Invent math (2009) 177: 509–532
DOI 10.1007/s00222-009-0186-6

Equidistribution of expanding translates of curves and Dirichlet's theorem on diophantine approximation

Nimish A. Shah

Received: 4 December 2007 / Accepted: 27 February 2009 / Published online: 2 April 2009

Abstract We show that for almost all points on any analytic curve on $\mathbb{R}^k$ which is not contained in a proper affine subspace, the Dirichlet's theorem on simultaneous approximation, as well as its dual result for simultaneous approximation of linear forms, cannot be improved. The result is obtained by proving asymptotic equidistribution of evolution of a curve on a strongly unstable leaf under certain partially hyperbolic flow on the space of unimodular lattices in $\mathbb{R}^{k+1}$. The proof involves Ratner's theorem on ergodic properties of unipotent flows on homogeneous spaces.

1 Introduction

The Dirichlet's theorem on simultaneous approximation of any k real numbers $\xi_1, \dots, \xi_k$ says the following:

(A) For any positive integer N there exist integers $q_1, \dots, q_k, p$ such that

$$|q_1\xi_1 + \cdots + q_k\xi_k - p| \le N^{-k} \quad \text{and} \quad 0 < \max_{1\le i\le k} |q_i| \le N;$$

(B) For any positive integer N there exist integers $q, p_1, \dots, p_k$ such that

$$\max_{1\le i\le k} |q\xi_i - p_i| \le N^{-1} \quad \text{and} \quad 0 < |q| \le N^k.$$

After Davenport and Schmidt [6] we say that the D.Th. (A) (respectively, (B)) cannot be improved for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ if for any $0 < \mu < 1$ the following holds:

Dedicated to my inspiring teacher Professor A.R. Rao (VASCSC, Ahmedabad) on his 100th birthday.

Research supported in part by Swarnajayanti Fellowship.

N.A. Shah (✉)
Tata Institute of Fundamental Research, Mumbai 400005, India
e-mail: nimish@math.tifr.res.in

(Aμ) There are infinitely many positive integers N for which the pair of inequalities

$$|(q_1\xi_1 + \cdots + q_k\xi_k) - p| \le \mu N^{-k} \quad \text{and} \quad 0 < \max_{1\le i\le k} |q_i| \le \mu N$$

are insoluble in integers $q_1, \ldots, q_k, p$ (respectively,

(Bμ) there are infinitely many positive integers N for which the pair of inequalities

$$\max_{1\le i\le k} |q\xi_i - p_1| \le \mu N^{-1} \quad \text{and} \quad 0 < |q| \le \mu N^k$$

are insoluble in integers $q, p_1, \ldots, p_k$).

In [6], Davenport and Schmidt proved that D.Th. (A) and (B) cannot be improved for almost all $\boldsymbol{\xi} \in \mathbb{R}^k$.

One says that D.Th. (A) (respectively, (B)) cannot be *μ-improved* for $\xi \in \mathbb{R}^k$ if (Aμ) (respectively, (Bμ)) holds. In [7] Davenport and Schmidt showed that D.Th. (A) cannot be (1/4)-improved for the pair (ξ, ξ^2) for almost all $\xi \in \mathbb{R}$. This result was generalized by Baker [1] for almost all points on 'smooth curves' in $\mathbb{R}^2$, by Dodson, Rynne and Vickers [8] for almost all points on 'higher dimensional curved submanifolds' of $\mathbb{R}^k$, and by Bugeaud [2] for almost all points on the curve $(\xi, \xi^2, \ldots, \xi^k)$; in each case (A$\mu$) holds for some small value of $\mu < 1$ depending on the curvature of the smooth submanifold. Their proofs typically involve the technique of regular system introduced in [6].

Recently the problem was recast in the language of flows on homogeneous spaces by Kleinbock and Weiss [15] using observations due to Dani [3], as well as Kleinbock and Margulis [13]. In [15] it was shown that D.Th. (A) and (B), as well as its various generalizations, cannot be μ-improved for almost all points on any *non-degenerate curve* on $\mathbb{R}^k$ for some small $\mu < 1$ depending on the curve. In this article, we shall strengthen such results for all $0 < \mu < 1$:

Theorem 1.1 *Let $\varphi : [a, b] \to \mathbb{R}^k$ be an analytic curve such that its image is not contained in a proper affine subspace. Then Dirichlet's theorem* (A) *and* (B) *cannot be improved for $\varphi(s)$ for almost all $s \in [a, b]$.*

This result will be deduced from a result about limiting distributions of certain expanding sequence of curves on the space of lattices in $\mathbb{R}^n$ where $n = k + 1$. A refinement of Theorem 1.1 is obtained in Theorem 1.4.

Notation Let $G = \mathrm{SL}(n, \mathbb{R})$, $n \ge 2$. For $t \in \mathbb{R}$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, define

$$a_t = \begin{bmatrix} e^{(n-1)t} & & & \\ & e^{-t} & & \\ & & \ddots & \\ & & & e^{-t} \end{bmatrix} \quad \text{and} \quad a'_t = \begin{bmatrix} e^{t} & & & \\ & \ddots & & \\ & & e^{t} & \\ & & & e^{-(n-1)t} \end{bmatrix}, \tag{1.1}$$

$$u(\boldsymbol{\xi}) = \begin{bmatrix} 1 & \xi_1 & \ldots & \xi_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad u'(\boldsymbol{\xi}) = \begin{bmatrix} 1 & & & -\xi_{n-1} \\ & \ddots & & \vdots \\ & & 1 & -\xi_1 \\ & & & 1 \end{bmatrix}. \tag{1.2}$$

The main goal of this article is to prove the following:

Theorem 1.2 *Let $\varphi : [a,b] \to \mathbb{R}^{n-1}$ be an analytic curve whose image is not contained in a proper affine subspace. Let Γ be a lattice in G. The for any $x_0 \in G/\Gamma$ and any bounded continuous function f on G/Γ,*

$$\lim_{t\to\infty} \frac{1}{|b-a|} \int_a^b f(a_t u(\varphi(s))x_0)\, ds = \int_{G/\Gamma} f\, d\mu_G, \tag{1.3}$$

where μ_G is the G-invariant probability measure on G/Γ. Similarly,

$$\lim_{t\to\infty} \frac{1}{|b-a|} \int_a^b f(a'_t u'(\varphi(s))x_0)\, ds = \int_{G/\Gamma} f\, d\mu_G. \tag{1.4}$$

This result corroborates [12, Sect. 4.3] and [15, Sect. 4.1]. In fact, we will prove the following more general statement.

Theorem 1.3 *Let L be a Lie group and Λ a lattice in L. Let $\rho : G \to L$ be a continuous homomorphism. Let $x_0 \in L/\Lambda$ and H be a minimal closed subgroup of L containing $\rho(G)$ such that the orbit Hx_0 is closed, and admits a finite H-invariant measure, say μ_H. Then for any bounded continuous function f on L/Λ the following holds*:

$$\lim_{t\to\infty} \frac{1}{|b-a|} \int_a^b f(\rho(a_t u(\varphi(s)))x_0)\, ds = \int_{Hx_0} f\, d\mu_H. \tag{1.5}$$

The above results are also valid for maps φ from boxes in R^d ($d \geq 1$) to $\mathbb{R}^{n-1}$, see Theorem 1.8.

The first part of Theorem 1.2 follows from Theorem 1.3 by taking $L = G$, ρ the identity map, and $\Lambda = \Gamma$; in this case $H = G$.

Let $\mathfrak{w} \in \mathrm{GL}(n, \mathbb{R})$ be such that $\mathfrak{w}(e_i) = e_{n+1-i}$ for $1 \leq i \leq n$, where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$. We define an involutive automorphism σ of G by $\sigma(g) = \mathfrak{w}({}^{\mathrm{t}}g^{-1})\mathfrak{w}^{-1}$ for all $g \in G$. Then $\sigma(a_t) = a'_t$ and $\sigma(u(\boldsymbol{\xi})) = u'(\boldsymbol{\xi})$. Now the second part of Theorem 1.2 follows from Theorem 1.3 by taking $L = G$, $\rho = \sigma$ and $\Lambda = \Gamma$; we observe that $H = L$ in this case.

1.0.1

Next we take $L = G \times G$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $\Lambda = \Gamma \times \Gamma$ and $\rho(g) = (g, \sigma(g))$ for all $g \in G$. Note that $\sigma(\Gamma) = \Gamma$, and hence $\rho(\Gamma) \subset \Lambda$ and $\rho(\Gamma)$ is a lattice in $\rho(G)$. Therefore if we let $x_0 = \Lambda$, then $\rho(G)x_0$ is closed and admits a finite $\rho(G)$-invariant measure; in other words, we have $H = \rho(G)$. Now using Theorem 1.3, in the next section we will deduce the following enhancement of Theorem 1.1.

Theorem 1.4 *Let $\varphi = (\varphi_1, \ldots, \varphi_k) : I = [a,b] \to \mathbb{R}^k$ be an analytic curve whose image is not contained in a proper affine subspace. Let $\mathcal{N}$ be an infinite set of positive integers. Then for almost every $s \in I$ and any $\mu < 1$, there exist infinitely many $N \in \mathcal{N}$ such that both the following pairs of inequalities are simultaneously insoluble*:

$$|q_1\varphi_1(s) + \cdots + q_k\varphi_k(s) - p| \leq \mu N^{-k} \quad \text{and} \quad \max_{1\leq i\leq k} |q_i| \leq \mu N \tag{1.6}$$

for $(q_1, \dots, q_k) \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$ *and* $p \in \mathbb{Z}$; *and*

$$\max_{1 \le i \le k} |q\varphi_i(s) - p_i| \le \mu N^{-1} \quad \textit{and} \quad |q| \le \mu N^k \tag{1.7}$$

for $q \in \mathbb{Z} \setminus \{0\}$ *and* $(p_1, \dots, p_k) \in \mathbb{Z}^k$.

Now suppose we let $I = [a_1, b_1] \times \cdots \times [a_d, b_d] \subset \mathbb{R}^d$, where $a_i < b_i$ for all $1 \le i \le d$, and let $\varphi : I \to \mathbb{R}^k$ to be an analytic map such that $\varphi(I)$ is not contained in a proper affine subspace. Then there exists a line L in $\mathbb{R}^d$ such that $\varphi((x + L) \cap I)$ is not contained in a proper affine subspace of R^k for almost every $x \in I$. Due to this observation we can deduce the following consequence of Theorem 1.4.

Corollary 1.5 *Let M be an analytically immersed submanifold of $\mathbb{R}^k$ which affinely spans $\mathbb{R}^k$. Then Dirichlet's theorem* (A) *and* (B) *cannot be improved for almost every vector on M, with respect to the smooth measure class on M.*

1.0.2

Due to the effective equidistribution result proved by Kleinbock and Margulis [14], it should be possible to prove that Dirichlet's theorem (A) and (B) cannot be improved for almost all $\boldsymbol{\xi} \in \mathbb{R}^k$ where the constant μ is replaced by an explicitly given function $\mu(N) < 1$ with $\mu(N) \to 1$ as $N \to \infty$. However proving such a result for almost all points on curves would require proving a version of Theorem 1.2 with explicit rate of convergence. This is currently beyond the scope of our methods.

1.0.3 Some background on the equidistribution results for translates of measures

In [9], using powerful techniques of harmonic analysis on $L^2(G/\Gamma)$, Duke, Rudnick and Sarnak proved that if H is a symmetric subgroup of a noncompact simple group G and μ_H is the Haar probability measure on a closed orbit of H on G/Γ, then the translated measures $g\mu_H$ get equidistributed in G/Γ as $g \to \infty$ in G modulo H. Eskin and McMullen [10] gave a simpler proof of the result using mixing property of geodesic flows. The limiting distributions for translates $g\mu_H$ for more general subgroups H were studied by Mozes and Shah [16], Shah [20], and Eskin, Mozes and Shah [11] using Ratner's classification theorem for measures invariant under unipotent flows.

From a different view point, in [13] Kleinbock and Margulis proved a quantitative non-divergence theorem for limiting distributions of translates of the parameter measure on any 'non-degenerate' curve on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$; like $u(\varphi([a, b]))x_0$ as in the statement of Theorem 1.2. Our results extend their work to show that the limiting measures are in fact G-invariant.

Unlike the limiting measures for translates of Haar measures on closed orbits of subgroups, the limiting distributions for translates of measures on curves or manifolds are not a priori invariant under non-trivial unipotent subgroups. However in [21], the author has introduced a technique which allows us to start applying Ratner's classification and linearization technique to this question. In this article we add the following 'linear dynamical' component to the linearization technique to prove the homogeneity of limiting measures in the above special setting.

Theorem 1.6 *Let $\varphi : [a,b] \to \mathbb{R}^{n-1}$ be a differentiable curve, whose image is not contained in a union of finitely many proper affine subspaces of $\mathbb{R}^{n-1}$. Let V be a finite dimensional normed linear space on which G acts linearly. Further suppose that there is no nonzero G-fixed vector in V. Then given $C > 0$ there exists $t_0 > 0$ such that*

$$\sup_{s\in[a,b]} \|a_t u(\varphi(s))v\| \geq C\|v\|, \quad \forall v \in V \text{ and } t > t_0. \tag{1.8}$$

The essential ingredient in this result is what we will call the 'Basic Lemma' (Proposition 4.2). For its proof we develop a method to understand dynamical interactions between linear actions of various $\mathrm{SL}(2,\mathbb{R})$'s contained in $\mathrm{SL}(n,\mathbb{R})$ acting on a finite dimensional vector space. This in turn uses linear dynamical properties of individual actions of $\mathrm{SL}(2,\mathbb{R})$ as studied in [21].

1.0.4 Translates by more general elements

In fact, in [13] one also considers translates by elements of the form

$$a_{\boldsymbol{t}} := \begin{bmatrix} e^{t_1+\cdots+t_{n-1}} & & & \\ & e^{-t_1} & & \\ & & \ddots & \\ & & & e^{-t_{n-1}} \end{bmatrix} \tag{1.9}$$

as $\|\boldsymbol{t}\| \to \infty$, where $\boldsymbol{t} = (t_1, \ldots, t_{n-1})$ and $t_i \geq 0$. In order to obtain similar equidistribution results in this case using the method of this article, one needs to obtain the analogue of Theorem 1.6, and the corresponding basic lemma for the above $a_{\boldsymbol{t}}$. This has been carried out in [23] using the basic lemma of this article to analyze dynamical interaction between various $\mathrm{SL}(m,\mathbb{R})$'s contained in $\mathrm{SL}(n,\mathbb{R})$. This generalization implies non-improvability of Minkowski's multiplicative version of Dirichlet's theorem [23].

Obtaining analogues of the basic lemma corresponding to translates by diagonal elements more general than those described in (1.9) is a challenging problem. We are still not able to provide the optimal conjectural description for the algebraic restriction on the curves living on associated horospherical subgroups to ensure non-divergence, or more generally equidistribution, of limits of translates of measures on trajectories of such curves.

1.1 Sketch of the proof of Theorem 1.3

Let $I = [a,b]$. We will treat G as a subset of L via the homomorphism ρ. We consider the normalized parameter measure, say ν, on the segment $\{u(\varphi(s))x_0 : s \in I\}$ on L/Λ. Take any sequence $t_i \to \infty$. Let $a_{t_i}\nu$ denote the translate of ν concentrated on the curve $a_{t_i}u(\varphi(I))x_0$.

We claim that, given any $\epsilon > 0$ there exists a large compact set $\mathcal{F} \subset L/\Lambda$ such that

$$(a_{t_i}\nu)(\mathcal{F}) := \frac{1}{|b-a|}\{s \in [a,b] : a_{t_i}u(\varphi(s))x_0 \in \mathcal{F}\} \geq 1-\epsilon.$$

If this claim fails to hold, then by Dani-Margulis criterion for nondivergence the following algebraic condition holds: There exists a finite dimensional representation V of L and a nonzero vector $v \in V$ such that after passing to a subsequence

$$\lim_{i\to\infty} \sup_{s\in I} \|a_{t_i}u(\varphi(s))v\| = 0.$$

But this cannot happen because of Theorem 1.6, and the claim is proved.

Therefore there exists a probability measure μ on L/Λ such that after passing to a subsequence, $a_{t_i}\nu \xrightarrow{i\to\infty} \mu$ with respect to the weak*-topology.

At this stage, we note that $u(\varphi(I))x_0$ is contained in a strongly unstable leaf for the action of a_t on L/Λ. Then for each $s_0 \in I$ if $\dot{\varphi}(s_0)$ denotes the derivative of φ at s_0, then

$$\varphi(s) - \varphi(s_0) = (s - s_0)\dot{\varphi}(s_0) + O((s - s_0)^2). \tag{1.10}$$

Therefore for any large $t > 0$, the translated curve

$$\{a_t u(\varphi(s))x_0 = u(e^{nt}(\varphi(s) - \varphi(s_0)))(a_t u(\varphi(s_0))x_0) : |s - s_0| < \delta_t\} \tag{1.11}$$

stays very close to the unipotent trajectory

$$\{u(e^{nt}(s - s_0)\dot{\varphi}(s_0))(a_t u(\varphi(s_0))x_0) : |s - s_0| < \delta_t\}, \tag{1.12}$$

if we choose $\delta_t > 0$ such that $e^{nt}\delta_t \to \infty$, but $e^{nt}\delta_t^2 \to 0$ as $t \to \infty$. Note that the length of the unipotent trajectory (1.12) is about $e^{nt}\delta_t\|\dot{\varphi}(s_0)\|$ and hence it is very long if $\dot{\varphi}(s_0) \neq 0$. Our difficultly is that the direction $\dot{\varphi}(s_0)$ of the flow varies with s_0.

In order to take care of this problem, instead of translating the original curve, we twist it by $z(s) \subset \mathrm{Z}_G(A) \cong A \times \mathrm{SL}(n-1, \mathbb{R})$ so that $z(s)u(\dot{\varphi}(s))z(s)^{-1} = u(e_1)$ for all $s \in I$; here $A = \{a_t : t \in \mathbb{R}\}$, e_1 is a fixed nonzero vector in $\mathbb{R}^{n-1}$, and we assume that $\dot{\varphi}(s) \neq 0$ for all $s \in I$. We take another curve: $\{z(s)u(\varphi(s))x_0 : s \in I\}$ and associate a normalized parameter measure ν' on it. Since $z(I)$ is contained in a compact set, we conclude that after passing to a subsequence $a_{t_i}\nu'$ converges to a probability measure μ' as $i \to \infty$. Now one can show that μ' is invariant under the flow $\{u(se_1) : s \in \mathbb{R}\}$. Then we use Ratner's theorem [17] and linearization technique (avoidance criterion) [5, 16, 18, 19] to show that there exist a finite dimensional representation V of L and a discrete set D of nonzero vectors in V which are algebraically associated to x_0 and $R > 0$ such that for each i there exists $v_i \in D$ such that

$$\sup_{s\in I}\|a_{t_i}u(\varphi(s))v_i\| \le R.$$

Therefore from Theorem 1.6 we conclude that for some i_0, the vector v_{i_0} is fixed by G. From this very restrictive situation, as v_{i_0} is algebraically related to x_0, we will deduce that the measure μ' is H-invariant, where H is the smallest closed subgroup of L containing G such that the orbit Hx_0 is closed. Since the modification of ν to obtain ν' was only by elements centralizing A in G, and since we have shown that μ' is invariant under $\mathrm{Z}_G(A)$, we obtain that $\mu = \mu'$, and hence μ is H-invariant. This will prove Theorem 1.3.

Remark 1.7 We assume φ to be analytic instead of just smooth because the Dani-Margulis nondivergence criterion, as well as the linearization technique, makes use of certain 'uniform' growth properties of the space of functions $s \mapsto \langle\varphi(s)v, w\rangle$ from $I \to \mathbb{R}$ for all $v, w \in V$. If φ is analytic, this space is finite dimensional and the required growth properties hold ([13]). If we assume that φ is just smooth and 'nondegenerate' as considered in [13], it is far from clear if the required growth properties will be valid for a general representation V of G. To overcome this difficulty one would require a much stronger form of Theorem 1.6 where the length of the interval I shrinks to 0 as $t \to 0$ [22, Sect. 7].

1.2 Variations of the equidistribution result

1.2.1 Expanding translates of any analytic subvariety

The following form of Theorem 1.3 is more appealing.

Theorem 1.8 *Let I be a bounded open subset of $\mathbb{R}^d$ $(d \geq 1)$ with zero boundary measure. Let $\boldsymbol{\psi} : I \to \mathrm{SL}(n, \mathbb{R})$ be an analytic map such that the image of the first row of this map is not contained in a proper subspace of $\mathbb{R}^n$. Let the notation be as in the statement of Theorem 1.3. Then*

$$\frac{1}{\mathrm{Vol}(I)} \int_I f(\rho(a_t \boldsymbol{\psi}(\boldsymbol{s}))x_0)\, d\boldsymbol{s} \xrightarrow{t\to\infty} \int_{Hx_0} f\, d\mu_H.$$

1.2.2 Uniform versions

First we state the basic uniform version of Theorem 1.2.

Theorem 1.9 *Let the notation be as in Theorem 1.2. Then given any compact set $\mathcal{K} \subset G/\Gamma$, a bounded continuous function f on G/Γ and an $\epsilon > 0$, there exists $t_0 > 0$ such that for any $x \in \mathcal{K}$ and $t \geq t_0$, we have*

$$\left| \int_a^b f(a_t u(\varphi(s))x)\, ds - \int f\, d\mu_G \right| \leq |b-a|\epsilon.$$

The following result is a general uniform version for Theorem 1.3.

Theorem 1.10 *Let I be any bounded open subset of $\mathbb{R}^d$ with boundary measure zero. Let $\boldsymbol{\psi} : I \to \mathrm{SL}(n, \mathbb{R})$ be an analytic map such that the image of the first row of this map is not contained in a proper subspace of $\mathbb{R}^n$. Let L be a Lie group, Λ a lattice in L and $\pi : L \to L/\Lambda$ the quotient map. Let $\rho : G \to L$ be a continuous homomorphism. Let $\mathcal{K}$ be a compact subset of L/Λ. Then given $\epsilon > 0$ and a bounded continuous function f on L/Λ, there exist finitely many proper closed subgroups $H_1, \ldots, H_r$ of L such that for each $1 \leq i \leq r$, $H_i \cap \Lambda$ is a lattice in H_i and there exists a compact set*

$$C_i \subset N(H_i, \rho(G)) := \{g \in L : \rho(G)g \subset gH_i\}$$

such that the following holds: *Given any compact set*

$$F \subset \mathcal{K} \Big\backslash \bigcup_{i=1}^{r} \pi(C_i)$$

there exists $t_0 > 0$ such that for any $x \in F$ and any $t \geq t_0$,.

$$\left| \frac{1}{\mathrm{Vol}(I)} \int_I f(\rho(a_t u(\varphi(\boldsymbol{s})))x)\, d\boldsymbol{s} - \int_{L/\Lambda} f\, d\mu_L \right| < \epsilon.$$

If L is an algebraic group then the sets $N(H_i, \rho(G))$ are algebraic subvarieties of L. Therefore unless H contains a proper normal subgroup of L containing G, the set

$\bigcup_{i=1}^{r} \pi(C_i)$ is contained in a finite union of relatively compact lower dimensional submanifolds of L/Λ.

In the next section we will deduce the number theoretic consequences from the equidistribution statement. Rest of the article closely follows the strategy laid out in Sect. 1.1. The basic lemma and its consequences are proved in Sect. 4.

2 Deduction of Theorem 1.4 from Theorem 1.3

The argument given below is based on [15, Sect. 2.1]. Let the homomorphism $\rho(g) = (g, \sigma(g))$ from G to $L = G \times G$, and other notation be as in Sect. 1.0.1. Let $n = k + 1$. Let $(q_1, \dots, q_k) \in \mathbb{Z}^k$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}$ and $(p_k, \dots, p_1) \in \mathbb{Z}^k$. Let $N \in \mathbb{N}$ and put $t = \log(N)$. We consider the standard action of $G \times G$ on $\mathbb{R}^n \times \mathbb{R}^n$. For $s \in [a, b]$, we have

$$\begin{aligned}(\boldsymbol{\zeta}(N,s), \boldsymbol{\eta}(N,s)) &:= \rho(a_t u(\varphi(s))) \left(\begin{bmatrix} -p \\ q_1 \\ \vdots \\ q_k \end{bmatrix}, \begin{bmatrix} p_k \\ \vdots \\ p_1 \\ q \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} N^k(-p + \sum_{i=1}^k q_i \varphi_i(s)) \\ N^{-1} q_1 \\ \vdots \\ N^{-1} q_k \end{bmatrix}, \begin{bmatrix} N(-q\varphi_k(s) + p_k) \\ \vdots \\ N(-q\varphi_1(s) + p_1) \\ N^{-k} q \end{bmatrix} \right).\end{aligned}$$

We now fix $0 < \mu < 1$. Let

$$B_\mu = \{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \max_{1 \le i \le n} \xi_i \le \mu\}.$$

Then (1.6) is equivalent to $\boldsymbol{\zeta}(N,s) \in B_\mu$ and (1.7) is equivalent to $\boldsymbol{\eta}(N,s) \in B_\mu$.

Let Ω denote the space of unimodular lattices in $\mathbb{R}^n$. Note that G acts transitively on Ω and the stabilizer of $\mathbb{Z}^n$ is Γ. Similarly L acts transitively on $\Omega \times \Omega$ and the stabilizer of $x_0 := (\mathbb{Z}^n, \mathbb{Z}^n)$ is Λ. Thus $G/\Gamma \cong \Omega$ and $L/\Lambda \cong \Omega \times \Omega$.

Let

$$K_\mu = \{\Delta \in \Omega : \Delta \cap B_\mu = \{\mathbf{0}\}\}.$$

As we observed above for any $s \in [a, b]$ and $N \in \mathbb{N}$, the inequalities (1.6) and (1.7) are simultaneously insoluble, if for $t = \log N$,

$$\rho(a_t u(\varphi(s))) x_0 \in K_\mu \times K_\mu. \tag{2.1}$$

As noted earlier there exists a $\rho(G)$-invariant probability measure on $\rho(G)x_0 \subset \Omega \times \Omega$, say λ. Since $\mu < 1$, $K_\mu \times K_\mu$ contains an open neighbourhood of x_0. Hence there exists $\epsilon > 0$ such that

$$\lambda(K_\mu \times K_\mu) > \epsilon.$$

Therefore there exists a continuous function $0 \le f \le 1$ on L/Λ such that

$$\operatorname{supp}(f) \subset K_\mu \times K_\mu \quad \text{and} \quad \int f \, d\lambda > \epsilon/2.$$

Let J be any subinterval of $[a, b]$ with nonempty interior. Then by Theorem 1.3, there exists $N_0 > 0$, such that for all $N \geq N_0$, and $t = \log N$,

$$\frac{1}{|J|}\int_{s\in J} f(\rho(a_t u(\varphi(s)))x_0)\,ds \geq \epsilon/4, \tag{2.2}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

Let $\mathcal{N} \subset \mathcal{N}$ be an infinite set. Let

$$E = \{s \in [a, b] : \rho(a_{(\log N)}u(\varphi(s)))x_0 \notin K_\mu \times K_\mu \text{ for all large } N \in \mathcal{N}\}.$$

By (2.2), for any subinterval $J \subset [a, b]$, we have $|J \cap E| \leq (1-\epsilon/4)|J|$. Therefore $|E| = 0$. In view of the observation associated to (2.1), this proves Theorem 1.4.

The Theorem 1.1 is a special case of Theorem 1.4. On the other hand, the proof of Theorem 1.1 can also be deduced directly from Theorem 1.2 in a similar way.

3 Non-divergence of the limiting distribution

Let $\varphi : I = [a, b] \to \mathbb{R}^{n-1}$ be an analytic map whose image is not contained in a proper affine subspace. Let $x_0 \in L/\Lambda$. Given a nontrivial continuous homomorphism $\rho : G \to L$, for the sake of simplicity of notation, without loss of generality, we will identify $g \in G$ with $\rho(g) \in L$. Therefore now onward we will treat G as a subgroup of L, ρ being an inclusion. We will assume that the orbit of x_0 under G is dense in L/Λ; that is $\overline{Gx_0} = L/\Lambda$. Let $t_i \to \infty$ be any sequence in $\mathbb{R}$. Let $x_i \to x_0$ be a convergent sequence in L/Λ. For any $i \in \mathbb{N}$ let μ_i be the measure on L/Λ defined by

$$\int_{L/\Lambda} f\,d\mu_i := \frac{1}{|I|}\int_I f(a_{t_i}u(\varphi(s))x_i)\,ds, \quad \forall f \in \mathrm{C_c}(L/\Lambda). \tag{3.1}$$

Theorem 3.1 *Given $\epsilon > 0$ there exists a compact set $\mathcal{F} \subset L/\Lambda$ such that $\mu_i(\mathcal{F}) \geq 1-\epsilon$ for all large $i \in \mathbb{N}$.*

It may be noted that in the case of $L = G$, ρ the identity map and $\Lambda = \mathrm{SL}(n, \mathbb{Z})$, the above result was proved by Kleinbock and Margulis [13, Proposition 2.3]. The rest of this section is devoted to obtaining the same conclusion in the case of arbitrary L and Λ.

3.1

Let $\mathcal{H}$ denote the collection of analytic subgroups H of G such that $H \cap \Lambda$ is a lattice in H, and a unipotent one-parameter subgroup of H acts ergodically with respect to the H-invariant probability measure on $H/H \cap \Lambda$. Then $\mathcal{H}$ is a countable collection [17, 19].

Let $\mathfrak{l}$ denote the Lie algebra associated to L. Let $V = \bigoplus_{d=1}^{\dim \mathfrak{l}} \wedge^d \mathfrak{l}$ and consider the $(\bigoplus_{d=1}^{\dim \mathfrak{l}} \wedge^d \mathrm{Ad})$-action of L on V. Given $H \in \mathcal{H}$, let $\mathfrak{h}$ denote its Lie algebra, and fix $p_H \in \wedge^{\dim \mathfrak{h}}\mathfrak{h} \setminus \{0\} \subset V$. Then

$$\mathrm{Stab}_L(p_H) = \mathrm{N}_L^1(H) := \{g \in \mathrm{N}_L(H) : \det((\mathrm{Ad} g)|_{\mathfrak{h}}) = 1\}.$$

Proposition 3.2 [5, Theorem 3.4] *The orbit $\Lambda \cdot p_H$ is a discrete subset of V.*

We may note that when L is a real algebraic group defined over $\mathbb{Q}$ and $\Lambda = L(\mathbb{Z})$ then the above countability result and the discreteness of the orbit are straightforward to prove.

3.2 Functions with growth factor C and growth order α

Let $\mathcal{F}$ denote the $\mathbb{R}$-span of all the coordinate functions of the map $\Upsilon : I \to \mathrm{End}(V)$ given by $\Upsilon(s) = (\bigoplus_{d=1}^{\dim \mathfrak{l}} \wedge^d \mathrm{Ad})(u(\varphi(s)))$ for all $s \in \mathbb{R}$. As explained in [21, Sect. 2.1], due to [13, Proposition 3.4] the family $\mathcal{F}$ has the following growth property for some $C > 0$ and $\alpha > 0$: for any subinterval $J \subset I$, $\xi \in \mathcal{F}$ and $r > 0$,

$$|\{s \in J : |\xi(s)| < r\}| < C \left(\frac{r}{\sup_{s \in J} |\xi(s)|} \right)^{\alpha} J.$$

As a direct consequence of this property, we have the following [13]: Fix any norm $\|\cdot\|$ on V.

Proposition 3.3 *Let $\epsilon > 0$. Then given any $r > 0$ there exists $R > 0$ (or given any $R > 0$ there exists $r > 0$) such that for any $h_1, h_2 \in L$ and a subinterval $J \subset I$, one of the following holds*:

(I) $\sup_{t \in J} \|h_1 u(\varphi(t)) h_2 p_H\| < R$.
(II) $|\{t \in J : \|h_1 u(\varphi(t)) h_2 p_H\| \le r\}| \le \epsilon |\{t \in J : \|h_1 u(\varphi(t)) h_2 p_H\| \le R\}|$.

3.3 The non-divergence criterion

Proposition 3.4 *There exist closed subgroups $W_1, \ldots, W_r \in \mathcal{H}$ (depending only on L and Λ) such that the following holds: Given $\epsilon > 0$ and $R > 0$, there exists a compact set $\mathcal{F} \subset L/\Lambda$ such that for any $h_1, h_2 \in L$ and a subinterval $J \subset I$, one of the following conditions is satisfied*:

(I) *There exists $\gamma \in \Lambda$ and $i \in \{1, \ldots, r\}$ such that*

$$\sup_{s \in J} \|h_1 u(\varphi(s)) h_2 \gamma p_{W_i}\| < R.$$

(II) $\frac{1}{|J|} |\{s \in J : (h_1 u(\varphi(s)) h_2)\Lambda/\Lambda \in \mathcal{F}\}| \ge 1 - \epsilon$.

Proof The result follows from the argument as in [20, Theorem 2.2] using the earlier results of Dani and Margulis [4]; as well as its extensions due to Kleinbock and Margulis [13]. The main difference is that instead of growth properties of polynomial functions, one uses the similar properties of functions in $\mathcal{F}$ as given by Proposition 3.3. □

Proof of Theorem 3.1 Take any $\epsilon > 0$. Take a sequence $R_k \to 0$ as $k \to \infty$. For each $k \in \mathbb{N}$, let $\mathcal{F}_k \subset L/\Lambda$ be a compact set as determined by Proposition 3.4 for these ϵ and R_k. If the theorem fails to hold, then for each $k \in \mathbb{N}$ we have $\mu_i(\mathcal{F}_k) < 1 - \epsilon$ infinitely may $i \in \mathbb{N}$. Therefore after passing to a subsequences of $\{\mu_i\}$, we may assume that $\mu_i(\mathcal{F}_i) < 1 - \epsilon$ for all i. Then by Proposition 3.4, after passing to a subsequence, we may assume that there exists $W \in \mathcal{H}$ such that for each i there exists $\gamma_i \in \Lambda$ such that

$$\left\| \sup_{s \in I} a_{t_i} u(\varphi(s)) \gamma_i p_W \right\| \le R_i \xrightarrow{i \to \infty} 0.$$

By Proposition 3.2, there exists $r_0 > 0$ such that $\|\gamma_i p_W\| \ge r_0$ for each i. We put $v_i = \gamma_i p_W / \|\gamma_i p_W\|$. Then $v_i \to v \in V$ and $\|v\| = 1$. Therefore

$$\sup_{s \in I} \|a_{t_i} u(\varphi(s)) v_i\| \le R_i / r_0 \to 0 \quad \text{as } i \to \infty. \tag{3.2}$$

Define

$$V^{-} = \{w \in V : \lim_{t\to\infty} a_t w = 0\},$$

$$V^{+} = \{w \in V : \lim_{t\to\infty} (a_t)^{-1} w = 0\},$$

$$V^{0} = \{w \in V : a_t w = w, \quad \forall t \in \mathbb{R}\}.$$

Since $\{a_t\}$ acts on V via commuting $\mathbb{R}$-diagonalizable matrices, we have that $V = V^{+} \oplus V^{0} \oplus V^{-}$. Let $\pi_0 : V \to V^0$ denote the associated projection. Then from (3.2) we conclude that

$$u(\varphi(s))v \subset V^{-}, \quad \forall s \in I. \tag{3.3}$$

The 'Basic Lemma' (Proposition 4.2) proved in the next section states that for any finite dimensional linear representation V of G, any $v \in V\backslash\{0\}$ and any subset $\mathcal{B}$ of $\mathbb{R}^{n-1}$ which is not contained in a proper affine subspace, if

$$u(\varphi(e))v \in V^{-} + V^{0}, \quad \forall e \in \mathcal{B}, \tag{3.4}$$

then

$$\pi_0(u(\varphi(e))v) \neq 0, \quad \forall e \in \mathcal{B}. \tag{3.5}$$

By our hypothesis (3.3) implies (3.4) but contradicts its consequence (3.5). □

As a consequence of Theorem 3.1 we deduce the following:

Corollary 3.5 *After passing to a subsequence, $\mu_i \to \mu$ in the space of probability measures on L/Λ with respect to the weak*-topology.*

Before we proceed further from here, we will give a proof of the Basic lemma and obtain its consequence, which will be used in the later sections.

4 Dynamics of linear actions of intertwined SL(2, $\mathbb{R}$)'s

A triple (X, H, Y) of elements of a Lie algebra is called an $\mathfrak{sl}_2$-*triple* if

$$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y.$$

The following observation on linear dynamics of SL_2-action played a crucial role in understanding limiting distributions of expanding translates of curves under the geodesic flows on hyperbolic manifolds [21].

Lemma 4.1 [21, Lemma 2.3] *Let W be a finite dimensional irreducible representation of an $\mathfrak{sl}_2$-triple (X, H, Y). Let W^- (respectively, W^+) denote the sum of strictly negative (respectively, strictly positive) eigenspaces of H. Let $\pi_+ : W \to W^+$ denote the projection parallel to the eigenspaces of H. Then*

$$v \in W^{-}\backslash\{0\} \implies \pi_{+}(\exp(X)v) \neq 0.$$

The main goal of this section is to obtain a similar result on linear dynamics of $\mathrm{SL}(n, \mathbb{R})$-actions by considering intertwined actions of copies of $\mathrm{SL}(2, \mathbb{R})$'s contained in $\mathrm{SL}(n, \mathbb{R})$.

Notation Let I_{n-1} denote the $(n-1)\times(n-1)$ identity matrix. Define

$$\mathcal{A}=\begin{bmatrix}(n-1) & \\ & -I_{n-1}\end{bmatrix}=\operatorname{diag}((n-1),-1,\dots,-1)\in\mathfrak{sl}(n,\mathbb{R}).$$

Then $a_t=\exp(t\mathcal{A})$ for all $t\in\mathbb{R}$. Define $A=\{a_t:t\in\mathbb{R}\}$ and $\mathfrak{a}=\operatorname{Lie}(A)=\mathbb{R}\cdot\mathcal{A}$. Consider a linear representation of G on a finite dimensional vector space V. For $\mu\in\mathbb{R}$, define

$$V^\mu=\{v\in V:\mathcal{A}v=\mu v\}.$$

Let $\pi_\mu:V\to V^\mu$ denote the projection parallel to the eigen spaces of $\mathcal{A}$. Put

$$V^-=\sum_{\mu<0}V^\mu,\qquad V^+=\sum_{\mu>0}V^\mu$$
$$\pi_-=\sum_{\mu<0}\pi_\mu,\qquad \pi_+=\sum_{\mu>0}\pi_\mu.$$

An *affine basis* of $\mathbb{R}^{n-1}$ is a set $\mathcal{B}\subset\mathbb{R}^{n-1}$ such that for any $e\in\mathcal{B}$, the set $\{e'-e:e'\in\mathcal{B}\setminus\{e\}\}$ is a basis of $\mathbb{R}^{n-1}$.

Proposition 4.2 (Basic Lemma) *Given an affine basis $\mathcal{B}$ of $\mathbb{R}^{n-1}$ and a nonzero vector $v\in V$, suppose that*

$$u(e)v\in V^0+V^-,\quad \forall e\in\mathcal{B}. \tag{4.1}$$

Then

$$\pi_0(u(e)v)\neq 0,\quad \forall e\in\mathcal{B}. \tag{4.2}$$

Proof By (4.1) there exists $\mu_0\le 0$ and $e_0\in\mathcal{B}$ such

$$\pi_{\mu_0}(u(e_0)v)\neq 0,\quad \text{and}$$
$$\pi_\mu(u(e)v)=0,\quad \forall\mu>\mu_0\quad\text{and}\quad\forall e\in\mathcal{B}.$$

We write the basis $\{e-e_0:e\in\mathcal{B}\setminus\{e_0\}\}$ of $\mathbb{R}^{n-1}$ as $\{e_1,\dots,e_{n-1}\}$. Put $v_0=u(e_0)v$. Then

$$\begin{aligned}\pi_{\mu_0}(v_0)&\neq 0 && \text{and}\\ \pi_\mu(u(e_i)v_0)&=0 && \text{for all } \mu>\mu_0 \text{ and } 1\le i\le n-1.\end{aligned} \tag{4.3}$$

To prove (4.2) we need to show that

$$\mu_0=0\quad\text{and} \tag{4.4}$$
$$\pi_0(u(e_i)v_0)\neq 0,\quad \forall 1\le i\le n-1. \tag{4.5}$$

Let the set $\{f_1,\dots,f_{n-1}\}$ consisting of real $((n-1)\times 1)$ column matrices be the dual to the basis $\{e_1,\dots,e_{n-1}\}$ of $\mathbb{R}^{n-1}$ consisting of $(1\times(n-1))$-row matrices; that is,

$$e_if_j=\delta_{i,j},\quad \forall i,j\in\{1,\dots,n-1\}. \tag{4.6}$$

For $i \in \{1, \ldots, n-1\}$, let

$$X_i := X(e_i) = \begin{bmatrix} 0 & e_i \\ & \mathbf{0}_{n-1} \end{bmatrix} \quad \text{and} \quad Y_i := Y(f_i) = \begin{bmatrix} 0 & \\ f_i & \mathbf{0}_{n-1} \end{bmatrix},$$

where $\mathbf{0}_{n-1}$ is the $((n-1) \times (n-1))$-zero matrix. Then $u(e_i) = \exp(X_i)$. Let

$$H_i := [X_i, Y_i] = \begin{bmatrix} e_i f_i & \\ & -f_i e_i \end{bmatrix} = \begin{bmatrix} 1 & \\ & -f_i e_i \end{bmatrix} \in \mathfrak{sl}(n, \mathbb{R}).$$

Then (X_i, H_i, Y_i) is an $\mathfrak{sl}_2$-triple. Let $\mathfrak{g}_i = \operatorname{span}\{X_i, H_i, Y_i\} \subset \mathfrak{sl}(n, \mathbb{R})$. Then

$$H_1 + \cdots + H_{n-1} = \sum_{i=1}^{n-1} \begin{bmatrix} 1 & \\ & -f_i e_i \end{bmatrix} = \begin{bmatrix} (n-1) & \\ & -I_{n-1} \end{bmatrix} = \mathcal{A},$$

because by (4.6), $(\sum_{i=1}^{n-1} f_i e_i) f_j = \sum_{i=1}^{n-1} \delta_{ij} f_i = f_j$. Also

$$\mathfrak{b} := \operatorname{span}\{H_i : i = 1, \ldots, n-1\} \tag{4.7}$$

is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{sl}(n, \mathbb{R})$. We can verify that

$$[H, \mathfrak{g}_i] = \mathfrak{g}_i, \quad \forall H \in \mathfrak{b}.$$

Thus $\mathfrak{b} + \mathfrak{g}_i$ is a reductive Lie algebra which is isomorphic to $\mathbb{R}^{n-2} \bigoplus \mathfrak{sl}_2$. Note that the Lie groups associated to these $\mathfrak{g}_i$'s are our intertwined copies of SL_2's, and we want to study their joint linear dynamics.

For a linear functional $\delta \in \mathfrak{b}^*$, let

$$V(\delta) = \{v \in V : Hv = \delta(H)v\}.$$

The set $\Delta = \{\delta \in \mathfrak{b}^* : V(\delta) \neq 0\}$ is finite and $V = \bigoplus_{\delta \in \Delta} V(\delta)$. Let $q_\delta : V \to V(\delta)$ be the associated projection.

Claim 4.2.1 *Let $\delta \in \Delta$ be such that $v(\delta) := q_\delta(\pi_{\mu_0}(v_0)) \neq 0$. Then $\delta(H_i) \geq 0$ for all $1 \leq i \leq n-1$.*

To prove the claim, take any $1 \leq i \leq n-1$. Consider the decomposition

$$V = W_1 \oplus \cdots \oplus W_s,$$

where W_j's are irreducible subspaces for the action of the Lie subalgebra $\mathfrak{b} + \mathfrak{g}_i$ and $s \in \mathbb{N}$. Therefore each W_j is an irreducible representation of the $\mathfrak{sl}_2$-triple (X_i, H_i, Y_i). Let $P_j : V \to W_j$ denote the associated projection. We note that

$$\pi_\mu \circ P_j = P_j \circ \pi_\mu, \quad \text{for all } 1 \leq j \leq s \text{ and } \mu \in \mathbb{R}, \text{ and} \tag{4.8}$$

$$q_\delta \circ P_j = P_j \circ q_\delta, \quad \text{for all } 1 \leq j \leq s. \tag{4.9}$$

There exists $1 \leq j \leq s$ such that

$$P_j(v(\delta)) \neq 0, \tag{4.10}$$

we take any such j. In particular, by (4.8) and (4.9),

$$W_j \cap V_{\mu_0} \ni P_j(\pi_{\mu_0}(v_0)) \neq 0. \tag{4.11}$$

By the standard description of finite dimensional representations of $\mathfrak{sl}_2$, let $k \geq 0$ and $w_{-k} \in W_j$ be such that

$$Y_i \cdot w_{-k} = 0 \quad \text{and} \quad H_i \cdot w_{-k} = -k \cdot w_{-k}.$$

For any $r \geq 0$, put $w_{-k+2r} := X_i^r \cdot w_{-k}$. Then

$$H_i \cdot w_{-k+2r} = (-k+2r)w_{-k+2r}$$

and $W_j = \operatorname{span}\{w_{-k}, \dots, w_k\}$. Since $[H_i, \mathfrak{b}] = 0$ and W_j is $\mathfrak{b}$-invariant, for each $0 \leq r \leq k$, there exists $\delta_r \in \Delta$ such that $w_{-k+2r} \in V(\delta_r)$ and

$$\delta_r \neq \delta_{r'}, \quad \text{if } r \neq r'. \tag{4.12}$$

Put $\lambda = \delta_0(\mathcal{A})$. Then

$$\mathcal{A} \cdot w_{-k} = \lambda \cdot w_{-k}.$$

Since $[\mathcal{A}, X_i] = n$, we have

$$\mathcal{A} \cdot w_{-k+2r} = (\lambda + nr)w_{-k+2r}, \quad \forall 0 \leq r \leq k.$$

Thus, if $P_j(V_\mu) \neq 0$ for any μ, then $P_j(V_\mu) \subset \mathbb{R} \cdot w_{-k+2r}$ for some $r \geq 0$ such that $\lambda + nr = \mu$.

Therefore by (4.11) there exists $r_0 \geq 0$ such that

$$\begin{aligned} \mu_0 &= \lambda + nr_0 \quad \text{and} \\ W_j \cap V_{\mu_0} &= \mathbb{R} \cdot w_{-k+2r_0}. \end{aligned} \tag{4.13}$$

Recall that $u(e_i) = \exp(X_i)$. By (4.3) and (4.8), for all $\mu > \mu_0$, we have

$$\begin{aligned} \pi_\mu(P_j(v_0)) &= P_j(\pi_\mu(v_0)) = 0, \quad \text{and} \\ \pi_\mu(\exp(X_i)P_j(v_0)) &= \pi_\mu(P_j(\exp(X_i)v_0)) = P_j(\pi_\mu(\exp(X_i)v_0)) = 0. \end{aligned}$$

Therefore

$$\{P_j(v_0), \exp(X_i)P_j(v_0)\} \subset \operatorname{span}\{w_{-k}, \dots, w_{-k+2r_0}\}. \tag{4.14}$$

Therefore by Lemma 4.1 applied to the $\mathfrak{sl}_2$-triple (X_i, H_i, Y_i), since $P_j(v_0) \neq 0$, we have

$$-k + 2r_0 \geq 0. \tag{4.15}$$

By (4.10), (4.12) and (4.13),

$$0 \neq P_j(v(\delta)) = P_j(q_\delta(\pi_{\mu_0}(v_0))) = q_\delta(P_j(\pi_{\mu_0}(v_0))) = P_j(\pi_{\mu_0}(v_0)).$$

Also

$$H_i(P_j(v(\delta))) = P_j(H_i(v(\delta))) = P_j(\delta(H_i)v(\delta)) = \delta(H_i) \cdot P_j(v(\delta)),$$

and

$$H_i(P_j(\pi_{\mu_0}(v_0))) = (-k+2r_0)P_j(\pi_{\mu_0}(v_0)).$$

Therefore by (4.15)

$$\delta(H_i) = -k+2r_0 \geq 0. \tag{4.16}$$

This completes the proof of the Claim 4.2.1.

Since $\pi_{\mu_0}(v_0) \neq 0$, there exists $\delta \in \Delta$ such that

$$v(\delta) := q_\delta(\pi_{\mu_0}(v_0)) \neq 0.$$

Now since $\mathcal{A} \cdot v(\delta) = \mu_0 v(\delta)$ and $\mathcal{A} = H_1 + \cdots + H_{n-1}$, by Claim 4.2.1

$$0 \geq \mu_0 = \sum_{i=1}^{n-1} \delta(H_i) \geq 0.$$

Therefore

$$\mu_0 = 0 \quad \text{and} \quad \delta(H_i) = 0, \quad \forall 1 \leq i \leq n-1. \tag{4.17}$$

Thus (4.4) is verified.

Going back to the representation W_j of $\mathfrak{g}_i$ considered above, by (4.16) and (4.17) $-k + 2r_0 = \delta(H_i) = 0$ and $\mathcal{A} \cdot w_0 = 0$. Therefore by (4.14), we have

$$\{P_j(v_0), \exp(X_i)P_j(v_0)\} \subset \operatorname{span}\{w_{-k}, \ldots, w_0\}.$$

Therefore, since $P_j(v_0) \neq 0$, by Lemma 4.1

$$\exp(X_i)P_j(v_0) \not\subset \operatorname{span}\{w_{-k}, \ldots, w_{-2}\}.$$

Hence by (4.13) and (4.8),

$$P_j(\pi_0(\exp(X_i)v_0)) = \pi_0(P_j(\exp(X_i)v_0)) = \pi_0(\exp(X_i)P_j(v_0)) \neq 0.$$

Therefore $\pi_0(\exp(X_i)v_0) \neq 0$. Thus (4.5) is verified. □

Consider the linear action of $\mathrm{Z}_G(A)$ on $\mathbb{R}^{n-1}$ such that

$$u(g \cdot e) = gu(e)g^{-1}, \quad \forall g \in \mathrm{Z}_G(A), \quad \forall e \in \mathbb{R}^{n-1}.$$

Note that under this action $\mathrm{Z}_G(A)$ maps onto $\mathrm{GL}(n-1, \mathbb{R})$. We also note that for any basis $\mathcal{C}$ of $\mathbb{R}^{n-1}$, the set

$$D_{\mathcal{C}} := \{g \in \mathrm{Z}_G(A) : \text{each } e \in \mathcal{C} \text{ is an eigenvector of } g\}$$

is a maximal $\mathbb{R}$-diagonalizable subgroup of G.

Corollary 4.3 *Let the notation be as in Proposition 4.2. Then for any $e \in \mathcal{B}$,*

$$g\pi_0(u(e)v) = \pi_0(u(e)v), \quad \forall g \in D_{\mathcal{C}}, \tag{4.18}$$

where $\mathcal{C} = \{e' - e : e' \in \mathcal{B} \setminus \{e\}\}$.

Proof As a consequence of Proposition 4.2, $\mu_0 = 0$ and $\pi_{\mu_0}(u(e)v) \neq 0$. Let the notation be as in the proof of Proposition 4.2. Then $u(e)v = v_0$. Now $\mathcal{C} = \{e_1, \dots, e_{n-1}\}$ and for any i, we have $H_i e_i = 2e_i$ and $H_i e_j = e_j$ if $j \neq i$. Therefore by (4.7), $D_{\mathcal{C}} = \exp(\mathfrak{b})$.

For any $\delta \in \Delta$, if $q_\delta(\pi_{\mu_0}(v_0)) \neq 0$ then by (4.17)

$$H_i q_\delta(\pi_{\mu_0}(v_0)) = \delta(H_i) q_\delta(\pi_{\mu_0}(v_0)) = 0, \quad \forall 1 \leq i \leq n-1.$$

Therefore $H_i \pi_{\mu_0}(v_0) = \sum_{\delta \in \Delta} H_i q_\delta(\pi_{\mu_0}(v_0)) = 0$ for all i. Hence $\mathfrak{b} \cdot \pi_0(v_0) = 0$. Therefore $D_{\mathcal{C}} \pi_0(v_0) = \pi_0(v_0) = \pi_0(u(e)v)$ and we obtain (4.18). □

Corollary 4.4 *Let a set $\mathcal{E} \subset \mathbb{R}^{n-1}$ and $e \in \mathcal{E}$ be such that the set $\mathcal{E}_e := \{e' - e : e' \in \mathcal{E}\}$ is not contained a union of $n-1$ proper subspaces of $\mathbb{R}^{n-1}$. Suppose that $v \in \mathbb{R}^{n-1}$ is such that*

$$u(e')v \in V^0 + V^-, \quad \forall e' \in \mathcal{E}.$$

Then

$$\begin{aligned} &\pi_0(u(e)v) \neq 0 \quad \textit{and} \\ &\mathrm{Z}_G(A) \subset \mathrm{Stab}_G(\pi_0(u(e)v)). \end{aligned} \tag{4.19}$$

Proof We note that the first conclusion follows from Proposition 4.2.

Replacing v by $u(e)v$ and $\mathcal{E}$ by $\mathcal{E}_e$, without loss of generality we may assume that $e = 0 \in \mathcal{E}$ and we only need to prove that

$$\mathrm{Z}_G(A) \subset \mathrm{Stab}_G(\pi_0(v)). \tag{4.20}$$

By our hypothesis there exists a basis $\{b_1, \dots, b_{n-1}\}$ of $\mathbb{R}^{n-1}$ contained in $\mathcal{E}$. Let $\{e_i : i = 1, \dots, n-1\}$ denote the standard basis of $\mathbb{R}^{n-1}$. We put $e_0 = 0$. Then there exists $z \in \mathrm{Z}_G(A)$ such that $zb_i = e_i$ for $1 \leq i \leq n-1$. Now by (4.4),

$$z(u(b)v) = (zu(b)z^{-1})(zv) = u(z \cdot b)(zv) \in V^0 + V^-, \quad \forall b \in \mathcal{E}.$$

Also $\pi_0(zw) = z\pi_0(w)$ for all $w \in V$. Therefore to prove the result, without loss of generality, we can replace $\mathcal{E}$ with $z \cdot \mathcal{E}$ and v with zv, and assume that

$$\mathcal{B} := \{e_i : 0 \leq i \leq n-1\} \subset \mathcal{E}.$$

Let $\mathcal{C} = \{e_1, \dots, e_{n-1}\}$. Let D denote the maximal diagonal subgroup of $\mathrm{SL}(n, \mathbb{R})$. Then $D_{\mathcal{C}} = D$ and by Corollary 4.3,

$$D \subset \mathrm{Stab}(\pi_0(v)). \tag{4.21}$$

By our hypothesis on $\mathcal{E}$, there exists $e_1' \in \mathcal{E}$ such that

$$e_1' = \sum_{i=1}^{n-1} \lambda_i e_i \quad \text{and} \quad \lambda_i \neq 0, \quad \forall 1 \leq i \leq n-1.$$

For $\boldsymbol{x} = (x_1, \ldots, x_{n-2}) \in \mathbb{R}^{n-2}$, let

$$w(\boldsymbol{x}) = \begin{bmatrix} 1 & & & & & \\ & 1 & x_2 & \ldots & x_{n-2} \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \tag{4.22}$$

Put $\boldsymbol{x} = (-\lambda_1^{-1}\lambda_2, \ldots, -\lambda_1^{-1}\lambda_{n-1})$. Then $w(\boldsymbol{x})e_1 = \lambda_1^{-1}e_1'$ and $w(\boldsymbol{x})e_i = e_i$ for $2 \le i \le n-1$. Let $\mathcal{C}' = \{e_1', e_2, \ldots, e_{n-1}\}$. Then

$$D_{\mathcal{C}'} = w(\boldsymbol{x})^{-1} D w(\boldsymbol{x}). \tag{4.23}$$

Since $\mathcal{B}' := \{0, e_1', e_2, \ldots, e_{n-1}\} \subset \mathcal{E}$, by Corollary 4.3 applied to $\mathcal{B}'$ in place of $\mathcal{B}$, we obtain that

$$w(\boldsymbol{x})^{-1} D w(\boldsymbol{x}) = D_{\mathcal{C}'} \subset \mathrm{Stab}(\pi_0(v)). \tag{4.24}$$

Since each coordinate of $\boldsymbol{x}$ is nonzero, the group generated by D and $w(\boldsymbol{x})^{-1}Dw(\boldsymbol{x})$ contains $W := \{w(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{R}^{n-2}\}$. Then by (4.21) and (4.24), $DW \subset \mathrm{Stab}_G(\pi_0(v))$. Let $\alpha > 1$. Then W is the expanding horospherical subgroup of $\mathrm{Z}_G(A)$ associated to

$$g_0 = \mathrm{diag}(1, \alpha^{n-2}, \alpha^{-1}, \ldots, \alpha^{-1}) \in D.$$

Note that $W^- := \{{}^{\mathrm{t}}w(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{R}^{n-2}\}$ is the contracting horospherical subgroup of $\mathrm{Z}_G(A)$ associated to g_0. Therefore $\pi_0(v)$ is stabilized by W^- [20, Lemma 5.2]. Since $\mathrm{Z}_G(A) \cong A \cdot \mathrm{SL}(n-1, \mathbb{R})$, one verifies that $\mathrm{Z}_G(A)$ is generated by W, W^- and D. Therefore $\mathrm{Z}_G(A)$ stabilizes $\pi_0(v)$; that is (4.20) holds. ☐

Lemma 4.5 *Let $x_i \to x$ be a convergent sequence in $\mathbb{R}^{n-1}$. Suppose there exists $v \in V$ such that*

$$u(x_i)v \in V^0 + V^-, \quad \forall i \in \mathbb{N}, \tag{4.25}$$

and a sequence $\delta_i \to 0$ of nonzero reals such that $f = \lim_{i\to\infty} (x_i - x)/\delta_i$ exists. Then

$$u(f) \in \mathrm{Stab}(\pi_0(u(x)v)). \tag{4.26}$$

Proof For any sequence $t_i \to \infty$ and $w_i \to w$ in $V^- + V^0$, we have $a_{t_i}w_i \to \pi_0(w)$ as $i \to \infty$. In particular,

$$a_{t_i}u(x_i)v \xrightarrow{i\to\infty} \pi_0(u(x)v). \tag{4.27}$$

Put $t_i = (1/n)\log(\delta_i^{-1})$ for all i. Then

$$\begin{aligned} a_{t_i}u(x_i) &= a_{t_i}u(x_i - x)u(x)v \\ &= u(e^{nt_i}(x_i - x))(a_{t_i}u(x)v) \xrightarrow{i\to\infty} u(f)\pi_0(u(x)v). \end{aligned}$$

Therefore $u(f)$ stabilizes $\pi_0(u(x)v)$. ☐

Corollary 4.6 *Let $\varphi : I = [a, b] \to \mathbb{R}^{n-1}$ be a differentiable curve which is not contained in a union of finitely many proper affine subspaces of $\mathbb{R}^{n-1}$. Let $0 \neq v \in V$ be such that*

$$u(\varphi(s))v \in V^0 + V^-, \quad \forall s \in I. \tag{4.28}$$

Then v is stabilized by G.

Proof We apply Corollary 4.4 to the set $\mathcal{E} = \{\varphi(s) : s \in I\}$ and conclude that $\pi_0(u(\varphi(s))v) \neq 0$ and it is stabilized by $Z_G(A)$ for all $s \in I$. Now let $s_0 \in I$ be such that $\dot{\varphi}(s_0) \neq 0$. Let $\delta_i \to 0$ be a sequence of nonzero reals. Then $\dot{\varphi}(s_0) = \lim_{i\to\infty}(\varphi(s_i) - \varphi(s_0))/\delta_i$. Therefore by Lemma 4.5 $\pi_0(u(\varphi(s_0))v)$ is stabilized by $u(\dot{\varphi}(s_0))$. Now the subgroup, say Q, generated by $u(\dot{\varphi}(s_0))$ and $Z_G(A)$ contains $\{u(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^{n-1}\}$. Therefore Q is a parabolic subgroup of G. Since Q stabilizes $\pi_0(u(\varphi(s_0))v)$, we conclude that G stabilizes $\pi_0(u(\varphi(s_0))v)$.

We put $v_0 = u(\varphi(s_0)v) - \pi_0(u(\varphi(s_0)v))$. Then

$$u(\varphi(s) - \varphi(s_0))v_0 = u(\varphi(s))v + \pi_0(u(\varphi(s_0))v) \in V^0 + V^-, \quad \forall s \in I.$$

We choose a finite subset $I_1 \subset I$ containing s_0 such that $\{\varphi(s) - \varphi(s_0) : s \in I_1\}$ is an affine basis of $\mathbb{R}^{n-1}$, and apply Proposition 4.2. Therefore if $v_0 \neq 0$ then $\pi_0(v_0) \neq 0$. Since by our choice $\pi_0(v_0) = 0$, we conclude that $v_0 = 0$. Therefore $u(\varphi(s_0))v$ is stabilized by G. Hence v is stabilized by G, because $u(\varphi(s_0)) \in G$. □

Proof of Theorem 1.6 If the conclusion of the theorem fails to hold then there exists $C > 0$ and a sequence $t_i \to \infty$ and convergent sequence $v_i \to v$ in V such that $\|v\| = 1$, and

$$\sup_{s\in I}\|a_{t_i}u(\varphi(s))v_i\| \le C\|v_i\|, \quad \forall i \in \mathbb{N}.$$

Therefore we conclude that for any $s \in I$,

$$\pi_+(u(\varphi(s)v)) = \lim_{i\to\infty} \pi_+(u(\varphi(s))v_i) = 0.$$

In other words,

$$u(\varphi(s))v \subset V^- + V^0, \quad \forall s \in I.$$

Then by Corollary 4.6, v is fixed by G. But this contradicts our hypothesis and the proof is complete. □

5 Ratner's theorem and dynamical behaviour of translated trajectories near singular sets

Our aim is to prove that μ, as obtained in Corollary 3.5, is L-invariant. As explained in Sect. 1.1, we will use a technique from [21].

5.1 Twisted curves and limit measure

Let $\dot{\varphi}(s)$ denotes the tangent to the curve φ at s. Now onward we shall assume that $\dot{\varphi}(s) \neq 0$ for all $s \in I$.

Fix $w_0 \in \mathbb{R}^{n-1}\setminus\{0\}$, and define

$$W = \{u(sw_0) : s \in \mathbb{R}\}. \tag{5.1}$$

Recall that $\mathrm{Z}_G(A)$ acts on $\mathbb{R}^{n-1}$ via the correspondence $u(z \cdot v) = zu(v)z^{-1}$ for all $z \in \mathrm{Z}_G(A)$ and $v \in \mathbb{R}^{n-1}$. This action is transitive on $\mathbb{R}^{n-1}\setminus\{0\}$. Therefore there exists an analytic function $z : I \to \mathrm{Z}_G(A)$ such that

$$z(s) \cdot \dot{\varphi}(s) = w_0. \tag{5.2}$$

From the first paragraph of Sect. 3 we recall that we have $\overline{Gx_0} = L/\Lambda$ and $x_i \to x_0$ in L/Λ. For any $i \in \mathbb{N}$, let λ_i be the probability measure on L/Λ defined by

$$\int_{L/\Lambda} f\, d\lambda_i := \frac{1}{|I|} \int_{s\in I} f(z(s)a_{t_i}u(\varphi(s))x_i)\, ds, \quad \forall f \in \mathrm{C_c}(L/\Lambda). \tag{5.3}$$

Corollary 5.1 *After passing to a subsequence, $\lambda_i \to \lambda$ with respect to the weak*-topology on the space of probability measures on L/Λ.*

Proof Given $\epsilon > 0$, by Theorem 3.1 there exists a compact set $\mathcal{F} \subset L/\Lambda$ such that $\mu_i(\mathcal{F}) \geq 1 - \epsilon$ for all $i > 0$. Since $z(I)$ is compact, there exists a compact set $\mathcal{F}_1 \supset z(I)\mathcal{F}$. Then $\lambda_i(\mathcal{F}_1) \geq 1 - \epsilon$ for all i. Now the corollary is deduced by standard arguments using the one-point compactification of L/Λ. □

5.2 Invariance under unipotent flow

Proposition 5.2 *Suppose that $\lambda_i \xrightarrow{i\to\infty} \lambda$ in the space of probability measures on L/Λ with respect to the weak*-topology. Then λ is W-invariant.*

Proof This statement can be deduced by an argument identical to that of the proof of [21, Theorem 3.1]. An idea of this proof is based on the explanation related to (1.10)–(1.12). □

Now we shall describe the measure λ using Ratner's [17] description of ergodic invariant measures for unipotent flows. Let $\pi : L \to L/\Lambda$ denote the natural quotient map. Let W be as defined in (5.1). For $H \in \mathcal{H}$, define

$$N(H, W) = \{g \in G : g^{-1}Wg \subset H\} \quad \text{and}$$
$$S(H, W) = \bigcup_{\substack{F\in\mathcal{H} \\ F\subsetneq H}} N(F, W).$$

Then by Ratner's theorem [17], as explained in [16, Theorem 2.2]:

Theorem 5.3 (Ratner) *Given a W-invariant probability measure λ on L/Λ, there exists $H \in \mathcal{H}$ such that*

$$\lambda(\pi(N(H, W)) > 0 \quad \textit{and} \quad \lambda(\pi(S(H, W)) = 0. \tag{5.4}$$

Moreover almost every W-ergodic component of λ on $\pi(N(H, W))$ is a measure of the form $g\mu_H$, where $g \in N(H, W) \setminus S(H, W)$ and μ_H is a finite H-invariant measure on $\pi(H) \cong H/H \cap \Lambda$. In particular if H is a normal subgroup of L then λ is H-invariant. □

5.3 Algebraic criterion for non-accumulation of measure on singular set

Let V be as in Sect. 3.1 and w_0 as in (5.1). Let $\mathcal{A} = \{v \in V : v \wedge w_0 = 0\}$. Then $\mathcal{A}$ is the image of a linear subspace of V. We observe that

$$N(H, W) = \{g \in L : g \cdot p_H \in \mathcal{A}\}. \tag{5.5}$$

Proposition 5.4 (Cf. [5]) *Given a compact set $\mathcal{C} \subset \mathcal{A}$ and $\epsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that given any neighbourhood Φ of $\mathcal{D}$ in V, there exists a neighbourhood Ψ of $\mathcal{C}$ in V contained in Φ such that for any $h \in G$, any $v \in V$ and any open interval $J \subset I$, one of the following holds*:

(I) $hz(t)u(\varphi(t))v \in \Phi$ *for all* $t \in J$.
(II) $|\{t \in J : hz(t)u(\varphi(t))v \in \Psi\}| \le \epsilon |\{t \in J : hz(t)u(\varphi(t))v \in \Phi\}|$.

Proof As noted in [21, Proposition 4.6], the argument in the proof of [5, Proposition 4.2] goes through with straightforward changes using the Proposition 3.3 instead of [5, Lemma 4.1]. □

The next criterion is the main outcome of the linearization technique.

Proposition 5.5 (Cf. [16]) *Let C be any compact subset of $N(H, W) \setminus S(H, W)$. Let $\epsilon > 0$ be given. Then there exists a compact set $\mathcal{D} \subset \mathcal{A}$ such that given any neighbourhood Φ of $\mathcal{D}$ in V, there exists a neighbourhood $\mathcal{O}$ of $\pi(C)$ in L/Λ such that for any $h_1, h_2 \in L$, and a subinterval $J \subset I$, one of the following holds*:

(a) *There exists $\gamma \in \Lambda$ such that $(h_1 z(t)u(\varphi(t))h_2\gamma)p_H \in \Phi$, $\forall t \in J$.*
(b) $|\{t \in J : h_1 z(t)u(\varphi(t))\pi(h_2) \in \mathcal{O}\}| \le \epsilon |J|$.

Proof Again this result and its proof are essentially same as those of [21, Proposition 4.7]. □

5.4 Applying the criterion and the basic lemma

Let $\{\lambda_i\}$ be the measures as defined in (5.3). By our assumption $\overline{Gx_0} = L/\Lambda$.

Theorem 5.6 *Suppose that $\lambda_i \xrightarrow{i\to\infty} \lambda$ in the space of probability measures on L/Λ with respect to weak*-topology. Then λ is L-invariant.*

Proof By Proposition 5.2, λ is invariant under the action of the nontrivial unipotent subgroup W. Therefore by Theorem 5.3 there exists $H \in \mathcal{H}$ such that

$$\lambda(\pi(N(H, W)) > 0 \quad \text{and} \quad \lambda(\pi(S(H, W)) = 0. \tag{5.6}$$

Let C be a compact subset of $N(H, W) \setminus S(H, W)$ such that $\lambda(C) > \epsilon$ for some $\epsilon > 0$. In other words, if we write $x_0 = \pi(g_0)$ for some $g_0 \in G$, then there exists a sequence $g_i \to g_0$ such that $x_i = \pi(g_i)$. Given any neighbourhood $\mathcal{O}$ of $\pi(C)$ in L/Λ, there exists $i_0 > 0$ such that for all $i \ge i_0$, we have $\lambda_i(\mathcal{O}) > \epsilon$ for all $i > i_0$ and hence

$$\frac{1}{|I|} |\{s \in I : z(s)a_{t_i}u(\varphi(s))x_i = \pi(a_{t_i}z(s)u(\varphi(s))g_i) \in \mathcal{O}\}| > \epsilon. \tag{5.7}$$

Let $\mathcal{D} \subset \mathcal{A}$ be as in the statement of Proposition 5.5. Choose any compact neighbourhood Φ of $\mathcal{D}$ in V. Then there exists a neighbourhood $\mathcal{O}$ of $\pi(C)$ in L/Λ such that one of the two possibilities of the Proposition 5.5 holds. Therefore due to (5.7), for all $i > i_0$ there exists $\gamma_i \in \Lambda$ such that

$$(z(s)a_{t_i}u(s)g_i\gamma_i)p_H = (a_{t_i}z(s)u(s)g_i\gamma_i)p_H \in \Phi, \quad \forall s \in I.$$

Let $\Phi_1 = \{z(s)^{-1} : s \in I\}\Phi$. Then Φ_1 is contained in a compact subset of V, and the following holds:

$$a_{t_i}u(s)(g_i\gamma_i)p_H \subset \Phi_1, \quad \forall s \in I,\ \forall i > i_0. \tag{5.8}$$

Now we express $V = W_0 \oplus W_1$, where W_0 is the subspace consisting of all G-fixed vectors and W_1 is its G-invariant complement. For $i \in \{0, 1\}$, let $P_i : V \to W_i$ denote the associated projection. Consider any norm $\|\cdot\|$ on V such that

$$\|w\| = \max\{\|P_0(w)\|, \|P_1(w)\|\}, \quad \forall w \in V. \tag{5.9}$$

Let $R = \sup\{\|w\| : w \in \Phi_1\}$. By (5.8), for all $i \geq i_0$ and $s \in I$ we have

$$\|a_t u(\varphi(s))(g_i\gamma_i p_H)\| = \|P_0(g_i\gamma_i p_H)\| + \|a_t u(\varphi(s))P_1(g_i\gamma_i p_H)\| < R. \tag{5.10}$$

Therefore, by Theorem 1.6 applied to W_1 in place of V and $C = 1$, there exists $i_1 > i_0$ such that

$$\|P_1(g_i\gamma_i p_H)\| < R, \quad \forall i > i_1. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we have

$$\|g_i\gamma_i p_H\| < R, \quad \forall i \geq i_1. \tag{5.12}$$

The orbit $\Lambda \cdot p_H$ is discrete due to Proposition 3.2. And $g_i \to g_0$ as $i \to \infty$. Therefore by passing to a subsequence we may assume that $\gamma_i p_H = \gamma_{i_1} p_H$ for all $i \geq i_1$. Put $\delta_0 = \|P_1(g_0\gamma_{i_1} p_H)\| > 0$ and $C = 2R\delta_0^{-1}$. Then By Theorem 1.6, there exists $i_2 \geq i_1$ such that for all $i \geq i_2$ we have

$$\sup_{s\in I}\|a(t_i)u(\varphi(s))P_1(g_i\gamma_{i_1} p_H)\| \geq C\|P_1(g_i\gamma_{i_1} p_H)\| > R.$$

This contradicts (5.10) for all $i \geq i_2$, unless $P_1(g_0\gamma_{i_1} p_H) = 0$. Hence $g_0\gamma_{i_1} p_H$ is G-fixed. Since $\Lambda \cdot p_H$ is closed in V, $\Lambda\mathrm{Stab}(p_H) = \Lambda\mathrm{N}_L^1(H)$ is closed in L. Therefore by taking the inverse $\mathrm{N}_L^1(H)\Lambda$ is closed in L. Hence the orbit $\pi(\mathrm{N}_L^1(H))$ is closed in L/Λ. By [19, Theorem 2.3] there exists a closed subgroup H_1 of $\mathrm{N}_L^1(H)$ of L containing all Ad-unipotent one-parameter subgroups of L contained in $\mathrm{N}_L^1(H)$ such that $H_1 \cap \Lambda$ is a lattice in H_1 and $\pi(H_1)$ is closed. Now $\rho(G)$ is generated by unipotent one-parameter subgroups. Therefore if we put $F = g_0\gamma_{i_1}H_1(g_0\gamma_{i_1})^{-1}$, then $\rho(G) \subset F$. Also $Fx_0 = \underline{g_0\gamma_{i_1}}\pi(H_1)$ is closed and admits a finite F-invariant measure. Hence by our assumption that $\overline{\rho(G)x_0} = L/\Lambda$, we have $F = L$. Therefore $L = H_1 \subset \mathrm{N}_L^1(H)$. Therefore, since $N(H, W) \neq 0$, we have $N(H, W) = L$. In particular, $W \subset H$. Thus $H \cap \rho(G)$ is a normal subgroup of $\rho(G)$ containing W. Since $\rho(G)$ is a simple Lie group, $\rho(G) \subset H$. Since H is a normal subgroup of L and $\pi(H)$ is a closed orbit with finite H-invariant measure, every orbit of H on L/Λ is closed and admits a finite H-invariant measure. Since $\overline{\rho(G)x_0} = L/\Lambda$, we have $H = L$. Now in view of (5.6), by Theorem 5.3 we conclude that the measure λ is $H = L$-invariant. $\square$

Corollary 5.7 *The measure μ as in the statement of Corollary 3.5 is the unique L-invariant probability measure on L/Λ.*

Proof Since φ is analytic, the set of points where $\dot{\varphi}(s)=0$ is finite. Therefore it is enough to prove the theorem separately for each closed subinterval J of I, in place of I, under the additional hypothesis that $\dot{\varphi}(s)\neq 0$ for all $s\in J$. Since φ is analytic, if J is any subinterval of I with nonempty interior, then $\varphi(J)$ is not contained in a proper affine subspace of $\mathbb{R}^{n-1}$. Therefore without loss of generality we assume that $\dot{\varphi}(s)\neq 0$ for all $s\in I$. Let $z(s)\in \mathrm{Z}_G(A)$ be as defined in (5.2). Given $\epsilon>0$ there exists a neighbourhood O of the e in $\mathrm{Z}_G(A)$ such that $|f(zx)-f(x)|<\epsilon$ for all $x\in L/\Lambda$ and $z\in O$. We consider a partition $I=J_1\cup\cdots\cup J_k$ such that for any $s,s'\in J_j$, we have $z(s)^{-1}z(s')\in O$. For each $j\in\{1,\ldots,k\}$, choose $s_j\in J_j$, and define the function $f_j(x)=f(z(s_j)^{-1}x)$ for all $x\in L/\Lambda$. Then by Theorem 5.6, applied to the interval J_j in the place of I, there exists $i_j>0$ such that for all $i>i_j$, we have

$$\left|\int_{J_j} f_j(z(s)a_{t_i}u(\varphi(s))x_i)\,ds-|J_j|\int_{L/\Lambda} f_j(x)\,d\mu_L(x)\right|\leq \epsilon|J_j|. \tag{5.13}$$

Since μ_L is $\mathrm{Z}_G(A)$-invariant,

$$\int_{L/\Lambda} f_j(x)\,d\mu_L(x)=\int_{L/\Lambda} f(x)\,d\mu_L(x)=: S_j. \tag{5.14}$$

Now

$$\begin{aligned}
&\left|\int_{J_j} f(a_{t_i}u(\varphi(s))x_i)\,ds-|J_j|S_j\right|\\
&\quad=\left|\int_{J_j} f((z(s)^{-1}z(s_j))z(s_j)^{-1}z(s)a_{t_i}u(\varphi(s))x_i)\,ds-|J_j|S_j\right|\\
&\quad\leq\left|\int_{J_j} f_j(z(s)a_{t_i}u(\varphi(s))x_i)\,ds-|J_j|S_j\right|+\epsilon|J_j| &(5.15)\\
&\quad\leq 2\epsilon|J_j|, &(5.16)
\end{aligned}$$

where (5.15) follows from the choice of O and the partition of I into J_j's, and (5.16) follows from (5.13) and (5.14).

Therefore for any $i\geq\max\{i_1,\ldots,i_k\}$, we have

$$\begin{aligned}
|I|\cdot\left|\int f\,d\mu_i-\int f\,d\mu_L\right| &=\left|\int_I f(a_{t_i}u(\varphi(s))x_i)\,ds-|I|\int f(x)\,d\mu_L\right|\\
&\leq\sum_{j=1}^{k}\left|\int_{J_j} f(a_{t_i}u(\varphi(s))x_i)\,ds-|J_j|\int f\,d\mu_L\right|\\
&\leq 2\epsilon\sum_{j=1}^{k}|J_j|\leq 2\epsilon|I|.
\end{aligned}$$

This shows that μ is L-invariant. □

Proof of Theorem 1.3 By [19, Theorem 2.3] there exists the smallest subgroup H of L containing $\rho(G)$ such that the orbit Hx_0 is closed and admits a finite H-invariant measure.

Therefore replacing L by H and Λ by the stabilizer of x_0 in H, without loss of generality we may assume that $H = L$.

If (1.5) fails to hold then there exist $\epsilon > 0$ and a sequence $t_i \to \infty$ such that for each i,

$$\left| \frac{1}{|b-a|} \int_a^b f(\rho(a_{t_i} u(\varphi(s)))x_0)\, ds - \int_{L/\Lambda} f\, d\mu_L \right| \geq \epsilon.$$

If we put $x_i = x_0$ for each i, then in view of (3.1) and Corollary 3.5, this statement contradicts Corollary 5.7. □

Proof of Theorem 1.9 Note that if the theorem fails to hold then, there exist sequences $x_i \to x_0$ in $\mathcal{K}$ and $t_i \to \infty$ in $\mathbb{R}$ such that

$$\left| \frac{1}{|b-a|} \int_a^b f(a_{t_i} u(\varphi(s))x_i)\, ds - \int f\, d\mu_G \right| > \epsilon, \quad \forall i.$$

This statement contradicts Corollary 5.7. □

Proof of Theorem 1.8 Without loss of generality we may assume that I is a box in $\mathbb{R}^d$ with sides parallel to the coordinate axis. Since $\psi(I)$ is not contained in a proper affine subspace of $\mathbb{R}^n$ and ψ is analytic, there exists a line ℓ in $\mathbb{R}^d$ such that for almost every $\boldsymbol{s} \in I$ the set $\psi((\boldsymbol{s}+\ell) \cap I)$ is not contained in a proper affine subspace of R^n. It is enough to prove that the limiting distributions of translates of conditional measures on $\psi((\boldsymbol{s}+\ell) \cap I)x_0$ are same as μ_H for almost all $\boldsymbol{s} \in I$. Therefore without loss of generality it is enough to prove the theorem in the case of $d = 1$ and $I = [a, b]$.

Let $\psi_{i,j}(s)$ denote the (i, j)-th coordinate of $\boldsymbol{\psi}(s)$ for all $s \in I$. By our hypothesis, the set $\{t : \psi_{1,1}(t) = 0\}$ is finite. Therefore arguing as in the proof of Corollary 5.7, without loss of generality we may assume that $\psi_{1,1}(s) \neq 0$ for all $s \in I := [a, b]$. Define

$$\varphi(s) = \left(\frac{\psi_{1,2}(s)}{\psi_{1,1}(s)}, \ldots, \frac{\psi_{1,n}(s)}{\psi_{1,1}(s)} \right) \in \mathbb{R}^{n-1}, \quad \forall s \in I.$$

Let $U^- = \{g \in G : a_t g a_t^{-1} \xrightarrow{t\to\infty} e\}$. Then there exist continuous maps $\psi_- : I \to U^-$ and $\psi_0 : I \to \mathrm{Z}_G(A)$ such that

$$\psi(s) = \psi_-(s)\psi_0(s)u(\varphi(s)), \quad \forall s \in I.$$

We observe that the curve $\{\varphi(s) : s \in I\}$ is contained in a proper affine subspace of $\mathbb{R}^{n-1}$ if and only if the curve $\{(\psi_{1,j}(s))_{1\leq j \leq n}) : s \in I\}$ is contained in a proper subspace of $\mathbb{R}^n$. Given any $\epsilon > 0$ and $f \in \mathrm{C_c}(L/\Lambda)$, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $x \in L/\Lambda$, we have $|f(a_t\psi_-(s)x) - f(a_t x)| < \epsilon$. Therefore without loss of generality we may replace $\psi(s)$ by $\psi_0(s)u(\varphi(s))$ for all $s \in I$ to prove the theorem.

Now we apply the argument of the proof of Corollary 3.5 to $\psi_0(s)$ in place of $z(s)$, and Theorem 1.3 in place of Theorem 5.6, to complete the proof of the theorem. □

Proof of Theorem 1.10 The result can be obtained by following the general strategy of [5] and the method of this article. □

Acknowledgements I am very grateful to Shahar Mozes and Elon Lindenstrauss for the extensive discussions which have significantly contributed to the process of obtaining the 'Basic Lemma'. Thanks are due to D. Kleinbock, B. Weiss, Y. Bugeaud and W. Schmidt for their remarks on number theoretic implications of the results considered here. I would like to thank my wife for her encouragement.

Thanks are due to the referee whose comments have led to improvement of the exposition and the statements of the main results.

References

1. Baker, R.C.: Dirichlet's theorem on diophantine approximation. Math. Proc. Camb. Philos. Soc. **83**(1), 37–59 (1978)
2. Bugeaud, Y.: Approximation by algebraic integers and Hausdorff dimension. J. Lond. Math. Soc. (2) **65**(3), 547–559 (2002)
3. Dani, S.G.: Divergent trajectories of flows on homogeneous spaces and diophantine approximation. J. Reine Angew. Math. **359**, 55–89 (1985)
4. Dani, S.G., Margulis, G.A.: Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces. Proc. Indian Acad. Sci. Math. Sci. **101**(1), 1–17 (1991)
5. Dani, S.G., Margulis, G.A.: Limit distributions of orbits of unipotent flows and values of quadratic forms. In: I.M. Gelfand Seminar, pp. 91–137. Am. Math. Soc., Providence (1993)
6. Davenport, H., Schmidt, W.M.: Dirichlet's theorem on diophantine approximation. II. Acta Arith. **16**, 413–424 (1969/1970)
7. Davenport, H., Schmidt, W.M.: Dirichlet's Theorem on Diophantine Approximation, INDAM, Rome, 1968/1969. Symposia Mathematica, vol. IV, pp. 113–132. Academic Press, London (1970)
8. Dodson, M.M., Rynne, B.P., Vickers, J.A.: Dirichlet's theorem and diophantine approximation on manifolds. J. Number Theory **36**(1), 85–88 (1990)
9. Duke, W., Rudnick, Z., Sarnak, P.: Density of integer points on affine homogeneous varieties. Duke Math. J. **71**(1), 143–179 (1993)
10. Eskin, A., McMullen, C.: Mixing, counting, and equidistribution in Lie groups. Duke Math. J. **71**(1), 181–209 (1993)
11. Eskin, A., Mozes, S., Shah, N.: Unipotent flows and counting lattice points on homogeneous varieties. Ann. Math. (2) **143**(2), 253–299 (1996)
12. Gorodnik, A.: Open problems in dynamics and related fields. J. Mod. Dyn. **1**(1), 1–35 (2007)
13. Kleinbock, D.Y., Margulis, G.A.: Flows on homogeneous spaces and diophantine approximation on manifolds. Ann. Math. (2) **148**(1), 339–360 (1998)
14. Kleinbock, D.Y., Margulis, G.A.: On effective equidistribution of expanding translates of certain orbits in the space of lattices. arXiv:math/0702433
15. Kleinbock, D., Weiss, B.: Dirichlet's theorem on diophantine approximation and homogeneous flows. J. Mod. Dyn. (JMD) **2**(1), 43–62 (2008). arXiv:math/0612171
16. Mozes, S., Shah, N.A.: On the space of ergodic invariant measures of unipotent flows. Ergod. Theory Dyn. Syst. **15**(1), 149–159 (1995)
17. Ratner, M.: On Raghunathan's measure conjecture. Ann. Math. (2) **134**(3), 545–607 (1991)
18. Ratner, M.: Raghunathan's topological conjecture and distributions of unipotent flows. Duke Math. J. **63**(1), 235–280 (1991)
19. Shah, N.A.: Uniformly distributed orbits of certain flows on homogeneous spaces. Math. Ann. **289**(2), 315–334 (1991)
20. Shah, N.A.: Limit distributions of expanding translates of certain orbits on homogeneous spaces. Proc. Indian Acad. Sci. (Math. Sci.) **106**, 105–125 (1996)
21. Shah, N.A.: Limiting distributions of curves under geodesic flow on hyperbolic manifold. Duke Math. J. **148**(2), (2009). arXiv:0708.4093
22. Shah, N.A.: Asymptotic evolution of smooth curves under geodesic flow on hyperbolic manifolds. Duke Math. J. **148**(2), (2009). arXiv:0804.3749
23. Shah, N.A.: Expanding translates of curves and Dirichlet-Minkowski theorem on linear forms. 28 pages. arXiv:0804.1424