{\substack{F \in \mathcal{H}\\ F \subsetneq H}} N(F, W).$$

Note that $N(H,W)N_G(H) = N(H,W)$. By [11, Proposition 2.1, Lemma 2.4],

$$N(H,W) \cap N(H,W)\gamma \subset S(H,W), \quad \forall \gamma \in \Gamma \setminus N_G(H). \tag{38}$$

By Ratner's theorem [14, Theorem 1], as explained in [11, Theorem 2.2], we have the following.

THEOREM 4.1 (Ratner)
Given the W-invariant probability measure λ on G/Γ, there exists $H \in \mathcal{H}$ such that

$$\lambda\bigl(\pi(N(H,W))\bigr) > 0 \qquad \text{and} \qquad \lambda\bigl(\pi(S(H,W))\bigr) = 0. \tag{39}$$

Moreover, almost every W-ergodic component of λ on $\pi(N(H,W))$ is a measure of the form $g\mu_H$, where $g \in N(H,W) \setminus S(H,W)$, μ_H is a finite H-invariant measure on $\pi(H) \cong H/H \cap \Gamma$, and $g\mu_H(E) := \mu(g^{-1}E)$ for all Borel sets $E \subset G/\Gamma$.

Let V be as in Section 2.1. Let $d = \dim H$, and fix $p_H = \bigwedge^d \mathfrak{h} \setminus \{0\}$.

As in [11, Section 3], we observe that for any $g \in N_G(H)$, $gp_H = \det(\operatorname{Ad} g|_{\mathfrak{h}})p_H$. Hence, the stabilizer of p_H in G equals

$$N_G^1(H) := \bigl\{g \in N_G(H) : \det((\operatorname{Ad} g)|_{\mathfrak{h}}) = 1\bigr\}.$$

Recall that $\operatorname{Lie}(W) = \mathbb{R}w_0$. Let

$$\mathcal{A} = \{v \in V : v \wedge w_0 = 0\}.$$

Then $\mathcal{A}$ is a linear subspace of V. We observe that

$$N(H,W) = \bigl\{g \in G : w_0 \in \operatorname{Ad}(g)(\mathfrak{h})\bigr\} = \{g \in G : gp_H \in \mathcal{A}\}. \tag{40}$$

Now, assume that $H \in \mathcal{H}$. If $\gamma \in \Gamma \cap N_G(H)$, then $\gamma\pi(H) = \pi(H)$. Since $\operatorname{Vol}(H/H \cap \Gamma)$ is finite, $|\det((\operatorname{Ad}\gamma)|_{\mathfrak{h}})| = 1$. Therefore, $(\Gamma \cap N_G(H))p_H = p_H$ or $(\Gamma \cap N_G(H))p_H = \{p_H, -p_H\}$.

PROPOSITION 4.2 ([2, Theorem 3.4])
The orbit Γp_H is a discrete subset of V. □

Remark 4.1
Let H^{nc} denote the subgroup of H generated by unipotent one-parameter subgroups of G contained in H. Since $H \in \mathcal{H}$, we have $\overline{\pi(H^{\mathrm{nc}})} = \pi(H)$. Now $N_G(H) \subset N_G(H^{\mathrm{nc}})$. If $\gamma \in \Gamma \cap N_G(H^{\mathrm{nc}})$, then

$$\gamma\pi(H) = \overline{\gamma\pi(H^{\mathrm{nc}})} = \overline{\pi(H^{\mathrm{nc}})} = \pi(H).$$

Hence,

$$\Gamma \cap N_G(H^{\mathrm{nc}}) = \Gamma \cap N_G(H). \tag{41}$$

Also,

$$N(H^{\mathrm{nc}}, W) = N(H, W). \tag{42}$$

Hence, by (40),

$$N(H, W) = \{g \in G : gp_{H^{\mathrm{nc}}} \in \mathcal{A}\}. \tag{43}$$

PROPOSITION 4.3
If $G = \mathrm{SO}(n, 1)$ and if H is reductive, then $Gp_{H^{\mathrm{nc}}}$ is closed.

Proof
Note that H^{nc} is conjugate to $\mathrm{SO}(k, 1)$. Since $N_G(\mathrm{SO}(k, 1))^0$ is a symmetric subgroup of $\mathrm{SO}(n, 1)$ (i.e., a subgroup of finite index in the set of fixed points of an involution; see [5, pages 284–285]), $N_G(H^{\mathrm{nc}})$ is a symmetric subgroup of $\mathrm{SO}(n, 1)$. Since $N_G(H^{\mathrm{nc}})^0 p_{H^{\mathrm{nc}}} = p_{H^{\mathrm{nc}}}$, by [7, Corollary 4.7] the orbit $Gp_{H^{\mathrm{nc}}}$ is closed. □

COROLLARY 4.4
If $G = \mathrm{SO}(n, 1)$ and if $H \in \mathcal{H}$ is reductive, then $\Gamma p_{H^{\mathrm{nc}}}$ is discrete.

Proof
Since $H\Gamma$ is closed and since $N_G^1(H^{\mathrm{nc}})/H$ is compact, we have that $N_G^1(H^{\mathrm{nc}})\Gamma$ is closed. Therefore, $\Gamma N_G^1(H^{\mathrm{nc}})$ is closed. Since $Gp_{H^{\mathrm{nc}}}$ is closed and since $N_G^1(H^{\mathrm{nc}}) = \mathrm{Stab}(p_{H^{\mathrm{nc}}})$, we conclude that $\Gamma p_{H^{\mathrm{nc}}}$ is closed and hence discrete. □

4.1.1. Linearization
Let $p_0 = p_H$, or let $p_0 = p_{H^{\mathrm{nc}}}$, and suppose that Γp_0 is closed. Given any compact set $\mathcal{D} \subset \mathcal{A}$, we define

$$S(\mathcal{D}) = \big\{g \in N(H, W) : g\gamma p_0 \in \mathcal{D} \text{ for some } \gamma \in \Gamma \setminus N_G(H)\big\}.$$

PROPOSITION 4.5 ([11, Proposition 3.2])

(1) *We have $S(\mathcal{D}) \subset S(H, W)$ and $\pi(S(\mathcal{D}))$ is closed in G/Γ.*

(2) *For any compact set $\mathcal{K} \subset G/\Gamma \setminus \pi(S(\mathcal{D}))$, there exists a neighborhood Φ of $\mathcal{D}$ in V such that for any $g \in G$ and $\gamma_1, \gamma_2 \in \Gamma$,*

$$\text{if } \pi(g) \in \mathcal{K} \text{ and } \{g\gamma_1 p_0, g\gamma_2 p_0\} \subset \overline{\Phi}, \text{ then } \gamma_1 p_0 = \pm\gamma_2 p_0, \tag{44}$$

where $\overline{\Phi}$ denotes the closure of Φ in V.

Proof

The proof can be deduced using the following two observations. First, since Gp_0 is discrete, the map $G/\operatorname{Stab}(p_0) \cap \Gamma \to G/\Gamma \times V$ defined via $g \mapsto (\pi(g), gp_0)$ is proper. Second, if $gp_0 \in \mathcal{D}$ and if $g\gamma p_0 \in \mathcal{D}$ for some $\gamma \in \Gamma$, then by (40) or (43), $g \in N(H, W)$ and $g\gamma \in N(H, W)$. If $\gamma p_0 \neq \pm p_0$, then $\gamma \notin N_G(H^{\mathrm{nc}})$. Then by (41), $\gamma \notin N_G(H)$. Since $g \in N(H, W) \cap N(H, W)\gamma^{-1}$, by (38) we have $g \in S(H, W)$. □

Let $\mathscr{F}(G)$ be defined as in Section 2.1. We say that $S \subset V$ is symmetric if $S = -S$.

PROPOSITION 4.6 (cf. [2])

Given a symmetric compact set $\mathcal{C} \subset \mathcal{A}$ and $\epsilon > 0$, there exists a symmetric compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that, given a symmetric neighborhood Φ of $\mathcal{D}$ in V, there exists a symmetric neighborhood Ψ of $\mathcal{C}$ in V contained in Φ such that for any $\psi \in \mathscr{F}(G)$, for any $v \in V$, and for any interval $J \subset I$, one of the following holds:

(I) $\psi(t)v \in \Phi$ *for all* $t \in J$;

(II) $|\{t \in J : \psi(t)v \in \Psi\}| \leq \epsilon |\{t \in J : \psi(t)v \in \Phi\}|$.

Proof

The argument in the proof of [2, Proposition 4.2] goes through with straightforward changes. Since $\mathcal{A}$ is a linear subspace of V, one can describe the neighborhoods of subsets of $\mathcal{A}$ in V via linear functionals. Further, one uses the property (15) of the functions in $\mathscr{F}$ instead of [2, Lemma 4.1] in the proof. □

4.2. Linear presentation of dynamics in injective neighborhoods of singular sets

PROPOSITION 4.7

Suppose that an $\epsilon > 0$, a compact set $\mathcal{K} \subset G/\Gamma$, open symmetric subsets Φ and Ψ of V, a countable set $\Sigma \subset V$, a bounded interval J, and a continuous map $\psi : J \to G$ are chosen such that the following two conditions are satisfied.

(C1) *For any $g \in G$ and $v_1, v_2 \in \Sigma$,*

$$\text{if } gv_1, gv_2 \in \overline{\Phi} \text{ and if } \pi(g) \in \mathcal{K}, \text{ then } v_1 = \pm v_2. \tag{45}$$

(C2) *For any $v \in \Sigma$, if we define*

$$E_v = \{s \in J : \psi(s)v \in \Psi\} \qquad \text{and} \qquad F_v = \{s \in J : \psi(s)v \in \Phi\}, \tag{46}$$

then $|J_1 \cap E_v| \leq \epsilon |J_1|$ for any connected component J_1 of F_v.

Then

$$\left|\{t \in J : \psi(t)\Sigma \cap \Psi \neq \emptyset,\ \pi(\psi(t)) \in \mathcal{K}\}\right| \leq 2\epsilon |J|. \tag{47}$$

Proof
Let $J^* = \{t \in J : \pi(\psi(t)) \in \mathcal{K},\ \psi(t)\Sigma \cap \Psi \neq \emptyset\}$. Let $t \in J^*$. By (C1), there exists, up to a $\pm$ sign, a unique $v_t \in \Sigma$ such that $t \in E_{v_t}$. Let J_t be the component of F_{v_t} containing t. We claim that

$$J^* \cap J_t = E_{v_t}. \tag{48}$$

To verify this, let $s \in J^* \cap J_t$. Then $\psi(s)v_s \in \Psi$ and $\psi(s)v_t \in \Phi$. By (C1), $v_s = \pm v_t$. Hence, $\psi(s)v_t \in \Psi$; that is, $s \in E_{v_t}$. Therefore, by (C2), we get

$$|J^* \cap J_t| \leq |E_{v_t} \cap J_t| \leq \epsilon |J_t|. \tag{49}$$

As we have verified above, $J_s = J_t$ for all $s \in J^* \cap J_t$. Therefore, there exists a countable set $\mathcal{J}^* \subset J^*$ such that

$$J^* \subset \bigcup_{t \in \mathcal{J}^*} J_t, \tag{50}$$

and if $t_1 \neq t_2$ in $\mathcal{J}^*$, then $t_1 \notin J_{t_2}$.

In particular, if $t_1 < t_2$ in $\mathcal{J}^*$, then $J_{t_1} \cap J_{t_2} \subset (t_1, t_2)$. Therefore, if $t_1 < t_2 < t_3$ in $\mathcal{J}^*$, then

$$J_{t_1} \cap J_{t_2} \cap J_{t_3} = \emptyset.$$

Hence,

$$\sum_{t \in \mathcal{J}^*} |J_t| \leq 2 \Big\| \bigcup_{t \in \mathcal{J}^*} \Big\| J_t. \tag{51}$$

Now by (49), (50), and (51), we have

$$|J^*| \leq \epsilon \sum_{t \in \mathcal{J}^*} |J_t| \leq (2\epsilon)|J|. \qquad \square$$

4.3. Algebraic consequences of positive limit measure on singular sets
Let $\{a_i\} \subset A$ and $x_i \xrightarrow{i \to \infty} x$ be the sequences involved in the definition of λ_i (see (28)).

In view of (23), $V = V^+ \oplus V^0 \oplus V^-$. Let $q^+ : V \to V^+$ and $q^{+0} : V \to V^+ + V^0$ denote the corresponding projections.

We recall that after passing to a subsequence, $\lambda_i \to \lambda$ in the space of probability measures on G/Γ, and by Theorems 3.1 and 4.1, we know that there exists $H \in \mathcal{H}$ such that

$$\lambda\big(\pi(N(H, W) \setminus S(H, W))\big) > 0. \tag{52}$$

The goal of this section is to analyze this condition using Proposition 4.7 and Corollary 2.4 to obtain its following algebraic consequence.

PROPOSITION 4.8
The group H is reductive. Let $p_0 = p_H$, or let $p_0 = p_{H^{\mathrm{nc}}}$. Let $g \in G$ be such that $x = \pi(g)$. Then there exists $\gamma \in \Gamma$ such that

$$u\bigl(\varphi(t)\bigr)g\gamma p_0 \subset V^0 + V^-, \quad \forall t \in I. \tag{53}$$

Proof
Let $p_0 = p_H$ or let $p_0 = p_{H^{\mathrm{nc}}}$ such that Γp_0 is discrete. By (52), there exists a compact set $C \subset N(H, W) \setminus S(H, W)$ such that $\lambda(\pi(C)) > c_0 > 0$ for some constant $c_0 > 0$. Let $\mathcal{C} = Cp_0 \cup -Cp_0$. We fix $0 < \epsilon < c_0/2$. We apply Proposition 4.6 to obtain a symmetric compact set $\mathcal{D} \subset \mathcal{A}$. Therefore, by Proposition 4.5, there exists a compact set $\mathcal{K} \subset G/\Gamma \setminus \pi(S(\mathcal{D}))$ such that $\pi(C)$ is contained in the interior of $\mathcal{K}$. Also, there exists a symmetric neighborhood Φ of $\mathcal{D}$ in V such that property (44) holds. Then we obtain a symmetric neighborhood Ψ of $\mathcal{C}$ in V such that the conclusion of Proposition 4.6 holds. Let

$$\mathcal{O} := \bigl\{\pi(h) : \pi(h) \in \mathcal{K}, hp_0 \in \Psi,\ h \in G\bigr\}. \tag{54}$$

Then $\mathcal{O}$ is a neighborhood of $\pi(C)$ in G/Γ.

Choose $i_1 \in \mathbb{N}$ such that

$$\lambda_i(\mathcal{O}) > c_0 \quad \text{for all } i \geq i_1. \tag{55}$$

Since $x_i \xrightarrow{i\to\infty} x$ and since $\pi(g) = x$, there exists a convergent sequence $g_i \xrightarrow{i\to\infty} g$ in G such that $\pi(g_i) = x_i$ for all $i \in \mathbb{N}$. By (55) and (28), since $z(t) \in Z_G(A)$, we have

$$\bigl|\bigl\{t \in I : \pi(a_i z(t)u(\varphi(t))g_i) \in \mathcal{O}\bigr\}\bigr| > c_0|I|, \quad \forall i \geq i_1. \tag{56}$$

We fix $i \geq i_1$, and we let $\psi(t) = a_i z(t)u(\varphi(t))g_i$ for all $t \in I$. Then $\psi \in \mathscr{F}(G)$. Let

$$\Sigma = \bigl\{v \in \Gamma p_0 : \psi(I)v \not\subset \Phi\bigr\}.$$

For any $v \in \Sigma$, define E_v and F_v as in (46) for $J = I$. Then $\overline{\psi(F_v)} \not\subset \Phi$. Therefore, if F_1 is any connected component of F_v, then by our choice of Ψ, we have $|E_v \cap F_1| \leq \epsilon|F_1|$. Therefore, the conditions of Proposition 4.7 are satisfied. If $\Sigma = \Gamma p_0$, then by (47),

$$\bigl|\bigl\{t \in I : \pi(\psi(t)) \in \mathcal{K},\ \psi(t)\Gamma p_0 \cap \Psi \neq \emptyset\bigr\}\bigr| \leq 2\epsilon|I|. \tag{57}$$

In other words, since $2\epsilon < c_0$, we have

$$\left|\left\{t \in I : \pi(\psi(t)) \in \mathcal{O}\right\}\right| < c_0|I|,$$

a contradiction of (56). Therefore, for each $i \geq i_1$, there exists $\gamma_i \in \Gamma$ such that $\gamma_i p_0 \notin \Sigma$; that is,

$$a_i z(t) u\big(\varphi(t)\big) g_i \gamma_i p_0 \in \Phi, \quad \forall t \in I. \tag{58}$$

Since $\{z(t) : t \in I\}$ is contained in a compact set, there exists $R > 0$ such that $z(I)^{-1}\Phi$ is contained in $B(R)$, the ball of radius R centered at zero in V. Thus,

$$\left\|a_i u(\varphi(t)) g_i \gamma_i p_0\right\| \leq R, \qquad \forall t \in I. \tag{59}$$

Fix any $t_1 \in I$. Since φ is a nonconstant function, by Corollary 2.4 there exists a constant $\kappa > 0$ such that

$$\sup_{t \in I} \left\|q^{+0}(u(\varphi(t))v)\right\| \geq \kappa \left\|u(\varphi(t_1))v\right\|, \quad \forall v \in V. \tag{60}$$

Let $t \in I$. Then by (59),

$$\left\|q^{+0}(u(\varphi(t)) g_i \gamma_i p_0)\right\| = \left\|a_i q^{+0}(u(\varphi(t)) g_i \gamma_i p_0)\right\| \leq \left\|a_i u(\varphi(t)) g_i \gamma_i p_0\right\| \leq R.$$

Therefore, by (60),

$$\|g_i \gamma_i p_0\| \leq \kappa^{-1} \left\|u(\varphi(t_1))^{-1}\right\| R. \tag{61}$$

Since Γp_0 is discrete and since $g_i \xrightarrow{i \to \infty} g$, due to (61) the set $\{\gamma_i p_0 : i \in \mathbb{N}\}$ is finite. Therefore, by passing to a subsequence, there exists $\gamma \in \Gamma$ such that $\gamma_i p_0 = \gamma p_0$ for all $i \in \mathbb{N}$, and hence,

$$a_i z(t) u\big(\varphi(t)\big) g_i \gamma p_0 \in \Phi, \quad \forall i \in \mathbb{N}. \tag{62}$$

For each $i \in \mathbb{N}$, if $w_i^+ = q^+\big(z(t) u(\varphi(t)) g_i \gamma p_0\big) \in V^+$, then by (62) we have $\limsup_{i \to \infty} \|a_i w_i^+\| < \infty$. Since $\alpha(a_i) \xrightarrow{i \to \infty} \infty$, we conclude that $w_i^+ \xrightarrow{i \to \infty} 0$. Since $g_i \xrightarrow{i \to \infty} g$, we have

$$q^+\big(z(t) u(\varphi(t)) g \gamma p_0\big) = \lim_{i \to \infty} q^+\big(z(t) u(\varphi(t)) g_i \gamma p_0\big) = \lim_{i \to \infty} w_i^+ = 0.$$

Hence, $z(t) u(\varphi(t)) g \gamma p_0 \subset V^0 + V^-$. Since $Z_G(A)(V^0 + V^-) = V^0 + V^-$ and $z(t) \in Z_G(A)$, (53) follows.

Due to (56) there exists $t \in I$ such that $a_i z(t)u(\varphi(t))g\gamma p_H \to gp_H \in V^0$ for some $g \in N(H, W)$. Hence, $A \subset g N_G(H)g^{-1}$. Since $G = \mathrm{SO}(n, 1)$, this condition implies that H is reductive. □

4.4. Intersection of N-orbits with weak stable subspace

PROPOSITION 4.9

Suppose that H is reductive, that $H \neq G$, and that $p_0 = p_{H^{nc}}$. For $v \in Gp_0$, define

$$S_v = \{x \in \mathbb{R}^{n-1} : u(x)v \in V^- + V^0\}. \tag{63}$$

Suppose also that $S_v \neq \emptyset$. Then there exist $g_1 \in G$ and a simple subgroup F of G containing A such that $(g_1 F g_1^{-1})v = v$ and

$$S_v = \{x \in \mathbb{R}^{n-1} : u(x) \in P^- F g_1^{-1}\}, \tag{64}$$

which is a subsphere of a sphere in $\mathbb{R}^{n-1}$ or a proper affine subspace of $\mathbb{R}^{n-1}$.

Proof

Let $x_1 \in S_v$. Since $Gv = Gp_0$ is closed, there exists $h_1 \in G$ such that $p_1 := \lim_{i\to\infty} a_i u(x_1)v = h_1 p_0$. Then p_1 is fixed by A. Let $F = h_1 H h_1^{-1}$. Then $N_G(F) = \mathrm{Stab}(p_1)$, and since $N_G(F)/F$ is compact, $A \subset F$. Now $g N_G(F) \mapsto gp : G/N_G(F) \to Gp$ is a homeomorphism. Let $g_1 \in G$ be such that $v = g_1 p_1$. Then for every $x \in S_v$ there exists $\xi(x) \in G$ such that

$$a_i u(x) g_1 p_1 \xrightarrow{i\to\infty} \xi(x)p_1 \qquad \text{and} \qquad a_i u(x) g_1 N_G(F) \xrightarrow{i\to\infty} \xi(x) N_G(F). \tag{65}$$

Since $A \subset F$, there exists a Weyl group "element" w of G contained in F such that $w = w^{-1}$, $waw^{-1} = a^{-1}$ for all $a \in A$, and G admits a Bruhat decomposition (see [12, Section 12.14]):

$$G = P^- w P^- \cup P^- = P^- N \cup P^- w. \tag{66}$$

Let $x \in S_v$. Since $w \in F$, and since $P^- = N^- AM$, by (66) there exist $n^-(x) \in N^-$, $\zeta(x) \in M$, $b(x) \in A$, and $X \in \mathfrak{n}$ such that

$$u(x) g_1 F = n^-(x)\zeta(x)b(x)\exp(X)F. \tag{67}$$

Therefore,

$$a_i u(x) g_1 p_1 = \big(a_i n^-(x) a_t^{-1}\big)\zeta(x)b(x)\exp\big(\mathrm{Ad}\, a_i(X)\big)p_1. \tag{68}$$

Now $a_i n^-(x) a_i^{-1} \to e$ as $i \to \infty$. Therefore, by (65),

$$\exp\big(\mathrm{Ad}\, a_i(X)\big)p_1 \xrightarrow{t\to\infty} b(x)^{-1}\zeta(x)^{-1}\xi(x)p_1. \tag{69}$$

Since N is a unipotent group, the orbit U^+p_1 is a closed affine variety, and hence the map

$$h\big(N \cap N_G(F)\big) \mapsto hp_1 : N/\big(N \cap N_G(F)\big) \to V \tag{70}$$

is proper. If $\exp(X) \notin N_G(F)$, then $\exp(\operatorname{Ad} a_i(X))(N \cap N_G(F))$ escapes to infinity in $N/(N \cap N_G(F))$ as $i \to \infty$. On the other hand, Mp_1 is compact. Therefore, from (69) we conclude that $\exp(X) \in N_G(F)$. Since $N_G(F)/F$ is compact, $\exp(X) \in F$. Therefore, by (67) we have

$$u(x)g_1F = n^-(x)\zeta(x)F \subset P^-F. \tag{71}$$

Therefore, $S_v \subset \{x \in \mathbb{R}^{n-1} : u(x) \in P^-Fg_1^{-1}\}$. The converse inclusion holds because $Fg_1^{-1}v = p_1 \in V^0$ and $P^-V^0 \subset V^0 + V^-$. Thus, (64) holds.

Let $\mathcal{I} : G \to P^-\backslash G \cong \mathbb{S}^{n-1}$ be the map as defined in Section 1.1. Let $\mathcal{S} : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$ be the map defined by $\mathcal{S}(x) = \mathcal{I}(u(x))$ for all $x \in \mathbb{R}^{n-1}$. Since F is a proper noncompact simple subgroup of $G = \mathrm{SO}(n,1)$ containing A, we conclude that $F = m\,\mathrm{SO}(k,1)m^{-1}$ for some $2 \le k \le n-1$ and that $m \in M$. Hence, $\mathcal{I}(Fg_1^{-1}) = \mathcal{I}(\mathrm{SO}(k,1))mg_1^{-1}$. Now $\mathcal{I}(\mathrm{SO}(k,1)) \cong \mathbb{S}^{k-1}$, and the G-action on $\mathbb{S}^{n-1} \cong P^-\backslash G$ from the right is conformal. Therefore, from (64) we deduce that

$$S_v = \mathcal{S}^{-1}\big(\mathcal{I}(Fg_1^{-1})\big) = \mathcal{S}^{-1}(\mathbb{S}^{k-1}mg_1^{-1}) \tag{72}$$

is a subsphere of a sphere in $\mathbb{R}^{n-1}$ or a proper affine subspace of $\mathbb{R}^{n-1}$. □

COROLLARY 4.10

Let notation be as in Proposition 4.9. Let $g \in G$ be such that $v = gp_0$. Then there exists $h_1 \in G$, and for every $x \in S_{gp_0}$ there exists $\zeta(x) \in M$ such that $A \subset h_1Hh_1^{-1}$,

$$u(x)gH \subset N^-\zeta(x)h_1H, \tag{73}$$

and

$$\lim_{i\to\infty} a_iu(x)gH = \zeta(x)h_1H. \tag{74}$$

Proof

By (71), $u(x)g_1 = n^-(x)\zeta(x)F$. Therefore,

$$\lim_{i\to\infty} a_iu(x)g_1F = \zeta(x)F \quad \text{in } G/F. \tag{75}$$

Now $gp_0 = v = g_1p_1 = g_1h_1p_0$ and $F = h_1Hh_1^{-1}$. Therefore, $g^{-1}g_1h_1 \in N_G(H)$. Hence, replacing h_1 by an appropriate element of $h_1N_G(H)$, we may assume that

$g^{-1}g_1h_1 \in H$. Thus, we have $gH = g_1h_1H$ and $h_1H = Fh_1$. Therefore, (75) and (74) are equivalent. □

4.5. Algebraic description of the measure λ

If $H = G$, then λ is G-invariant. Therefore, we may assume that $\dim(H) < \dim(G)$. By Proposition 4.8, H is reductive and $u(\varphi(I))g\gamma p_{H^{\mathrm{nc}}} \subset V^0 + V^-$. Hence, by Proposition 4.9, $\varphi(I)$ is contained in a proper affine subspace or a subsphere of $\mathbb{R}^{n-1}$. Therefore, if φ does not satisfy this condition, then λ must be G-invariant.

Now, suppose that $\varphi(I)$ is contained in a proper affine subspace or a sphere in $\mathbb{R}^{n-1}$. Then we have assumed that $x_i = x$ for all i.

Consider the natural quotient map $q : G/H \cap \Gamma \to G/\Gamma \times G/H$. For $i = 1, 2$, let q_i denote the projection on the ith factor. Let $\tilde{\lambda}_i$ be the probability measure on $G/H \cap \Gamma$ defined by

$$\int_{G/H\cap\Gamma} f(y)\tilde{\lambda}_i(y) := \frac{1}{|I|}\int_{t\in I} f\big(z(t)a_iu(\varphi(t))x\big)\,\mathrm{d}t, \quad \forall f \in \mathrm{C_c}(G/H\cap\Gamma). \quad (76)$$

Then $(q_1)_*(\tilde{\lambda}_i) = \lambda_i$. In view of Corollary 4.10, let ν denote the probability measure on G/H such that

$$\int_{G/H} f\,\mathrm{d}\nu = \frac{1}{|I|}\int_{t\in I} f\big(z(s)\zeta(\varphi(s))h_1\big)\,\mathrm{d}s.$$

Then by (74), we have $(q_2)_*(\tilde{\lambda}_i) \to \nu$. Therefore, there exists a probability measure $\tilde{\lambda}$ on $G/H \cap \Gamma$ such that, by passing to a subsequence, $\tilde{\lambda}_i \to \tilde{\lambda}$. As we observed previously, $\tilde{\lambda}_i$ is W-invariant. Therefore, ν is W-invariant. Hence, support of ν is pointwise fixed by W. Therefore, ν is supported on $N(H, W)/H$. Hence, we conclude that $\tilde{\lambda}$ is supported on $N(H, W)/(H \cap \Gamma)$. Therefore, λ is concentrated on $\pi(N(H, W)) \smallsetminus \pi(S(H, W))$. Therefore, by Ratner's theorem, almost every ergodic component of $\tilde{\lambda}$ is of the form $g\mu_H$ for some $g \in N(H, W)$. From this we conclude that for any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\int_{G/\Gamma} f\,\mathrm{d}\lambda = \frac{1}{|I|}\int_{t\in I}\Big(\int_{G/\Gamma} f\big(z(t)\zeta(\varphi(t))h_1y\big)\,\mathrm{d}\mu_H(y)\Big)\,\mathrm{d}t. \quad (77)$$

As a consequence of this description of the measure λ, we deduce the following. Recall that by (14) and Corollary 2.5, after passing to a subsequence, $\mu_i \to \mu$ as $i \to \infty$.

PROPOSITION 4.11

For any $f \in \mathrm{C_c}(G/\Gamma)$*, we have*

$$\int_{G/\Gamma} f\,\mathrm{d}\mu = \frac{1}{|I|}\int_{t\in I}\int_{G/\Gamma} f\big(\zeta(\varphi(t))h_1y\big)\,\mathrm{d}\mu_H(y)\,\mathrm{d}t. \quad (78)$$

In particular, if $\varphi(I)$ is not contained in a proper affine subspace or a sphere in $\mathbb{R}^{n-1}$, then $H = G$ and hence $\mu = \mu_G$.

Proof
Let $f \in C_c(G/\Gamma)$ and $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if J is any subinterval of I with $0 < |J| < \delta$, then

$$\left| f(z(t_0) \in z(t)y) - f(y) \right| \leq \epsilon, \quad \forall t_0, t \in J. \tag{79}$$

We define the measures λ_i^J as in (28) for J in place of I. Then after passing to a subsequence, we may assume that $\lambda_i^J \to \lambda^J$ as $i \to \infty$ and λ^J is W-invariant. We note that if $\lambda\big(\pi(N(F, W))\big) = 0$, then $\lambda^J\big(\pi(N(F, W))\big) = 0$ for any $F \in \mathcal{H}$. Moreover, since φ is an analytic map, we also obtain that (77) holds for J in place of I and λ^J in place of λ.

Fix any $t_0 \in J$, and let $f_0(y) = f(z(t_0)^{-1}y)$ for all $y \in G/\Gamma$. By (79),

$$\begin{aligned} &\int_{t\in J} f\big(a_i\, u(\varphi(t))x_i\big)\,\mathrm{d}t \\ &\qquad \overset{\epsilon}{\approx} \int_{t\in J} f\big(z(t_0)^{-1}z(t)a_i u(\varphi(t))x_i\big)\,\mathrm{d}t \\ &\qquad = |J| \int_{G/\Gamma} f_0(y)\,\mathrm{d}\lambda_i^J(y) \\ &\qquad \overset{\epsilon}{\approx} |J| \int_{G/\Gamma} f_0(y)\lambda^J(y) \quad (\forall i \geq i_J \text{ for some } i_J) \\ &\qquad = \int_{t\in J} \Big(\int_{y\in G/\Gamma} f_0\big(z(t)\zeta(\varphi(t))h_1 y\big)\,\mathrm{d}\mu_H(y) \Big)\,\mathrm{d}t \quad \text{(by (77))} \\ &\qquad \overset{\epsilon}{\approx} \int_{t\in J} \Big(\int_{y\in G/\Gamma} f\big(\zeta(\varphi(t))h_1 y\big)\,\mathrm{d}\mu_H(y) \Big)\,\mathrm{d}t. \end{aligned} \tag{80}$$

Therefore, by partitioning I into finitely many subintervals J with $0 < |J| < \delta$, we get

$$|I| \int_{G/\Gamma} f\,\mathrm{d}\mu_i \overset{3\epsilon}{\approx} \int_{t\in I} \Big(\int_{G/\Gamma} f\big(\zeta(\varphi(t))h_1 y\big)\,\mathrm{d}\mu_H(y) \Big)\,\mathrm{d}t. \tag{81}$$

Since $\mu_i \to \mu$ as $i \to \infty$, (78) follows. □

5. Proofs of results stated in the introduction

Proof of Theorem 1.8
Since φ is analytic and nonconstant, the set $E := \{t \in I : \dot{\varphi}(t) = 0\}$ is finite. It is enough to prove the theorem for all compact intervals J contained in $I \setminus E$ with nonempty interiors.

Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be sequences in A^+ such that for any $X \in \mathrm{Lie}(N)$, $\mathrm{Ad}(a_i)^{-1}X \to 0$ and $\mathrm{Ad}(b_i)^{-1}(X) \to 0$ as $i \to \infty$. We note that if J is a subinterval of I with a nontrivial interior, and if μ_i^J (and, resp., ν_i^J) denotes the measures as defined in (14) for J in place of I (and, resp., b_i in place of a_i), then by Corollary 2.5, after passing to subsequences, $\mu_i^J \to \mu^J$ and $\nu_i^J \to \nu^J$ as $i \to \infty$. By Proposition 4.11,

$$\mu^J = \int_{t\in J} \zeta_1(t)h_1\mu_{H_1}\,\mathrm{d}t \qquad \text{and} \qquad \nu^J = \int_{t\in J} \zeta_2(t)h_2\mu_{H_2}\,\mathrm{d}t, \tag{82}$$

where the h_i's and H_i's satisfy the conditions of Corollary 4.10 and they are independent of the choice of J. Again, due to Corollary 4.10, we see that ν^J is concentrated on $\zeta_1(J)h_1\pi(H_1)$. Therefore,

$$\zeta_2(J)h_2\pi(H_2) \subset \zeta_1(J)h_1\pi(H_1).$$

By symmetry, we have equality in the above expression. Since the interval J is arbitrary, we deduce that

$$\zeta_2(t)h_2\pi(H_2) = \zeta_1(t)h_1\pi(H_1), \quad \forall t \in I.$$

Hence, by (82), we have $\mu^I = \nu^I$. Combining this information with Corollary 4.10, it is straightforward to deduce all the conclusions of the theorem. □

Proof of Theorem 1.7

In view of the argument as in the proof of Theorem 1.8, the result follows from the first paragraph of Section 4.5. □

Proof of Theorem 1.2

Let $G = \mathrm{SO}(n,1)$, let $K = \mathrm{SO}(n)$, and let P^- be a maximal parabolic subgroup of G such that $P^- \cap K = \mathrm{SO}(n-1)$. Let A be the maximal $\mathbb{R}$-diagonalizable subgroup of G centralizing $P^- \cap K$. Then $A \subset P^-$. Now G admits a transitive right action on $T^1(\mathbb{H}^n)$ via isometries. We fix $\tilde{x}_0 \in \mathbb{H}^n$ such that $K = \mathrm{Stab}_G(\tilde{x}_0)$, and we fix $v_0 \in S_{\tilde{x}_0}(\mathbb{H}^n)$ such that

$$K_0 := \mathrm{Stab}_K(v_0) = Z_G(A) \cap K.$$

Thus, $T^1(\mathbb{H}^n) \cong K_0\backslash G$ and $S_{\tilde{x}_0}(\mathbb{H}^n) \cong K_0\backslash K$. Under this isomorphism, the geodesic flow $\{\tilde{g}_t\}$ on $T^1(\mathbb{H}^n)$ corresponds to the action of $\{a_t\} = A$ on $K_0\backslash G$ by left multiplications, where $\alpha(a_t) = e^{\tau t}$ for all $t \in \mathbb{R}$ and some $\tau > 0$.

There exists a discrete subgroup Γ of G such that $\pi : \mathbb{H}^n \to M$ factors through $\mathbb{H}^n/\Gamma$ and $M \cong \mathbb{H}^n/\Gamma$ as isometric Riemannian manifolds. Hence, $T^1(M) \cong K_0\backslash G/\Gamma$, and the geodesic flow $\{g_t\}$ on $T^1(M)$ corresponds to the left action of $\{a_t\}$ on $K_0\backslash G/\Gamma$.

There exists an analytic map $\theta : I \to G$ such that

$$\psi(t) = D\pi\big(v_0\theta(t)\big), \quad \forall t \in I.$$

Let $\varphi : I \to \mathbb{R}^{n-1}$ be the map such that $\theta(t) \subset P^- u(\varphi(t))$ for all $t \in I$. Then by Theorem 1.8 for $x = e\Gamma \in G/\Gamma$, there exist $H \in \mathcal{H}$, $h_1 \in G$, and $\gamma \in \Gamma$ such that $Ah_1 \subset h_1 H$, and by (13), $u(\varphi(I)) \subset P^- h_1 H \gamma^{-1}$. Therefore,

$$\theta(t) \subset P^- u\big(\varphi(t)\big) \subset P^- h_1 \gamma^{-1} = K_0 N^- h_1 H \gamma^{-1}, \quad \forall t \in I. \tag{83}$$

Therefore, since $\pi(v_0 K_0 g \Gamma) = \pi(v_0 g)$, we have

$$\pi\big(v_0 \theta(t)\big) \subset \pi(v_0 N^- h_1 H), \qquad A \subset h_1 H h_1^{-1}, \qquad \text{and} \qquad N \cap h_1 H h_1^{-1} \neq \{e\}.$$

Therefore, there exists $k_1 \in K_0$ such that

$$K_0\, k_1 h_1 H h_1^{-1} k_1^{-1} = K_0\, \mathrm{SO}(m, 1), \quad \text{where } 2 \leq m \leq n.$$

Now $v_0\, \mathrm{SO}(m, 1) \cong T^1(\mathbb{H}^m)$, where $\mathbb{H}^m$ is isometrically embedded in $\mathbb{H}^n$. Since $H\Gamma/\Gamma$ is a closed subset of G/Γ,

$$\pi(v_0 h_1 H) = \pi\big(v_0\, \mathrm{SO}(m, 1) k_1 h_1\big) = \pi\big(T^1(\mathbb{H}^m) h_2\big)$$

is a closed subset of M, where $h_2 = k_1 h_1 \in G$. Therefore, $K_0\, h_1 H/(H \cap \Gamma)$ corresponds to the embedding of $D\,\Phi(T^1(M_1))$ in $T^1(M)$, where Φ is a totally geodesic immersion of a hyperbolic manifold M_1 in M (see [16, Section 2] for the details). It may also be noted that the projection of $h_1 H h_1^{-1}$-invariant probability measure (say, μ_1) on $h_1 H/(H \cap \Gamma)$ onto $K_0 \backslash G/\Gamma \cong M$ (say, $\bar{\mu}_1$) is the same as the projection under $D\,\Phi$ of the normalized measure on $T^1(M_1)$ associated to the Riemannian volume form on M_1.

By (12) of Theorem 1.8, for any subinterval J of I with nonempty interior and for any $f \in \mathrm{C_c}(K_0 \backslash G/\Gamma)$, we have

$$\lim_{t\to\infty} \frac{1}{|J|} \int_J f\big(K_0\, a_t u(\varphi(s))\Gamma\big)\, \mathrm{d}s = \int_{K_0\, h_1 H\Gamma/\Gamma} f(y)\, \mathrm{d}\bar{\mu}_1(y). \tag{84}$$

Recall that $\theta(s) \in P^- u(\varphi(s))$ for all $s \in I$ and that $\bar{\mu}_1$ is $Z_G(A)$-invariant with respect to the left action. Therefore, by the uniform continuity argument as in the proof of Proposition 4.11, we deduce that

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f\big(K_0\, a_t \theta(s)\Gamma\big)\, \mathrm{d}s = \int_{K_0\, h_1 H\Gamma/\Gamma} f(y)\, \mathrm{d}\bar{\mu}_1(y). \tag{85}$$

Now, in view of the relation between the closed $h_1 H h_1^{-1}$-orbits with totally geodesic immersions of hyperbolic manifolds of finite volume as described above, (85) implies (3). $\square$

Proof of Corollary 1.5
Let $\tilde{x} \in \mathbb{H}^n$ such that $x = \pi(\tilde{x})$. We can identify $T^1{}_x(M)$, the unit tangent sphere at x, with $T^1{}_{\tilde{x}}(\mathbb{H}^n)$, which in turn identifies with the ideal boundary sphere $\partial\mathbb{H}^n$ via the visual map. Since all these identifications are conformal, we conclude that $\mathrm{Vis}(\tilde{\theta}(I))$ is not contained in any proper subsphere of $\partial\mathbb{H}^n$. Therefore, in terms of the notation in Remark 1.1, $\mathbb{S}^{k-1} = \partial\mathbb{H}^n$, and we conclude that $M_1 = M$ and that Φ is the identity map. Now the conclusion follows from Theorem 1.6. □

Proof of Theorem 1.4
The proof is similar to the proof of Corollary 1.5. □

Proof of Theorem 1.1
We identify $\mathbb{S}^{n-1}$ with a hyperbolic sphere of radius 1 centered at zero in $\mathbb{H}^n$ (in the unit ball B^n-model)—say, S—and treat $\bar{\psi}$ as a map from I to S. For any $s \in I$, let $v_s \in T^1{}_{\bar{\psi}(s)}(\mathbb{H}^n)$ be the unit vector normal to S which is also a tangent to the directed geodesic from zero to $\bar{\psi}(s)$. We define an analytic curve $\psi : I \to T^1(M)$ by

$$\psi(s) = \big(\pi(\bar{\psi}(s)), D\pi(v_s)\big), \quad \forall s \in I.$$

Therefore, the condition of Theorem 1.4 is satisfied because

$$\mathrm{Vis}\,\big(\tilde{\psi}(s)\big) = \mathrm{Vis}\,\big((\bar{\psi}(s), v_s)\big) = \bar{\psi}(s), \quad \forall s \in I,$$

and hence, $\mathrm{Vis}(\tilde{\psi}(s))$ is not contained in a proper subsphere of $\partial\mathbb{H}^n$.

For any $\alpha > 0$, we have $\pi(\alpha\bar{\psi}(s)) = g_{t(\alpha)}\pi(\psi(s))$ for some $t(\alpha) > 0$ such that $t(\alpha) \to \infty$ as $\alpha \to 1^-$. Therefore, (2) follows from Theorem 1.4. □

6. Scope for generalizations and applications

The results of this article lead to obvious similar questions about expanding translates of smooth curves on horospherical subgroups of general semisimple Lie groups. Especially, the affirmative answer to the following question has interesting applications to problems in Diophantine approximation (see [8], [9]).

QUESTION 6.1
Let $G = \mathrm{SL}(n+m, \mathbb{R})$, let $\Gamma = \mathrm{SL}(m+n, \mathbb{Z})$, and let μ_G denote the G-invariant probability measure on G/Γ. Let

$$u(v) = \begin{pmatrix} I_m & v \\ 0 & I_n \end{pmatrix}, \quad \forall v \in M_{m,n}(\mathbb{R}), \qquad \textit{and let} \qquad a(t) = \begin{pmatrix} e^{t/m} I_m & \\ & e^{-t/n} I_n \end{pmatrix}, \quad \forall t \in \mathbb{R}.$$

Let $\varphi : [0,1] \to M_{m,n}(\mathbb{R})$ be a smooth curve. What should be the algebraic or geometric condition on the image of the curve so that for any $x \in G/\Gamma$ and any

$f \in C_c(G/\Gamma)$,

$$\lim_{t\to\infty} \int_0^1 f\big(a(t)u(\varphi(s))x\big)\,\mathrm{d}s = \int_{G/\Gamma} f\,\mathrm{d}\mu_G\,? \tag{86}$$

When $m = 1$ and φ is analytic, we consider the condition that that $\varphi(I)$ is not contained in a proper affine subspace. In this case, [8, Proposition 2.3] provides a very good estimate for the rate of nondivergence of this translated measure. The method of this article is applicable to show that, after a suitable twist of the curve by elements from the centralizer of $\{a(t)\}$, the limiting measure is invariant under a unipotent one-parameter subgroup of the form $\{u(sw_0)\}$ for some $w_0 \in \mathbb{R}^n \smallsetminus \{0\}$. Also, the method to study behavior of expanded trajectories near the singular sets is applicable here. Obtaining an analogue of Lemma 2.3 in order to derive algebraic consequences of Proposition 4.7 is the main difficulty in this problem. Since the initial submission of this article, we have answered Question 6.1 in the case where $m = 1$ and φ is an analytic curve (see [19]).

In another direction, it is still an open question to prove the exact analogue of Theorem 1.4 for the actions of $\mathrm{SO}(n, 1)$ on homogeneous spaces of larger Lie group G containing $\mathrm{SO}(n, 1)$ (see [6]). Essentially, what we need is the analogue of Proposition 4.9 without any condition on V and p.

The generalizations of the main results of this article for smooth curves are considered in a subsequent article.

Acknowledgments. The precise geometric consequences of the equidistribution results on homogeneous spaces were formulated during our visits with Elon Lindenstrauss at Princeton University and Hee Oh at Brown University. We thank them for their hospitality and support. Special thanks are also owed the referees for the simplified proof of Theorem 3.1 as well as for many other useful suggestions.

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