EQUIDISTRIBUTION OF POLYNOMIALLY BOUNDED O-MINIMAL CURVES IN HOMOGENEOUS SPACES

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Abstract. We extend Ratner's theorem on equidistribution of individual orbits of unipotent flows on finite volume homogeneous spaces of Lie groups to trajectories of non-contracting curves definable in a polynomially bounded o-minimal structure.

To be precise, let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a continuous curve whose coordinate functions are definable in a polynomially bounded o-minimal structure; for example, rational functions. Suppose that φ is non-contracting; that is, for any linearly independent vectors $v_1,\dots,v_k$ in $\mathbb{R}^n$, $\varphi(t)\cdot(v_1\wedge\cdots\wedge v_k) \not\to 0$ as $t\to\infty$. Then, there exists a unique closed connected subgroup $H_\varphi \subseteq \mathrm{SL}(n,\mathbb{R})$ of smallest dimension such that H_φ is generated by unipotent one-parameter subgroups and such that $\varphi(t)H_\varphi \to g_0 H_\varphi$ in $\mathrm{SL}(n,\mathbb{R})/H_\varphi$ as $t\to\infty$, for some $g_0 \in \mathrm{SL}(n,\mathbb{R})$.

Let $\mathcal{G} \subseteq \mathrm{SL}(n,\mathbb{R})$ be a closed subgroup and let $\Gamma \subseteq \mathcal{G}$ be a lattice. Suppose that $\varphi([0,\infty)) \subseteq \mathcal{G}$. Then $H_\varphi \subseteq \mathcal{G}$, and for any $x \in \mathcal{G}/\Gamma$, the trajectory $\{\varphi(t)x : t\in[0,T]\}$ is equidistributed with respect to the measure $g_0\mu_{Lx}$ as $T\to\infty$, where $L \subseteq \mathcal{G}$ is a closed subgroup such that $\overline{H_\varphi x} = Lx$ and Lx admits a unique L-invariant probability measure, denoted by μ_{Lx}.

A crucial new ingredient is the proof that for any finite-dimensional representation V of $\mathrm{SL}(n,\mathbb{R})$, there exist $T_0>0$, $C>0$, and $\alpha>0$ such that for any $v\in V$, the map $t\mapsto\|\varphi(t)v\|$ is (C,α)-good on $[T_0,\infty)$.

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1. Introduction

Let $\mathcal{G}$ be a Lie group, let $\Gamma \subseteq \mathcal{G}$ be a lattice and consider $X := \mathcal{G}/\Gamma$. For a closed subgroup $H \subseteq \mathcal{G}$ and an $x \in X$, the orbit Hx is called *periodic* if the orbit Hx is closed and supports a unique H-invariant probability measure. We say that a probability measure μ on X is *homogeneous* if μ is supported on a periodic orbit Hx and is the H-invariant probability on that orbit. Marina Ratner, in proving Raghunathan's conjectures stated in [Dan81], showed that any Borel probability measure μ which is invariant and ergodic under the action of a $\mathrm{Ad}_{\mathcal{G}}$-unipotent one-parameter subgroup is homogeneous, see [Rat91a]. Using this measure classification and the non-divergence property of unipotent orbits due to Margulis and Dani [Mar71; Dan86], Ratner [Rat91b] proved the following equidistribution result for unipotent flows: Let $U := \{u(t) : t \in \mathbb{R}\}$ be a one-parameter $\mathrm{Ad}_{\mathcal{G}}$-unipotent subgroup of $\mathcal{G}$ and let $x \in X$. Then there exists a closed subgroup H of $\mathcal{G}$ containing U such that Hx is a periodic orbit, and for any $f \in C_c(X)$, $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u(t)x)dt = \int_X f\, d\mu_{Hx}$, where μ_{Hx} is the homogeneous probability measure on Hx.

More generally, for a continuous curve $\varphi : [0,\infty) \to \mathcal{G}$, $x \in X$, and $T > 0$, consider the probability measure $\mu_{T,\varphi,x}$ on X defined by:

$$\mu_{T,\varphi,x}(f) = \frac{1}{T}\int_0^T f(\varphi(t)x)\, dt,\ \forall f \in C_c(X). \tag{1.1}$$

In [Sha94], it was shown that the equidistribution of unipotent flows due to Ratner generalizes to polynomial curves. Namely, if $\mathcal{G} \subseteq \mathrm{SL}(n,\mathbb{R})$ and $\varphi : \mathbb{R} \to \mathcal{G}$ is a map whose coordinate functions are polynomials, then the measures $\mu_{T,\varphi,x}$ converge to $\varphi(0)\mu_{Hx}$ as $T \to \infty$, where $H \subseteq \mathcal{G}$ is the smallest closed subgroup such that $\{\varphi(0)^{-1}\varphi(t) : t \in \mathbb{R}\} \subseteq H$ and the orbit Hx is periodic.

More recently, Peterzil and Starchenko [PS21] studied such questions in the following setting: Let $\mathcal{G}$ be a closed subgroup of the group of the upper-triangular unipotent matrices in $\mathrm{SL}(n,\mathbb{R})$ and let $\Gamma \subseteq \mathcal{G}$ be a (cocompact) lattice. They showed that if $S \subseteq \mathcal{G}$ is definable in an o-minimal structure, then there exists a definable set $S' \subseteq \mathcal{G}$ whose image in $\mathcal{G}/\Gamma$ is the closure of the image of S in $\mathcal{G}/\Gamma$. Moreover, they proved the following [PS21, Theorem 1.6]: Let $\varphi : [0,\infty) \to \mathcal{G}$ be a curve definable in a polynomially bounded o-minimal structure, and let $x \in \mathcal{G}/\Gamma$ such that $\{\varphi(t)x : t \geq 0\}$ is dense in G/Γ. Then, $\mu_{T,\varphi,x}$ converges to the $\mathcal{G}$-invariant probability on $\mathcal{G}/\Gamma$ as $T \to \infty$.

Our goal in this paper is to generalize the above equidistribution results for curves definable in polynomially bounded o-minimal structures taking values in linear Lie groups, under the assumption that an additional non-contraction condition is satisfied.

1.1. **O-minimal curves.**

We now provide an overview of the notion of an o-minimal structure. In this paper, o-minimal structures refer to o-minimal

structures on the real field. We have provided a more detailed summary of the required definitions, results, and references in Appendix B.

The notion of o-minimality was introduced in [Dri86] as a generalization of semi-algebraic geometry. The semi-algebraic sets in $\mathbb{R}^d$ are formed by taking finite unions and intersections of algebraic sets $\{x \in \mathbb{R}^d : P(x) = 0\}$ and of sets of the form $\{x \in \mathbb{R}^d : P(x) > 0\}$, where P is a polynomial in d variables. These semi-algebraic sets form the smallest o-minimal structure, and our results are already new in the semi-algebraic setup.

More generally, an o-minimal structure is composed of sets in $\mathbb{R}^d$ for each $d \in \mathbb{N}$ such that several axioms are satisfied. The collection of semi-algebraic sets satisfy those axioms and form the smallest o-minimal structure. In a general o-minimal structure, the collection of sets are referred to as the *definable sets*, and a function is called a *definable function* if its graph is a definable subset. The definable sets and functions share many desirable properties, see Appendix B. Particularly, the collection of definable functions is both rich and flexible: for example, in any o-minimal structure, all rational functions are definable; these are functions of the form $f(t) = \frac{P(t)}{Q(t)}$, on the domain where $Q(t) \neq 0$, and where P and Q are polynomials. Moreover, if a definable function is injective, its inverse is also definable, and the composition of definable functions remains definable.

An o-minimal structure is called *polynomially bounded*, if for every definable function $f : \mathbb{R} \to \mathbb{R}$, there exists $r \in \mathbb{R}$ such that $f(x) = O(x^r)$, as $x \to \infty$. The semi-algebraic o-minimal structure is polynomially bounded. Larger polynomially bounded o-minimal structures include as definable functions the analytic functions restricted to compact sets, and the power functions $t \mapsto t^r$, for $r \in \mathbb{R}$; see [Dri86; Mil94a].

We note the following useful fact.

Proposition 1.1. [Mil94b]: *If $f : [0, \infty) \to \mathbb{R}$ is definable in a polynomially bounded o-minimal structure, then either f is eventually constantly zero, or there exists $r \in \mathbb{R}$ such that $\lim_{t\to\infty} f(t)/t^r = c \neq 0$. In the latter case, we define*

$$\deg(f) := r. \tag{1.2}$$

Remark 1.2. Our equidistribution results will be stated only for polynomially bounded o-minimal structures. As noted in [PS21], this setup is optimal in the following sense. Notice that the measures on the circle defined by

$$\frac{1}{T}\int_0^T f(\log(t+1) + \mathbb{Z})\, dt,\ \forall f \in C(\mathbb{R}/\mathbb{Z}),$$

do not converge as $T \to \infty$. By [Mil94b], if an o-minimal structure is not polynomially bounded, then $x \mapsto e^x$ is definable, and as a consequence, $x \mapsto \log(x)$ will also be definable.

We will denote by $\mathbf{G}$ an affine algebraic group defined over $\mathbb{R}$, and in what follows, we let $G := \mathbf{G}(\mathbb{R})$ be the real points of $\mathbf{G}$.

Definition 1.3. Let $\mathbb{R}[G]$ be the coordinate ring of real polynomial functions on G. We say that a curve $\varphi : [0,\infty) \to G$ is *definable* in an o-minimal structure, if the map $t \mapsto f(\varphi(t))$ on $[0,\infty)$ is definable, for all $f \in \mathbb{R}[G]$.

We observe that if V is a finite-dimensional rational representation, and $\varphi : [0,\infty) \to G$ is definable in $\mathscr{S}$, then the curve $t \mapsto \varphi(t) \cdot v$ is definable in $\mathscr{S}$ for all $v \in V$. So, if G is embedded in $\mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{M}(n,\mathbb{R})$, and

$$\varphi(t) = [\varphi_{i,j}(t)]_{1\le i,j\le n},\ \forall t \in [0,\infty), \tag{1.3}$$

then φ is definable in $\mathscr{S}$ if and only if each $\varphi_{i,j} : [0,\infty) \to \mathbb{R}$ is definable in $\mathscr{S}$.

1.1.1. *The non-contraction property.* We now introduce the non-contraction property, which narrows down the polynomially bounded o-minimal curves to a class of curves whose orbits on rational linear representations of G have the (C,α)-good growth property introduced by Kleinbock and Margulis [KM98].

Definition 1.4. We say that a definable curve $\varphi : [0,\infty) \to G$ is *non-contracting* for G if for all rational finite-dimensional representations V of G defined over $\mathbb{R}$, it holds that $\lim_{t\to\infty} \varphi(t) \cdot v \neq 0$ for all $v \in V \setminus \{0\}$.

We note that for any definable curve $\varphi : [0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ and any $v \in \mathbb{R}^n$, $\lim_{t\to\infty} \varphi(t) \cdot v$ exists in $V \cup \{\infty\}$ by the monotonicity theorem B.5.

Remark 1.5. Suppose that G is a unipotent group and $\varphi : [0,\infty) \to G$ is a definable curve. Then φ is non-contracting. This is so because, for any algebraic action of an algebraic unipotent group on a finite-dimensional vector space, every orbit of a nonzero vector is Zariski closed, and in particular, the closure of the orbit does not contain the origin. See [Bir71, Theorem 12.1].

We now give a practical criterion that allows one to verify the non-contraction property for curves in $\mathrm{SL}(n,\mathbb{R})$, see also Proposition 2.22. Consider the exterior representation of $\mathrm{SL}(n,\mathbb{R})$ on $\bigwedge^k \mathbb{R}^n$ defined by

$$g \cdot (v_1 \wedge v_2 \wedge \cdots \wedge v_k) := gv_1 \wedge gv_2 \wedge \cdots \wedge gv_k, \quad \forall g \in \mathrm{SL}(n,\mathbb{R}).$$

Proposition 1.6. *Suppose that $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ is a continuous curve definable in an o-minimal structure. Then φ is non-contracting if and only if for all $1 \le k \le n-1$ and linearly independent vectors $v_1, ..., v_k \in \mathbb{R}^n$, it holds that*

$$\lim_{t\to\infty} \varphi(t) \cdot (v_1 \wedge \cdots \wedge v_k) \neq 0. \tag{1.4}$$

1.2. **The main equidistribution result.** To state our main result, we need to introduce an important subgroup associated with our curve. Let $\varphi : [0,\infty) \to G$ be a non-contracting, continuous curve definable in an o-minimal structure. In Theorem 2.5, we will show that there exists a smallest closed connected subgroup $H_\varphi \subseteq G$ *generated by unipotent one-parameter subgroups* such that the image of $\varphi([0,\infty))$ in G/H_φ is bounded. The subgroup H_φ will be called the *hull* of φ. We also establish the existence of a bounded

definable curve $\beta : [0, \infty) \to G$, which will be called a *correcting curve*, such that $\beta(t)\varphi(t) \in H_\varphi$ for all t. Since β is bounded and definable, $\lim_{t\to\infty} \beta(t) = \beta_\infty \in G$ by the monotonicity theorem, see Theorem B.5.

Now suppose that φ takes its values in a connected unimodular Lie subgroup $\mathcal{G} \subseteq G$ and the o-minimal structure is polynomially bounded. Then $H_\varphi \subseteq \mathcal{G}$, see Proposition 5.13, and hence β takes its values in $\mathcal{G}$. Suppose that Γ is a lattice in $\mathcal{G}$. Then, by Ratner's theorem [Rat91a], since H_φ is generated by one-parameter unipotent subgroups, for any $x_0 \in \mathcal{G}/\Gamma$ there exists a closed connected subgroup $L \subseteq \mathcal{G}$ such that Lx_0 is a periodic orbit, and $\overline{H_\varphi x_0} = Lx_0$.

Theorem 1.7. *Let $\mathcal{G} \subseteq G$ be a connected Lie subgroup and let $\Gamma \subseteq \mathcal{G}$ be a lattice. Let $\varphi : [0, \infty) \to G$ be a non-contracting, continuous curve definable in a polynomially bounded o-minimal structure such that $\varphi([0, \infty)) \subseteq \mathcal{G}$. Let $x_0 \in \mathcal{G}/\Gamma$. Then,*

$$\lim_{T\to\infty} \mu_{T,\varphi,x_0} = \beta_\infty^{-1}\mu_{Lx_0},$$

in the weak-$*$ topology, where $Lx_0 = \overline{H_\varphi x_0}$ is periodic, and μ_{Lx_0} is the L-invariant probability measure on Lx_0.

As mentioned above, Theorem 1.7 generalizes Shah's result on polynomial curves [Sha94], and generalizes the result of Peterzil and Starchenko on dense curves in nilmanifolds [PS21].

Remark 1.8. The non-contraction assumption in Theorem 1.7 is essential. Indeed, if φ is a contracting curve in $\mathrm{SL}(n, \mathbb{R})$, then, there exists $x_0 \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ such that $\lim_{t\to\infty} \varphi(t)x_0 = \infty$. To see this, we note that if φ is contracting, then by Proposition 1.6, there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^n$ such that $\varphi(t) \cdot (v_1 \wedge \cdots \wedge v_k) \to 0$ as $t \to \infty$. Let $g_0 \in \mathrm{SL}(n, \mathbb{R})$ satisfy $v_1 \wedge \cdots \wedge v_k = g_0 \cdot (e_1 \wedge \cdots \wedge e_k)$. Since the orbit $\mathrm{SL}(n, \mathbb{Z}) \cdot (e_1 \wedge \cdots \wedge e_k)$ is discrete and does not contain 0, given a compact set $C \subseteq \mathrm{SL}(n, \mathbb{R})$, it holds that $\varphi(t)g_0 \notin C\mathrm{SL}(n, \mathbb{Z})$ for all large t.

In the next result, proved in Section 7, we give an example of a definable curve in $\mathrm{SL}(n + 1, \mathbb{R})$ whose hull is $\mathrm{SL}(n + 1, \mathbb{R})$. As a consequence, every trajectory of such a curve in the space of unimodular lattice in $\mathbb{Z}^{n+1}$ gets equidistributed (Corollary 1.10).

Proposition 1.9. *Let $f_0, ..., f_n$, and $h_1, \dots, h_n$ be continuous real valued functions on $[0, \infty)$ definable in a polynomially bounded o-minimal structure. For $t \geq 0$, define*

$$\varphi(t) := \begin{bmatrix} f_0(t) & f_1(t) & \cdots & f_n(t) \\ & h_1(t)^{-1} & & \\ & & \ddots & \\ & & & h_n(t)^{-1} \end{bmatrix} \tag{1.5}$$

Suppose that the functions h_i's and f_i's defining φ satisfy the following conditions:

(1) $f_0 = h_1 \cdots h_n$, $\deg f_0 = n$, *and* $\deg h_1 \geq \cdots \geq \deg h_n > 0$.
(2) $\deg(f_0 + g) \geq n$ *for any* g *in the linear span of* $f_1, \ldots, f_n$.
(3) *For any* $i \in \{1, \ldots, n\}$, $\deg(f_i + g) > n - i$ *for any* g *in the linear span of* $f_{i+1}, \ldots, f_n$.

Then the hull of φ *is* $H_\varphi = \mathrm{SL}(n+1, \mathbb{R})$.

For example, let $f_0(t) = t^n$. Suppose $f_1, \ldots, f_n$ are rational functions such that $\deg f_1 > \cdots > \deg f_n$, and $\deg f_i \neq n$ and $\deg f_i > n - i$ for all $1 \leq i \leq n$. Let $h_i(t) = t^{r_i}$ for some $r_1 \geq \cdots \geq r_n > 0$ such that $\sum_{i=1}^n r_i = n$. Then, these functions satisfy the conditions of Proposition 1.9.

Using Theorem 1.7, we obtain the following:

Corollary 1.10. *Let* $\mathcal{G}$ *be a Lie group and* Γ *be a lattice in* $\mathcal{G}$. *Suppose that* $\rho : \mathrm{SL}(n+1, \mathbb{R}) \to \mathcal{G}$ *is a continuous homomorphism. Let* $\varphi : [0, \infty) \to \mathrm{SL}(n+1, \mathbb{R})$ *be a curve satisfying the conditions of Proposition 1.9. Then, for any* $x \in \mathcal{G}/\Gamma$, *and any bounded continuous function* f *on* $\mathcal{G}/\Gamma$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi(t) x_0) dt = \mu_X(f),$$

where μ_X *is the homogeneous probability measure on the homogeneous space* $\overline{\rho(\mathrm{SL}(n+1, \mathbb{R}))x}$.

1.3. **Growth of o-minimal functions.** We now describe the (C, α)-good property, introduced by Kleinbock and Margulis [KM98], concerning the growth of certain families of functions definable in polynomially bounded o-minimal structures. This property is well known for polynomials [DM93, 4.1 Lemma], but is new in the o-minimal setting. It is crucial for the techniques used in this paper.

Definition 1.11. Let C and α be positive constants. We say that a function $f : [0, \infty) \to \mathbb{R}$ is (C, α)-*good* on $[T_0, \infty) \subseteq [0, \infty)$, if for every $\epsilon > 0$, and for every bounded interval $I \subseteq [T_0, \infty)$, it holds that:

$$|\{t \in I : |f(t)| \leq \epsilon\}| \leq C \left(\frac{\epsilon}{\|f\|_I} \right)^\alpha \cdot |I|. \tag{1.6}$$

For the following statement, we say that a subset $\mathscr{F}$ of a finite-dimensional vector space $\mathcal{V}$ over $\mathbb{R}$ is a *definable cone*, if $\mathscr{F}$ is a cone in the vector space, and $\mathscr{F}$ is a definable set when identifying $\mathcal{V} \cong \mathbb{R}^n$ by choosing a basis.

Theorem 1.12. ((C, α)-good property). *Let* $\mathcal{V}$ *be a finite-dimensional real vector space spanned by functions* $f : [0, \infty) \to \mathbb{R}$, *each definable in the same polynomially bounded o-minimal structure. Suppose that* $\mathscr{F} \subseteq \mathcal{V}$ *is a closed definable cone such that for all* $f \in \mathscr{F} \setminus \{0\}$ *it holds that*

$$\lim_{t \to \infty} f(t) \neq 0.$$

Then, there exist $C > 0$, $\alpha > 0$ *and* $T_0 \geq 1$ *such that for any* $f \in \mathscr{F} \setminus \{0\}$, f is (C, α)-good on $[T_0, \infty)$.

We obtain the following statement as a direct corollary of Theorem 1.12.

Proposition 1.13. *Let $\varphi : [0, \infty) \to G$ be a non-contracting curve definable in a polynomially bounded o-minimal structure. Suppose that V is a finite-dimensional rational representation of G. Fix a norm on $\|\cdot\|$ on V. Then, there exist $T_0 > 0$, $C > 0$, and $\alpha > 0$ such that for all $v \in V \setminus \{0\}$ the function $\Theta_v(t) := \|\varphi(t) \cdot v\|$ is (C, α)-good in $[T_0, \infty)$.*

Proposition 1.13 is obtained from Theorem 1.12 as follows. By choosing a basis for the G-representation V, we identify $V \cong \mathbb{R}^m$. Let $\psi : [0, \infty) \to \mathrm{GL}(m, \mathbb{R})$ represent the image of φ in the representation V. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^m$. Then, the vector space of functions

$$\mathcal{V} := \mathrm{Span}_{\mathbb{R}}\{t \mapsto \|\psi(t)\mathbf{v}\|^2,\ t \geq 0 : \mathbf{v} \in \mathbb{R}^m\}$$

is finite dimensional, and $\mathscr{F} := \{t \mapsto \|\psi(t)\mathbf{v}\|^2,\ t \geq 0 : \mathbf{v} \in \mathbb{R}^m\}$ is a closed definable cone.

1.4. **Proof ideas and structure of the paper.** Our main theorem will be proved by the following strategy employed in many works on equidistribution in homogeneous spaces, for example [DM93; MS95; Sha94]:

(1) Proving that there is no escape of mass (Section 4) – that is, any limiting measure μ of the probability measures $\mu_{T,\varphi,x}$ is a probability measure.
(2) Showing that any limiting measure μ of the measures $\mu_{T,\varphi,x}$ is invariant under a group generated by nontrivial unipotent one-parameter subgroups (Section 5).
(3) Studying the ergodic decomposition of a limiting measure μ (Section 6). By Ratner's theorem [Rat91a], every ergodic component appearing in the ergodic decomposition of μ is homogeneous. The theorem is proved once it is verified that μ is equal to exactly one uniquely determined ergodic component. For this step, we will use the linearization technique of [DM93] to prove avoidance of singular sets that contain ergodic components of various types.

To establish unipotent invariance, we will use an idea similar to the one used for polynomial curves [Sha94]. This idea generalizes to the o-minimal setting via the notion of the Peterzil-Steinhorn group (see Section 5), which was also used in [PS21].

The (C, α)-good property will be used to prove the non-escape of mass, which is avoidance of the point at infinity, and to apply the linearization technique.

1.5. **Notation and conventions.** We will denote the Zariski closure of a subset A of $\mathbb{R}^N$ by $\mathrm{zcl}(A)$. For a linear Lie group H, we denote by H_u the subgroup of H generated by all the one-parameter unipotent subgroups contained in H. We note that if $H \subseteq \mathrm{GL}(n, \mathbb{R})$, then H_u equals the connected component of the identity in $\mathrm{zcl}(H_u)$.

Acknowledgement. We thank Chris Miller for helpful discussions. We thank anonymous referees for their careful reading and valuable suggestions which helped us improve the readability of this article.

2. On the non-contracting property and the hull

This section is divided into two main parts. In the first part, we prove Proposition 1.6, which gives a convenient criterion for verifying the non-contraction property. The second part of the section concerns the existence of the hull and the correcting curve, and discusses their behavior under homomorphisms.

2.1. The criterion for the non-contracting property.

Our main goal here is to prove Proposition 1.6, namely to reduce the verification of the non-contracting property for any vector in any representation to decomposable vectors in the exterior of the standard representation of $\mathrm{SL}(n,\mathbb{R})$. Our proof follows from the observations in [SY24, Section 2], which uses Kempf's numerical criteria on geometric invariant theory [Kem78].

We recall the following preliminaries. Let S denote the full diagonal subgroup of $\mathrm{SL}(n,\mathbb{R})$. The group of the unipotent upper triangular matrices in $\mathrm{SL}(n,\mathbb{R})$, denoted by N, is a maximal unipotent subgroup of $\mathrm{SL}(n,\mathbb{R})$ normalized by S. Let $X^*(S)$ denote the group of algebraic homomorphisms from S to $\mathbb{R}^*$ defined over $\mathbb{R}$. Then $X^*(S)$ is a free abelian group with $(n-1)$ generators, and we treat it as an additive group. For each $i \in \{1,\dots,n-1\}$, let $\alpha_i \in X^*(S)$ be defined by $\alpha_i(\mathrm{diag}(t_1,\dots,t_n)) = t_i/t_{i+1}$. Then $\Delta = \{\alpha_i : 1 \le i \le n-1\}$ is the set of simple roots on S corresponding to the choice of the maximal unipotent subgroup N. For each $1 \le i \le n-1$, let $\mu_i \in X^*(S)$ be defined by $\mu_i(\mathrm{diag}(t_1,\dots,t_n)) = t_1 \cdots t_i$. Then $\{\mu_i : 1 \le i \le n-1\}$ is the set of fundamental characters (weights) of S with respect to our choice of simple roots. For each $1 \le i \le n-1$, let W_i denote the i-th exterior of the standard representation of $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$, called a *fundamental representation* of $\mathrm{SL}(n,\mathbb{R})$. So $W_i = \wedge^i\mathbb{R}^n$, and for any $v_1 \wedge \cdots \wedge v_i \in W_i$ and $g \in \mathrm{SL}(n,\mathbb{R})$, $g \cdot (v_1 \wedge \cdots \wedge v_i) = gv_1 \wedge \cdots \wedge gv_i$. Let $w_i = e_1 \wedge \cdots \wedge e_i$. Then w_i is fixed by N, and S acts on the line $\mathbb{R}w_i$ via the fundamental character μ_i.

A *dominant* integral character (weight) is a non-negative integral combination of fundamental characters. Suppose $\chi = m_1\mu_1 + \cdots + m_{n-1}\mu_{n-1}$, where $m_i \in \mathbb{Z}_{\ge 0}$. For each i, let $W_i^{\otimes m_i}$ denote the tensor product of m_i-copies of W_i, and $w_i^{m_i} := w_i \otimes \cdots \otimes w_i \in W_i^{\otimes m_i}$. Consider the tensor product representation of $\mathrm{SL}(n,\mathbb{R})$ on $W_\chi = W_1^{\otimes m_1} \otimes \cdots \otimes W_{n-1}^{\otimes m_{n-1}}$, and let $w_\chi = w_1^{m_1} \otimes \cdots \otimes w_{n-1}^{m_{n-1}}$. Then w_χ is fixed by N, and S acts on the line $\mathbb{R}w_\chi$ via the character χ. We endow W_χ with a $\mathrm{SO}(n,\mathbb{R})$-invariant norm.

Proof of Proposition 1.6. Let V be a finite dimensional representation of $G := \mathrm{SL}(n,\mathbb{R})$ endowed with some norm. Suppose $v \in V$ is a nonzero vector such that $\varphi(t)\cdot v \to 0$ as $t \to \infty$. Then $\overline{G\cdot v} \ni 0$, so v is a *G-unstable*

vector in V. So, by [SY24, Remark 2.3], there exist $g_0 \in G$, a dominant integral character $\chi \in X^*(S)$, and constants $\beta > 0$ and $C > 0$ such that for any $g \in G$,

$$\|gg_0 \cdot w_\chi\| \leq C\|g \cdot v\|^\beta. \tag{2.1}$$

Therefore, $\lim_{t\to\infty} \varphi(t)g_0 \cdot w_\chi = 0$. Let $A = S^0$, the component of the identity in S. Let $K = \mathrm{SO}(n, \mathbb{R})$. Then by Iwasawa decomposition, $G = KAN$. For each t, we express $\varphi(t)g_0 = k_t a_t n_t$, where $k_t \in K$, $a_t \in A$ and $n_t \in N$. Since the norm on W_χ is K-invariant and N fixes w_χ,

$$\|\varphi(t)g_0 \cdot w_\chi\| = \|a_t \cdot w_\chi\| = \chi(a_t)\|w_\chi\|.$$

By (2.1), $\lim_{t\to\infty} \varphi(t)g_0 \cdot w_\chi = 0$. Therefore, $\lim_{t\to\infty} \chi(a_t) = 0$. Now

$$\chi(a_t) = \mu_1(a_t)^{m_1} \cdots \mu_{n-1}(a_t)^{m_{n-1}}.$$

Since $m_i \geq 0$, after passing to a subsequence, we can pick $i \in \{1, \ldots, n-1\}$ and a sequence $t_l \to \infty$ such that $\lim_{l\to\infty} \mu_i(a_{t_l}) = 0$. Therefore, since N fixes w_i, and the norm is K-invariant,

$$\|\varphi(t_l)g_0 \cdot w_i\| = \|a_{t_l} \cdot w_i\| = \mu_i(a_{t_l})\|w_i\| \to 0, \text{ as } l \to \infty.$$

Since the coordinate functions of $t \to \varphi(t)g_0 \cdot w_i$ are definable in an o-minimal structure, we get $\lim_{t\to\infty} \varphi(t)g_0 \cdot w_i = 0$. Let $v_j = g_0 e_j$ for each j. Since $w_i = e_1 \wedge \cdots \wedge e_i$, $\varphi(t) \cdot (g_0 \cdot w_i) = \varphi(t) \cdot (v_1 \wedge \cdots \wedge v_i) \to 0$ as $t \to \infty$. This contradicts the assumption that the curve is non-contracting. □

We record the following statement. It will be used in the proof of Lemma 7.1, which is in turn needed for proving Proposition 1.9.

Lemma 2.1. *Let $\varphi : [0, \infty) \to \mathrm{SL}(n, \mathbb{R})$ is a map such that*

$$\lim_{t\to\infty} \varphi(t) \cdot (v_1 \wedge \cdots \wedge v_k) = \infty, \tag{2.2}$$

for all $1 \leq k \leq n-1$ and linearly independent vectors $v_1, ..., v_k \in \mathbb{R}^n$. Then for any finite-dimensional rational representation V and any $v \in V$, if $\{\varphi(t) \cdot v : t \geq 0\}$ is bounded in V, then $\mathrm{SL}(n, \mathbb{R}) \cdot v$ is Zariski closed.

Proof. Suppose that V is a finite-dimensional rational representation of $G = \mathrm{SL}(n, \mathbb{R})$ and let $v \in V$ such that $G \cdot v$ is not Zariski closed. So there exists a finite-dimensional rational representation W of G and a G-equivariant polynomial map $P : V \to W$ such that $P(v) \neq 0$ and $\overline{G \cdot P(v)} \ni 0$, see Lemma A.1. Therefore, by [SY24, Remark 2.3], there exist a $g_0 \in G$, a nontrivial dominant integral character $\chi \in X^*(S)$, and constants $\beta > 0$ and $C > 0$ such that for any $g \in G$,

$$\|gg_0 \cdot w_\chi\| \leq C\|g \cdot P(v)\|^\beta = C\|P(g \cdot v)\|^\beta,$$

cf. [SY24, Corollary 2.5]. Since P is continuous, given any $R > 0$, there exists $C_R > 0$ such that for any $g \in G$,

$$\text{if } \|g \cdot v\| \leq R, \text{ then } \|gg_0 \cdot w_\chi\| \leq C_R\|g \cdot v\|^\beta. \tag{2.3}$$

Now suppose that $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ is a map satisfying the diverging property (2.2) and assume for contradiction that $\{\varphi(t)\cdot v : t \geq 0\}$ is bounded in V. Then, by (2.3), we get that $\{\|\varphi(t)g_0 \cdot w_\chi\| : t > 0\}$ is bounded. As in the proof of Proposition 1.6, by Iwasawa decomposition, for all $t > 0$, $\varphi(t)g_0 = k_t a_t n_t$, where $k_t \in K$, $a_t \in A$ and $n_t \in N$, and

$$\|\varphi(t)g_0 \cdot w_\chi\| = \|a_t \cdot w_\chi\| = \chi(a_t)\|w_\chi\|.$$

We claim that $\lim_{t\to\infty} \chi(a_t) = \infty$, which is a contradiction. Indeed, we have

$$\chi(a_t) = \mu_1(a_t)^{m_1} \cdots \mu_{n-1}(a_t)^{m_{n-1}},$$

for some $(m_1,\dots,m_{n-1}) \in \mathbb{Z}_{\geq 0}$ and $m_i > 0$ for some i. Now

$$\|\varphi(t)g_0 \cdot w_i\| = \|a_t \cdot w_i\| = \mu_i(a_t)\|w_i\|,$$

where $w_i = e_1 \wedge \cdots \wedge e_i$ and $1 \leq i \leq n-1$. And by (2.2), we have $\lim_{t\to\infty}\|\varphi(t)g_0 \cdot w_i\| = \infty$. So, $\lim_{t\to\infty} \mu_i(a_t) \to \infty$. □

2.2. **On the hull and the correcting curve.** In what follows, $G := \mathbf{G}(\mathbb{R})$ is the real points of an algebraic group $\mathbf{G}$ over $\mathbb{R}$. To define the hull and obtain its useful properties, we will need the following notion.

Definition 2.2. We say that a Zariski closed subgroup H is *observable* in G if there is a finite-dimensional rational representation V over $\mathbb{R}$ and $v \in V$ such that $H = \{g \in G : g \cdot v = v\}$.

The notion of observable groups was introduced [BHM63]. More precisely, [BHM63] defines observable groups as the algebraic subgroups of algebraic groups for which every finite-dimensional rational representation extends to a finite-dimensional rational representation of the ambient group. The above definition turns out to be equivalent to the extension property [BHM63, Theorem 8] (see [TB05, Theorem 9] for this statement for non-algebraically closed fields). The following known result will be useful, see e.g. [Gro97, Corollary 2.8].

Lemma 2.3. *Let H be an algebraic subgroup of an algebraic group G over $\mathbb{R}$. If the radical of H is unipotent, then H is an observable subgroup of G.*

Remark 2.4. Suppose that L is a closed subgroup of $\mathrm{SL}(n,\mathbb{R})$ generated by unipotent one-parameter subgroups. Then $L = \mathrm{zcl}(L)^0$ and the radical of $\mathrm{zcl}(L)$ is unipotent (see [Sha91, Lemma 2.9]). Hence, by Lemma 2.3, $\mathrm{zcl}(L)$ is observable in $\mathrm{SL}(n,\mathbb{R})$.

In what follows, we say that a curve $\varphi([0,\infty)) \subseteq G$ is *bounded* modulo a closed subgroup $H \subseteq G$ if there exists a compact subset $K \subseteq G$ such that $\varphi([0,\infty)) \subseteq KH$.

We fix an o-minimal structure $\mathscr{S}$ for the rest of the following two subsections. We stress that the assumption of polynomial boundedness on $\mathscr{S}$ is unnecessary here.

Here is our main theorem in this section.

Theorem 2.5. *Let $\varphi : [0,\infty) \to G$ be a continuous, non-contracting curve definable in $\mathscr{S}$. Then, there exists a unique closed connected (in the Hausdorff topology) subgroup $H_\varphi \subseteq G$ of the smallest dimension such that H_φ is generated by one-parameter unipotent subgroup and such that φ is bounded modulo H_φ. Moreover, the following properties hold:*

1. *The subgroup H_φ is minimal in the following sense: if H' is an observable group such that φ is bounded modulo H', then $H_\varphi \subseteq H'$. Moreover, if V is a rational representation, and $v \in V$ is such that $\{\varphi(t) \cdot v : t \geq 0\}$ is bounded in V, then φ is bounded modulo the isotropy group of v and $H_\varphi \cdot v = \{v\}$.*
2. *There is a bounded, continuous curve $\beta : [0,\infty) \to G$ definable in $\mathscr{S}$ such that $\beta(t)\varphi(t) \in H_\varphi$ for all $t \geq 0$.*
3. *If V is a rational finite-dimensional representation of G over $\mathbb{R}$, then for any $v \in V$, either $H_\varphi \cdot v = \{v\}$ or $\lim_{t\to\infty} \beta(t)\varphi(t) \cdot v = \infty$.*

In view of the above statement, we can now define the hull.

Definition 2.6. Let $\varphi : [0,\infty) \to G$ be a continuous non-contracting curve definable in $\mathscr{S}$. We define the *hull* H_φ to be the smallest connected closed subgroup generated by unipotent elements such that φ is bounded modulo H_φ, and we call a curve $\beta : [0,\infty) \to G$ a *correcting curve* if $\beta(t)\varphi(t) \in H_\varphi$ for all $t \geq 0$.

Remark 2.7. In view of Remark 2.4, $H_\varphi = \mathrm{zcl}(H_\varphi)^0$ and $\mathrm{zcl}(H_\varphi)$ is the observable subgroup of smallest dimension modulo which φ is bounded.

Remark 2.8. We note that if φ is as in the above theorem and β is a definable correcting curve, then the hull of $\psi(t) := \beta(t)\varphi(t)$ is the same as the hull of φ, and the correcting curve of ψ can be chosen to be the constant identity matrix. This follows from the uniqueness of the hull.

Remark 2.9. Suppose that $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ is a definable curve whose entry functions $\varphi_{i,j}$ generate an algebra which consists of functions that are either constant or diverging to ∞, e.g., when $t \mapsto \varphi_{i,j}(t)$ are polynomials for all i, j. Then, by Theorem 2.5, $\varphi(0)^{-1}\varphi([0,\infty)) \subseteq H_\varphi$. A correcting curve is given by the constant curve $\beta(t) = \varphi(0)^{-1}$ for all $t \geq 0$.

Before proving Theorem 2.5, we establish that the hull is intrinsically defined.

Lemma 2.10. *Let G_1 and G_2 be the real points of algebraic groups over $\mathbb{R}$, and let $f : G_1 \to G_2$ be a rational homomorphism. Suppose that $\varphi : [0,\infty) \to G_1$ is a non-contracting curve definable in $\mathscr{S}$. Then $f(H_\varphi) = H_{f\circ\varphi}$.*

Proof. We start by showing that $H_{f\circ\varphi} \subseteq f(H_\varphi)$. Let $\beta : [0,\infty) \to G_1$ be a correcting curve of φ. Since H_φ is generated by unipotents, we get that $f(H_\varphi)$ is generated by unipotents. Moreover, the curve $f \circ \varphi$ is bounded modulo $f(H_\varphi)$. In fact, $t \mapsto f(\beta(t))$ is bounded, and for all $t \geq 0$:

$$f(\beta(t)\varphi(t)) = f(\beta(t))f(\varphi(t)) \in f(H_\varphi).$$

Thus, by minimality of the hull, we conclude that $H_{f\circ\varphi} \subseteq f(H_\varphi)$.

We proceed to show the other inclusion. Since $\mathrm{zcl}(H_{f\circ\varphi})$ is observable, we choose a rational representation $\rho : G_2 \to \mathrm{GL}(V)$ and a $v \in V$ such that

$$\mathrm{zcl}(H_{f\circ\varphi}) = \{g \in G_2 : \rho(g)v = v\}.$$

Since $f \circ \varphi$ is bounded modulo $H_{f\circ\varphi}$, the trajectory,

$$\rho((f \circ \varphi)(t))v = (\rho \circ f)(\varphi(t))v,\ t \geq 0,$$

is bounded in V. Now, $\eta := \rho \circ f : G_1 \to \mathrm{GL}(V)$ is a representation of G_1 such that $t \mapsto \eta(\varphi(t))v,\ t \geq 0$ is bounded. By Theorem 2.5(1), we obtain that $\rho(f(H_\varphi))v = \eta(H_\varphi)v = \{v\}$. That is, $f(H_\varphi)$ is contained in the isotropy group of v which is equal to $\mathrm{zcl}(H_{f\circ\varphi})$. Since $f(H_\varphi)$ is generated by unipotent one-parameter subgroups,

$$f(H_\varphi) \subseteq \mathrm{zcl}(H_{f\circ\varphi})^0 = H_{f\circ\varphi}.$$

□

2.3. **Proving Theorem 2.5.** We begin with a "curve correcting" lemma.

Lemma 2.11. *Consider an algebraic action of G on the $\mathbb{R}$ points of an affine or projective variety $\mathcal{Z}$ over $\mathbb{R}$. Let $\varphi : [0,\infty) \to G$ be a continuous curve definable in $\mathscr{S}$. Suppose that $x \in \mathcal{Z}$ and $g_0 \in G$ are such that*

$$\lim_{t\to\infty} \varphi(t) \cdot x = g_0 \cdot x.$$

Then, there exists a continuous $\mathscr{S}$-definable curve $\delta : [0,\infty) \to G$ such that

$$\lim_{t\to\infty} \delta(t) = e \text{ and } \delta(t)\varphi(t) \cdot x = g_0 \cdot x,\ \forall t \geq 0.$$

Proof. Identify G as a closed subgroup of $\mathrm{SL}(n,\mathbb{R})$. Consider the following definable set:

$$\mathcal{D} := \{(t,g) : t > 0,\ g \in G, \|g - I_n\| < 1 \text{ and } g_0^{-1}\varphi(t) \cdot x = g \cdot x\}.$$

Here $\|\cdot\|$ is a definable norm (e.g., the sum of squares norm). Since the orbit map is an open map (maps open sets in G to open sets in $G \cdot x$ with respect to the induced topology from $\mathcal{Z}$, see [PRR23, Corollary 3.9]), and since

$$\lim_{t\to\infty} g_0^{-1}\varphi(t) \cdot x = x,$$

we get that for all t large enough, say $t \geq T_0$, there exists $g \in G$ such that $(t,g) \in \mathcal{D}$. Using the choice function theorem, see Theorem B.7, there is a definable curve $\tilde{\delta}(t) : [T_0,\infty) \to G$ such that $(t,\tilde{\delta}(t)) \in \mathcal{D}$ for all $t \geq T_0$. In particular,

$$\tilde{\delta}(t) \cdot x = g_0^{-1}\varphi(t) \cdot x,\ \forall t \geq T_0,$$

and $\tilde{\delta}$ is bounded. Since the curve $\tilde{\delta}(t)$ is bounded, by the monotonicity theorem, see Theorem B.5, we have that $\lim_{t\to\infty} \tilde{\delta}(t) = \delta_0$. Note that $\delta_0 \cdot x = x$. Let $\delta(t) := g_0(\tilde{\delta}(t)\delta_0^{-1})^{-1}g_0^{-1}$ for all $t \geq T_0$. Then,

$$\delta(t)\varphi(t) \cdot x = g_0 \cdot x, \forall t \geq T_0.$$

By the monotonicity theorem, $\delta(t)$ is continuous in $[T_0', \infty)$, for some $T_0' \geq T_0$. We now extend δ to the interval $[0, T_0']$ by putting

$$\delta(t) := \delta(T_0')\varphi(T_0')\varphi(t)^{-1}, \forall t \leq T_0'.$$

Then $\delta(t)\varphi(t) \cdot x = g_0 \cdot x, \forall t \geq 0$. Since φ is continuous and definable, $\delta : [0, \infty) \to G$ is continuous, definable and bounded. □

Lemma 2.12. *Suppose that $\varphi : [0, \infty) \to G$ is a non-contracting, continuous curve, definable in $\mathscr{S}$. Suppose that V is a finite-dimensional rational representation of G and $v \in V \setminus \{0\}$ is such that $\varphi(t) \cdot v$ is bounded. Then, there exists $g_0 \in G$ such that*

$$\lim_{t\to\infty} \varphi(t) \cdot v = g_0 \cdot v.$$

In particular, there exists a bounded, continuous curve $\beta : [0, \infty) \to G$ definable in $\mathscr{S}$, such that $\beta(t)\varphi(t) \cdot v = v$ for all $t \geq 0$.

Proof. Since $\varphi : [0, \infty) \to G$ is definable, and since V is a rational representation, $\varphi(t) \cdot v$ is a bounded definable curve. By the monotonicity theorem (see Theorem B.5), $\lim_{t\to\infty} \varphi(t) \cdot v = v' \in V$. We claim that $v' \in G \cdot v$. Denote $S := \mathrm{zcl}(G \cdot v) \setminus G \cdot v$, and suppose for contradiction that $v' \in S$. Then, by Lemma A.1, which is a slight modification of [Kem78, Lemma 1.1(b)], there exists a rational representation W of G and G-equivariant polynomial map $P : V \to W$ such that $P(v) \neq 0$ and $P(S) = 0$. In particular, $P(v') = 0$. Then

$$\lim_{t\to\infty} \varphi(t) \cdot P(v) = P(\lim_{t\to\infty} \varphi(t) \cdot v) = P(v') = 0,$$

which is a contradiction, as φ is non-contracting. Thus, we conclude that there exists $g_0 \in G$ such that

$$\lim_{t\to\infty} \varphi(t) \cdot v = g_0 \cdot v.$$

The existence of the bounded curve β is obtained by Lemma 2.11. □

Lemma 2.13. *Suppose that $\varphi : [0, \infty) \to G$ is a non-contracting, continuous curve, definable in $\mathscr{S}$. Let H_1 and H_2 be observable groups of G such that the image of φ is bounded in G/H_i for $i \in \{1, 2\}$. Then, the image of φ is bounded in $G/(H_1 \cap H_2)$.*

Proof. For $i \in \{1, 2\}$, let V_i be a finite-dimensional rational representation such that $H_i = \{g \in G : g \cdot v_i = v_i\}$ where $v_i \in V_i$. Consider the direct-sum representation $V_1 \oplus V_2$, and notice that

$$H_1 \cap H_2 = \{g \in G : g \cdot (v_1, v_2) = (v_1, v_2)\}.$$

Since the image of φ is bounded in G/H_i, $\varphi(t) \cdot v_i$ is bounded for all $i \in \{1, 2\}$. In particular, $\varphi(t) \cdot (v_1, v_2)$ is bounded in $V_1 \oplus V_2$. Then, by Lemma 2.12, there is a bounded continuous curve $\delta : [0, \infty) \to G$ such that $\delta(t)\varphi(t) \cdot (v_1, v_2) = (v_1, v_2)$ for all $t \geq 0$. Namely, $\delta(t)\varphi(t) \in H_1 \cap H_2$ for all $t \geq 0$, and therefore the image of φ is bounded in $G/(H_1 \cap H_2)$. □

Corollary 2.14. *Suppose that $\varphi : [0,\infty) \to G$ is a non-contracting, continuous curve definable in $\mathscr{S}$. Then there exists a unique observable subgroup $\tilde{H}_\varphi \subseteq G$ of minimal dimension such that φ is bounded in $G/\tilde{H}_\varphi$. Moreover, if H is an observable subgroup such that φ is bounded in G/H, then $\tilde{H}_\varphi \subseteq H$.*

Proof. Let H_0 be a Zariski connected observable group of the smallest dimension such that φ is bounded in G/H_0. If H is any connected observable group such that φ is bounded in G/H, then by Lemma 2.13, we get that φ is bounded in $G/(H_0 \cap H)$. If $H_0 \not\subseteq H$, then $H \cap H_0$ is a proper subgroup of dimension strictly smaller than $\dim H_0$, which leads to a contradiction. □

Corollary 2.15. *Suppose that $\varphi : [0,\infty) \to G$ is a non-contracting, continuous curve, definable in $\mathscr{S}$. Let $\tilde{H}_\varphi \subseteq G$ be the minimal observable group such that φ is bounded modulo $\tilde{H}_\varphi$. Denote $H_\varphi := \tilde{H}_\varphi^0$, the identity component of $\tilde{H}_\varphi$.*

Then, there exists a bounded, continuous definable curve $\beta : [T_0,\infty) \to G$ such that $\beta(t)\varphi(t) \in H_\varphi$ for all $t \geq 0$. Moreover, if V is a finite-dimensional rational representation of G and $v \in V$ is such that $\varphi(t) \cdot v$ is bounded, then $\beta(t)\varphi(t) \cdot v = v$ for all $t \geq T_0$.

Proof. By Lemma 2.12, we obtain a continuous definable curve $\tilde{\beta} : [0,\infty) \to G$ such that $\tilde{\beta}(t)\varphi(t) \in \tilde{H}_\varphi$ for all $t \geq 0$. Since φ is continuous, it is contained in a connected component of $\tilde{H}_\varphi$. By choosing a suitable $h_0 \in \tilde{H}_\varphi$ we get that for $\beta(t) := h_0\tilde{\beta}(t)$ it holds $\beta(t)\varphi(t) \in \tilde{H}_\varphi^0 = H_\varphi$, for all $t \geq 0$. Now, if V is a finite-dimensional rational representation such that $\varphi(t) \cdot v$ is bounded, then by Lemma 2.12, we conclude that φ is bounded modulo the observable group $H := \{g \in G : g \cdot v = v\}$. Hence, by minimality of $\tilde{H}_\varphi$, we get that $\beta(t)\varphi(t) \in H_\varphi \subseteq H$. □

The last property in Theorem 2.5 remaining to prove is that the identity component of the observable group of smallest dimension is generated by one-parameter unipotent subgroups. This is given by the following.

Lemma 2.16. *Let $\varphi : [0,\infty) \to G$ be a $\mathscr{S}$-definable, non-contracting, continuous curve, and let $\tilde{H}_\varphi$ be the smallest observable subgroup such that φ is bounded modulo $\tilde{H}_\varphi$. Then $H_\varphi = \tilde{H}_\varphi^0$ is generated by one-parameter unipotent subgroups.*

Proof. Let $\psi(t) := \beta(t)\varphi(t)$ for all $t \geq 0$, where β is a correcting curve of φ. Let $L \subseteq \tilde{H}_\varphi$ be the Zariski closure of the subgroup generated by all one-parameter unipotent subgroups of $\tilde{H}_\varphi$. Then, L is observable in G, see Remark 2.4. We will now show that the image of φ in $\tilde{H}_\varphi/L$ is bounded, which implies that $\tilde{H}_\varphi = L$, and proves the claim, see Remark 2.7.

Since L is a normal subgroup of $\tilde{H}_\varphi$, we get that $\tilde{H}_\varphi/L$ is an affine algebraic group. Moreover, since L contains the unipotent radical of $\tilde{H}_\varphi$, we get that $\tilde{H}_\varphi/L$ is reductive. Then, there is an almost direct product decomposition

$\tilde{H}_\varphi/L = ZD$, where Z is the center, which is a torus, and D is the derived group, which is a normal semi-simple subgroup. Notice that each simple factor of D must be compact since a non-compact simple group is generated by unipotent one-parameter subgroups. Namely, D is compact. Let $q : \tilde{H}_\varphi \to \tilde{H}_\varphi/L$ be the natural quotient map. To finish the proof, we show below that for every character $\chi : \tilde{H}_\varphi/L \to \mathbb{R}^\times$ it holds that $\chi \circ q(\psi(t)) = 1$ for all $t \geq 0$. This will complete the proof since then $q \circ \psi([0,\infty)) \subseteq Z_a D$, where $Z_a \subseteq Z$ is the maximal anisotropic compact torus subgroup of Z.

Let $\chi : \tilde{H}_\varphi/L \to \mathbb{R}^\times$ be a character, and consider the representation of $\tilde{H}_\varphi$ on $\mathbb{R}^2$ defined by

$$h \cdot (x,y) := \left(\chi(q(h))x, \chi(q(h))^{-1}y\right), \forall (x,y) \in \mathbb{R}^2, \forall h \in \tilde{H}_\varphi. \tag{2.4}$$

Since $\tilde{H}_\varphi \subseteq G$ is observable, the above representation extends to a representation of G, see [TB05, Theorem 9]. Now, pick $x_0, y_0 \in \mathbb{R} \setminus \{0\}$. Because ψ is non-contracting, we get that $\psi(t) \cdot (x_0, 0)$ and $\psi(t) \cdot (0, y_0)$ are bounded away from 0. In view of (2.4), we deduce that $\psi(t) \cdot (x_0, y_0)$ is bounded. So, by Corollary 2.15, $\psi(t) \cdot (x_0, y_0) = (x_0, y_0)$. Thus, $\chi \circ q(\psi(t)) = 1$ for all $t \geq 0$. □

2.4. **The non-contracting property under homomorphisms.** In this subsection, we show that the homomorphic image of a definable non-contracting curve is non-contracting, Lemma 2.17, and also we show that if a definable curve in a homomorphic image is non-contracting, then it has a definable non-contracting lift, Lemma 2.21. Finally, we establish that the non-contracting property is intrinsic for curves in groups whose radical is unipotent, Proposition 2.22.

Lemma 2.17. *Let G_1 and G_2 be the real points of affine algebraic groups over $\mathbb{R}$, and let $f : G_1 \to G_2$ be a homomorphism of algebraic groups. Let $\varphi : [0,\infty) \to G_1$ be a continuous, non-contracting curve in G_1 definable in $\mathscr{S}$. Then $f \circ \varphi$ is a continuous, non-contracting curve in G_2 definable in $\mathscr{S}$.*

Proof. Let $\rho : G_2 \to \mathrm{GL}(V)$ be a rational representation of G_2. Let $v \in V$. Then $\rho \circ f : G_1 \to \mathrm{GL}(V)$ is a rational representation of G_1. Let $v \in V$. If $\rho(f \circ \varphi(t))v \to 0$ as $t \to \infty$, then $(\rho \circ f)(\varphi(t))v \to 0$ as $t \to \infty$. Since φ is non-contracting, we get $v = 0$. □

Lemma 2.18. *Suppose that $U \subseteq G$ is a normal, unipotent algebraic subgroup. Let $q : G \to G/U$ be the natural map. Then, a $\mathscr{S}$-definable curve $\varphi : [0,\infty) \to G$ is non-contracting in G if and only if $q \circ \varphi$ is non-contracting in G/U.*

Proof. By Lemma 2.17, it suffices to prove that if $q \circ \varphi$ is non-contracting, then φ is non-contracting. Suppose for contradiction that $q \circ \varphi$ is non-contracting and φ is contracting. Let V be a rational representation of G of the smallest dimension such that there exists a $v \in V \setminus \{0\}$ with

$\lim_{t\to\infty} \varphi(t)\cdot v = 0$. Since U is a unipotent group, the subspace of U-fixed points $V^U := \{x \in V : u\cdot x = x, \forall u \in U\}$ is of positive dimension. Since U is a normal subgroup, V^U is G-invariant. Then G acts on V/V^U, and the natural quotient map $\pi : V \to V/V^U$ is G-equivariant. Now,

$$\lim_{t\to\infty} \varphi(t)\cdot\pi(v) = \lim_{t\to\infty} \pi(\varphi(t)\cdot v) = 0.$$

Since $\dim(V/V^U) < \dim(V)$, we get that $\pi(v) = 0$. Namely, $v \in V^U$, and again, since V is of smallest dimension with a $\varphi(t)$-contracting nonzero vector, we get $V = V^U$. Since U acts trivially on V, the action of G on V factors through the action of G/U. Therefore, $\varphi(t)\cdot v = (q\circ\varphi(t))\cdot v$. Since $\varphi(t)\cdot v \to 0$ as $t\to\infty$, and $q\circ\varphi$ is non-contracting, we get a contradiction. □

Lemma 2.19. *Let G_1 and G_2 be the real points of algebraic groups over $\mathbb{R}$. Suppose that $f : G_1 \to G_2$ is a homomorphism of algebraic groups, and $\varphi : [0,\infty) \to f(G_1)$ is a curve definable in $\mathscr{S}$. Then there exists a* definable lift $\psi : [0,\infty) \to G_1$ *of φ. Namely, ψ is definable in $\mathscr{S}$, and $f\circ\psi = \varphi$.*

Proof. Consider the definable set

$$\mathcal{D} := \{(t,g) : g \in G_1,\ t \geq 0,\ f(g) = \varphi(t)\}.$$

By the choice function theorem (see Theorem B.7), there exists a definable curve $\psi : [0,\infty) \to G_1$ such that $(t,\psi(t)) \in \mathcal{D}$ for all $t \geq 0$. □

Lemma 2.20. *Let G_1 and G_2 be the $\mathbb{R}$-points of semi-simple connected algebraic groups over $\mathbb{R}$, and let $f : G_1 \to G_2$ be a surjective homomorphism with finite kernel. Let $\psi : [0,\infty) \to G_1$ be a definable curve. Suppose that $f\circ\psi$ is non-contracting for G_2, then ψ is non-contracting for G_1.*

Proof. Let V be a finite-dimensional irreducible rational representation of G_1. By Schur's lemma, the center of G_1 acts via a character on $V\otimes\mathbb{C}$. Then, $\ker f$, which is central in G_1, acts trivially on the n-fold tensor product $W = \otimes^n(V\otimes\mathbb{C})$, where n is the order of $\ker f$. Therefore, the action of G_1 on W factors through an action of $G_1/(\ker f) \cong G_2$ on W. Let $v \in V$. If $\psi(t)\cdot v \to 0$, then $f\circ\psi(t)\cdot(v\otimes\cdots\otimes v) \to 0$ as $t\to\infty$. But, since $f\circ\psi$ is non-contracting, we get that $v = 0$. □

Lemma 2.21. *Let G_1 and G_2 be the real points of affine algebraic groups over $\mathbb{R}$, and let $f : G_1 \to G_2$ be a surjective homomorphism of algebraic groups. Let $\varphi : [0,\infty) \to G_2$ be a non-contracting curve in G_2 definable in $\mathscr{S}$. Then, φ has a $\mathscr{S}$-definable lift $\psi : [0,\infty) \to G_1$ which is non-contracting in G_1.*

Proof. In this proof, all subgroups considered are the open subgroups of $\mathbb{R}$-points of algebraic groups defined over $\mathbb{R}$.

Since the hull is generated by one-parameter unipotent subgroups, by modifying φ with a bounded correcting curve and replacing G_2 by $(G_2)_u$, we may assume without loss of generality that the radical of G_2 is unipotent, and we denote it by U_2. Let $\bar{G}_2 = G_2/U_2$ and $q_2 : G_2 \to \bar{G}_2$ be the quotient

homomorphism. Let U_1 be the unipotent radical of G_1. Then, there exists a reductive subgroup M_1 of G_1 such that $G_1 = M_1U_1$ and $M_1 \cap U_1 = \{e\}$, see [Rag72, Page 11]. Since f is surjective, $f(U_1)$ is a unipotent normal subgroup of G_2, and hence $f(U_1) \subseteq U_2$. In fact, $f(U_1) = U_2$. This can be proved as follows. $f^{-1}(U_2) = (M_1 \cap f^{-1}(U_2))U_1$. Since $f^{-1}(U_2)$ is a normal subgroup of G_1, $M_1 \cap f^{-1}(U_2)$ is reductive. So $f(M_1 \cap f^{-1}(U_2))$ being a reductive subgroup of U_2, it is trivial. Since f is surjective, we get $f(U_1) = U_2$. Therefore, $q_2 \circ f : M_1 \to \bar{G}_2$ is surjective. Since $\bar{G}_2$ is semisimple, $q_2 \circ f([M_1, M_1]) = \bar{G}_2$. Let $L_2 = [M_1, M_1] \cap \ker(q_2 \circ f)$. Since L_2 is a normal subgroup of the semisimple group $[M_1, M_1]$, there exists a semisimple normal subgroup L_1 of $[M_1, M_1]$ such that $[M_1, M_1] = L_1L_2$, where $L_1 \cap L_2$ is a finite central subgroup of L_1. Therefore, $q_2 \circ f : L_1 \to \bar{G}_2$ is a surjective homomorphism whose kernel is $L_1 \cap L_2$. Also, since $\bar{G}_2$ is generated by unipotent one-parameter subgroups, it has no compact simple normal subgroup. Therefore, L_1 also does not have a compact normal subgroup, and L_1 is generated by unipotent one-parameter subgroups.

Let $\tilde{G}_1 = L_1U_1$. Since $q_2 \circ f(L_1) = G_2/U_2$ and $f(U_1) = U_2$, we have $f : \tilde{G}_1 \to G_2$ is a surjective map. Therefore, by Lemma 2.19, there exists a definable $\psi : [0, \infty) \to \tilde{G}_1$ such that $f \circ \psi = \varphi$. Let $q_1 : \tilde{G}_1 \to \tilde{G}_1/U_1$ be the natural quotient map. Since $q_1 : L_1 \to \tilde{G}_1/U_1$ is surjective (it is an isomorphism), let $\tilde{\psi} : [0, \infty) \to L_1$ be a definable curve such that $q_1 \circ \tilde{\psi} = q_1 \circ \psi$. Since $U_1 \subseteq \ker(q_2 \circ f)$, $q_2 \circ f$ factors through q_1. Therefore, we have $(q_2 \circ f) \circ \tilde{\psi} = (q_2 \circ f) \circ \psi = q_2 \circ \varphi$. Now $q_2 \circ f : L_1 \to \bar{G}_2$ is surjective with finite central kernel, and $q_2 \circ \varphi$ is definable and non-contracting in $\bar{G}_2$. Therefore, by Lemma 2.20, $\tilde{\psi}$ is non-contracting in L_1. Therefore, $q_1 \circ \psi = q_1 \circ \tilde{\psi}$ is non-contracting in $\tilde{G}_1/U_1$. Therefore, by Lemma 2.18, ψ is non-contracting in $\tilde{G}_1$. Finally, by Lemma 2.17 we get that ψ is non-contracting in G_1. □

The following observation extends Lemma 2.20.

Proposition 2.22. *Suppose that G_1 is a real algebraic group generated by unipotent one-parameter subgroups. Let $f : G_1 \to G_2$ be a homomorphism of real algebraic groups with finite kernel. Then a definable curve $\varphi : [0, \infty) \to G_1$ is non-contracting in G_1 if and only if $f \circ \varphi$ is non-contracting in G_2.*

Proof. If φ is non-contracting in G_1, then by Lemma 2.17, it follows that $f \circ \varphi$ is non-contracting in G_2.

Now suppose that $f \circ \varphi$ is non-contracting in G_2. Since G_1 is generated by unipotent one-parameter subgroups, $f(G_1)$ is a subgroup of G_2 generated by unipotent one-parameter subgroups. Therefore, the solvable radical of $f(G_1)$ is unipotent. Hence, $f(G_1)$ is an observable subgroup of G_2, see [Gro97, Corollary 2.8]. Therefore, given any finite-dimensional rational representation of V of $f(G_1)$, V is a subrepresentation of a finite-dimensional representation W of G_2 restricted to $f(G_1)$. Let $v \in V$. Suppose that $f \circ \varphi(t) \cdot v \to 0$. Since $v \in V \subseteq W$ and $f \circ \varphi$ is non-contracting in G_2, we conclude that $v = 0$. This proves that $f \circ \varphi$ is non-contracting for

$f(G_1)$. Therefore, replacing G_2 by $f(G_1)$, we may assume that f is surjective. Therefore, by Lemma 2.21 there exists a lift $\psi : [0,\infty) \to G_1$ of $f \circ \varphi$ which is definable and non-contracting in G_1. By the monotonicty theorem (see Theorem B.5), there is a $T_0 \geq 0$ such that φ and ψ are continuous on $[T_0,\infty)$. Let $z = \varphi(T_0)\psi(T_0)^{-1} \in \ker f$. So, $t \mapsto z\psi(t)$ and $t \mapsto \varphi(t)$ are two lifts of $f \circ \varphi$ from $[T_0,\infty)$ to G_1 from the point $\varphi(T_0) \in G_1$. Since $\ker f$ is finite, f is a covering map, we get $\varphi(t) = z\psi(t)$ for all $t \in [0,\infty)$. Since ψ is non-contracting, so is φ. □

3. (C,α)-GOOD PROPERTY

In our setting, a property more fundamental than the (C,α)-good property (Theorem 1.12) is the following.

Definition 3.1. Let $\delta \in (0,1]$. A family $\tilde{\mathscr{F}}$ of real functions defined on $[T_0,\infty)$ is called *δ-good* if there exists a constant $M(\delta)$ depending only on δ such that

$$\frac{\|f\|_I}{\|f\|_{I_\delta}} \leq M(\delta), \tag{3.1}$$

for all $f \in \tilde{\mathscr{F}}$ and all bounded sub-intervals $I_\delta \subseteq I \subseteq [T_0,\infty]$ satisfying $|I_\delta| = \delta|I|$.

Such inequalities are well-known in the literature as Remez-type inequalities; see [Rem36].

We will prove the following result, from which Theorem 1.12 will be deduced. For the rest of this section, we fix a *polynomially bounded* o-minimal structure $\mathscr{S}_{\mathrm{pbd}}$.

Theorem 3.2. *Let $\mathcal{V}$ be a finite-dimensional vector space of real functions defined on $[0,\infty)$ definable in $\mathscr{S}_{\mathrm{pbd}}$. Suppose that $\mathscr{F} \subseteq \mathcal{V}$ is a closed definable cone such that*

$$\lim_{t\to\infty} f(t) \neq 0, \ \forall f \in \mathscr{F} \setminus \{0\}.$$

Then, there exists $T_0 > 0$ such that the family of functions $\tilde{\mathscr{F}}$ of the nonzero functions in $\mathscr{F}$ restricted to $[T_0,\infty)$ is δ-good, for all $\delta \in (0,1]$.

Remark 3.3. The T_0 in Theorem 3.2 can be chosen as a T_0, which is obtained by the outcome of Lemma 3.8.

We now proceed to prove Theorem 1.12 by assuming Theorem 3.2, and the rest of the section will be dedicated to proving Theorem 3.2.

3.1. Proving Theorem 1.12 via δ-goodness.

We first note the following corollary of Theorem 3.2.

Corollary 3.4. *Let $\mathcal{V}$ be a finite-dimensional vector space of real functions on $[0,\infty)$ which are definable in $\mathscr{S}_{\mathrm{pbd}}$. Suppose that $\mathscr{F} \subseteq \mathcal{V}$ is a closed definable cone such that for all $f \in \mathscr{F} \setminus \{0\}$,*

$$\lim_{t\to\infty} f(t) \neq 0.$$

Let $T_0 \geq 1$ such that the outcome of Theorem 3.2 holds. Then, there exist $\mathrm{m}, r > 0$ *such that*:

$$\|f\|_I \leq \mathrm{m} x^r \|f\|_{I'}, \tag{3.2}$$

for all $x \geq 1$, $f \in \mathscr{F} \setminus \{0\}$, and bounded intervals $I' \subseteq I \subseteq [T_0, \infty)$ such that $\frac{|I|}{|I'|} \leq x$.

Proof. Consider the definable set (see Lemma B.3):

$$S := \left\{ (x, y) : \forall I' \subseteq I \subseteq [T_0, \infty),\ \frac{|I|}{|I'|} \leq x,\ \frac{\|f\|_I}{\|f\|_{I'}} \leq y,\ \forall 0 \neq f \in \mathscr{F} \right\}.$$

By Theorem 3.2, the projection of S to the first coordinate includes $[1, \infty)$. Then, by the definable choice function theorem (see Theorem B.7), there exists a function $\phi : [1, \infty) \to \mathbb{R}^2$ definable in the same polynomially bounded o-minimal structure, such that

$$(x, \phi(x)) \in S,\ \forall x \geq 1.$$

Thus, for all $x > 1$, $f \in \mathscr{F} \setminus \{0\}$, and intervals $I' \subseteq I \subseteq [T_0, \infty)$ such that $\frac{|I|}{|I'|} \leq x$, it holds that

$$\|f\|_I \leq \phi(x) \|f\|_{I'}.$$

Since ϕ is polynomially bounded, there exist $r > 0$ and $T_1 \geq 1$ such that $\frac{\phi(x)}{x^r} \leq c_0$ for all $x \geq T_1$, for some $c_0 > 0$. Finally, since $\frac{\phi(x)}{x^r}$ is bounded in $[1, T_1]$, the result follows. □

Our proof below is inspired by [KM98, proof of Proposition 3.2].

Proof of Theorem 1.12 assuming Theorem 3.2. We recall that the number of connected components of the family of definable sub-level sets

$$\{t \in I : |f(t)| \leq \epsilon\},\ \epsilon > 0,\ I \subseteq [T_0, \infty),\ f \in \mathscr{F} \setminus \{0\},$$

is bounded uniformly, say by K, see Theorem B.6. Now fix $I \subseteq [T_0, \infty)$, $\epsilon > 0$ and an $f \in \mathscr{F} \setminus \{0\}$. Let

$$I' \subseteq \{t \in I : |f(t)| \leq \epsilon\}$$

be an interval of maximum length. Then

$$L \leq K|I'|, \text{ where } L := |\{t \in I : |f(t)| \leq \epsilon\}|.$$

Therefore, $|I|/|I'| \leq |I|/(L/K)$. By Corollary 3.4, we get

$$\|f\|_I \leq \mathrm{m} \left(\frac{|I|}{L/K} \right)^r \|f\|_{I'} \leq \mathrm{m} \left(\frac{|I|}{L/K} \right)^r \epsilon.$$

Therefore,

$$|\{t \in I : |f(t)| \leq \epsilon\}| = L \leq \mathrm{m}^{\frac{1}{r}} K \left(\frac{\epsilon}{\|f\|_I} \right)^{\frac{1}{r}} |I|.$$

Thus, f is $(\mathrm{m}^{\frac{1}{r}} K, 1/r)$-good on $[T_0, \infty)$, see (1.6). □

3.2. **Proving Theorem 3.2.** The space of polynomials of bounded degrees is the prototypical example of δ-good functions. We record the following well-known result, as it will play a role in the proof of Theorem 3.2.

Proposition 3.5 ([DM93, Lemma 4.1])**.** *Fix $n \in \mathbb{N}$. Then, for every $\delta \in (0,1]$, nested intervals $I_\delta \subseteq I$ with $|I_\delta| = \delta|I| > 0$, and every nonzero polynomial $c_0 + c_1 x + \cdots + c_n x^n$, it holds that*:

$$\frac{\|c_0 + c_1 x + \cdots + c_n x^n\|_I}{\|c_0 + c_1 x + \cdots + c_n x^n\|_{I_\delta}} \leq (n+1)\frac{n^n}{\delta^n}. \tag{3.3}$$

We will be using the following convenient fact. Recall by the monotonicity theorem (Theorem B.5) that the derivative of a definable function $f(t)$ on an unbounded interval exists for all large t.

Lemma 3.6. [Mil94c, Proposition 3.1] *Let $f : [0,\infty) \to \mathbb{R}$ be a function definable in $\mathscr{S}_{\text{pbd}}$ such that $f(t) \sim t^r$ as $t \to \infty$, where $r \neq 0$. Then, $f'(t) \sim rt^{r-1}$, as $t \to \infty$.*

Lemma 3.7. *Let $F_i : [0,\infty) \to \mathbb{R}$ be functions definable in $\mathscr{S}_{\text{pbd}}$, for $i = 0, 1, \ldots, N$, and assume that none of the F_i are eventually zero. Suppose that the degrees $\deg(F_0), \ldots, \deg(F_N)$ are all distinct. Then, there exists a $T_0 > 0$ such that for any interval $I \subseteq [T_0, \infty)$ of positive length, it holds that*

$$\|c_0 F_0 + c_1 F_1 + ... + c_N F_N\|_I > 0$$

for all $(c_0, c_1, \ldots, c_N) \neq 0$.

Proof. By o-minimality (Theorem B.6), the definable set $F_0^{-1}(0)$ has finitely many connected components. Since F_0 is not eventually zero, for some $T_0 > 0$, $F_0(t) \neq 0$ for all $t \geq T_0$. Thus for all $I \subseteq [T_0, \infty)$ we have

$$\|c_0 F_0 + c_1 F_1 + \cdots + c_N F_N\|_I > 0 \iff \left\| c_0 + c_1 \frac{F_1}{F_0} + \cdots + c_N \frac{F_N}{F_0} \right\|_I > 0.$$

Since the degrees of $F_0, F_1, ..., F_N$ are distinct, the degrees of

$$1, F_1/F_0, ..., F_N/F_0,$$

are distinct. Thus, we are reduced to proving the statement under the assumption that $F_0(t) = 1$ for all $t \geq T_0$.

Pick $T_1 \geq T_0$ such that the functions $F_1, \ldots, F_N$ are differentiable in $[T_1, \infty)$, see Theorem B.5. By Lemma 3.6, $\deg(F_i') = \deg(F_i) - 1$ and F_i' is not eventually zero for $1 \leq i \leq N$. So, the degrees of $F_1', F_2', \cdots, F_N'$ are all distinct. Now, for any interval $I \subseteq [T_1, \infty)$ of positive length,

$$\text{if } \|c_0 + c_1 F_1 + ... + c_N F_N\|_I = 0, \text{ then } \|c_1 F_1' + ... + c_N F_N'\|_I = 0.$$

In view of the above observations, the result is straightforward to deduce by induction. □

Next, for definable functions $F_0(t), F_1(t), ..., F_N(t)$, we consider the *Wronskian matrix* $W(F_0, ..., F_N)(t)$, which is defined to be the $(N+1) \times (N+1)$

matrix whose k-th row is $(F_0^{(k)}(t), F_1^{(k)}(t), \dots, F_N^{(k)}(t))$ for $0 \leq k \leq N$. The Wronskian is well defined for all t large enough due to Theorem B.5.

Lemma 3.8. *Let the notation be as in Lemma 3.7. Then, there exists a $T_0 > 0$ such that the functions $F_0, ..., F_N$ are $(N+1)$-time continuously differentiable and $W(F_0, ..., F_N)(t)$ is non-singular for all $t \geq T_0$.*

Proof. Since $\det(W(F_0, ..., F_N)(t))$ is a definable function, either

$$|\det(W(F_0, ..., F_N)(t))| > 0,$$

for all large t or $\det(W(F_0, ..., F_N)(t)) = 0$ for all large t. We will prove the result by induction. Since F_0 is not eventually zero, the result follows for $N = 0$. Now suppose $N \in \mathbb{N}$ and $T_0 > 0$ such that $\det(W(F_0, ..., F_{N-1})(t)) \neq 0$ for all $t \geq T_0$. Now, suppose $\det(W(F_0, ..., F_N)(t)) = 0$ for all $t \geq T_1 \geq T_0$. Then, by a classical result of Bôcher [Bôc01, Theorem II], $F_0, \dots, F_N$ are linearly dependent on $[T_1, \infty)$. This contradicts Lemma 3.7. □

Lemma 3.9. *Let $\mathcal{V}$ be a real finite-dimensional vector space spanned by functions definable in $\mathscr{S}_{\text{pbd}}$, and suppose that the zero function is the only function in $\mathcal{V}$ which is eventually constantly zero. Then, there exists a basis $\{F_1, ..., F_N\}$ of $\mathcal{V}$ such that:*

$$\deg(F_1) < \cdots < \deg(F_N). \tag{3.4}$$

Proof. We argue by induction. The claim is trivial for $N = 1$. Now choose an arbitrary basis $\{H_1, ..., H_N\}$ of $\mathcal{V}$, and suppose without loss of generality that $\deg(H_i) \leq \deg(H_N)$ for all i. Moreover, we claim that there is no loss of generality in assuming that $\deg(H_i) < \deg(H_N)$, for all $i < N$. Indeed, for each $i < N$ such that $\deg(H_i) = \deg(H_N)$, we can find $c_i \neq 0$ such that $\deg(H_i - c_i H_N) < \deg(H_N)$. By replacing H_i with $H_i - c_i H_N$, we again get a basis for $\mathcal{V}$. Note that the vector space $\mathcal{U} := \text{Span}_{\mathbb{R}}\{H_1, ..., H_{N-1}\}$ is $(N-1)$-dimensional, and $\deg(f) < \deg(H_N)$ for all $f \in \mathcal{U}$. By induction, there is a basis $\{F_1, ..., F_{N-1}\}$ for $\mathcal{U}$ such that $\deg(F_i) < \deg(F_{i+1})$ for all i. Then, $\{F_1, ..., F_{N-1}, F_N\}$ with $F_N := H_N$ is the required basis for $\mathcal{V}$. □

Remark 3.10. If $\mathcal{V}$ is a finite-dimensional vector spanned by definable functions, then the subset of functions which are eventually constantly zero forms a finite-dimensional subspace $\mathcal{V}_0 \subseteq \mathcal{V}$. Then, there exists a uniform $T_0 > 0$ such that for all $f \in \mathcal{V}_0$, it holds that $f(t) = 0$, for all $t \geq T_0$. Thus, by restricting the functions in $\mathcal{V}$ to $[T_0, \infty)$, we may assume $\mathcal{V}_0 = \{0\}$.

Proof of Theorem 3.2. Let $\mathcal{V}$ be a finite-dimensional vector space of functions $f : [0, \infty) \to \mathbb{R}$ definable in a polynomially bounded o-minimal structure. By Remark 3.10, there is no loss in generality in assuming that only the zero function is eventually constantly zero. Using Lemma 3.9, we may further assume, by possibly enlarging $\mathcal{V}$, that $\mathcal{V}$ has a basis of the form $\{F_1^-, ..., F_n^-, F_0^+, F_1^+, ..., F_m^+\}$ such that

$$\deg(F_1^-) < \cdots < \deg(F_n^-) < 0 \leq \deg(F_0^+) < \cdots < \deg(F_m^+),$$

where $m, n \in \mathbb{N}$. Let $N = m + n + 1$. We will denote

$$r_i = \deg(F_i^+), \quad -\kappa_j := \deg(F_j^-), \tag{3.5}$$

where $r_i, \kappa_j \geq 0$. In addition, we assume for convenience that

$$\lim_{t\to\infty} F_i^+(t)/t^{r_i} = 1 \text{ and } \lim_{t\to\infty} F_j^-(t)/t^{-\kappa_j} = 1.$$

For any $\underline{c} := (c_1^-, \dots, c_n^-, c_0^+, \dots, c_m^+) \in \mathbb{R}^N$, we denote

$$f_{\underline{c}}^- := c_1^- F_1^- + \cdots + c_n^- F_n^-,\ f_{\underline{c}}^+ := c_0^+ F_0^+ + \cdots + c_m^+ F_m^+, \text{ and } f_{\underline{c}} = f_{\underline{c}}^- + f_{\underline{c}}^+.$$

Let $\mathscr{F} \subseteq \mathcal{V}$ be a closed definable cone such that

$$\lim_{t\to\infty} f(t) \neq 0,\ \forall f \in \mathscr{F} \setminus \{0\}.$$

Let $\tilde{\mathscr{F}} := \{\underline{c} \in \mathbb{R}^N : f_{\underline{c}} \in \mathscr{F}\} \cong \mathscr{F}$. Then $\tilde{\mathscr{F}}$ is a closed cone in $\mathbb{R}^N$.

Fix a $T_0 \geq 1$ for which the outcome of Lemma 3.8 holds, and suppose for contradiction that there exists a $\delta \in (0,1)$ such that $\mathscr{F} \setminus \{0\}$ is not δ-good on $[T_0, \infty)$. Consider the following definable subset (see Lemma B.3):

$$\mathcal{A} := \left\{ (s, \underline{c}, a, l, \alpha) : \begin{array}{c} s \geq 1,\ \underline{c} \in \tilde{\mathscr{F}} \setminus \{0\},\ a > T_0,\ l > 0, \\ a \leq \alpha \leq a + l - \delta l,\ \frac{\|f_{\underline{c}}\|_{[a,a+l]}}{\|f_{\underline{c}}\|_{[\alpha,\alpha+\delta l]}} > s \end{array} \right\}. \tag{3.6}$$

By the assumption for contradiction, the projection of $\mathcal{A}$ to the first coordinate equals $[1, \infty)$. By the choice function theorem (Theorem B.7), there exists a polynomially bounded definable curve

$$\phi(s) := (\underline{c}(s), a(s), l(s), \alpha(s)) \text{ for } s \geq 1,$$

such that $(s, \phi(s)) \in \mathcal{A},\ \forall s \geq 1$. In particular, there exists an $C_0 \neq 0$ and an $\kappa \in \mathbb{R}$ such that

$$\frac{a(s)}{l(s)} \sim C_0 s^\kappa, \text{ as } s \to \infty. \tag{3.7}$$

The remainder of the proof is organized by dividing the argument into cases based on the sign of κ and on whether $l(s)$ and $a(s)$ are bounded or unbounded. In each case, we derive a contradiction. To guide the reader, we begin with a brief overview.

If $\kappa \leq 0$, this implies that the length of the interval $I(s) = [a(s), a(s)+l(s)]$ grows faster than its starting point. The case where $l(s)$ is bounded is easier to handle, since in that case $I(s)$ remains a bounded interval. When $l(s)$ is unbounded, we exploit the fact that polynomially bounded definable functions $f(t)$ are asymptotically equivalent to functions of the form ct^r.

On the other hand, if $\kappa > 0$, the intervals $I(s)$ are short relative to their left endpoints. In this case, we can apply Taylor approximation, and use the δ-goodness of polynomials (see Proposition 3.5) to reach a contradiction.

Before proceeding to the case analysis, we record the following observation, which will be useful throughout the proof. Let $\|\cdot\|_1$ denote the L^1-norm on

$\mathbb{R}^N$, and define

$$\hat{\underline{c}}(s) := \frac{\underline{c}(s)}{\|\underline{c}(s)\|_1},\ s \geq 1. \tag{3.8}$$

The function $s \mapsto \hat{\underline{c}}(s)$ is a bounded definable function with $\|\hat{\underline{c}}(s)\|_1 = 1$. It follows that

$$\lim_{s\to\infty} \hat{\underline{c}}(s) = \mathbf{v}$$

for some $\mathbf{v} \in \mathbb{R}^N$ satisfying $\|\mathbf{v}\|_1 = 1$. Since $\underline{c}(s) \in \tilde{\mathscr{F}} \setminus \{0\}$ and $\tilde{\mathscr{F}}$ is a closed cone, we conclude that $\mathbf{v} \in \tilde{\mathscr{F}} \setminus \{0\}$.

Case 1: $\kappa \leq 0$. There are two sub-cases: $\{l(s) : s \geq 1\}$ is bounded or not.

Case 1.1. *$l(s)$ is bounded in s.* Then $a(s)$ is also bounded, and by o-minimality $l(s)$ and $a(s)$ converge as $s \to \infty$. We observe that

$$\lim_{s\to\infty} l(s) \neq 0.$$

Because, since $a(s) \geq T_0 \geq 1, \forall s \geq 1$, if $\lim_{s\to\infty} l(s) = 0$, then $\lim_{s\to\infty} \frac{a(s)}{l(s)} = \infty$, which contradicts the assumption that $\kappa \leq 0$. We also note that the end-points of the nested intervals

$$[\alpha(s), \alpha(s) + \delta l(s)] \subseteq [a(s), a(s) + l(s)],$$

converge to the end-points of intervals $I_\delta \subseteq I \subseteq [T_0, \infty)$ of positive length, where $\frac{|I_\delta|}{|I|} = \delta$. Hence $\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+l(s)]}$ is uniformly bounded in s. Now,

$$\begin{aligned}\frac{\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+l(s)]}}{\|f_{\hat{\underline{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}} &= \frac{\|f_{\underline{c}(s)/\|\underline{c}(s)\|_1}\|_{[a(s),a(s)+l(s)]}}{\|f_{\underline{c}(s)/\|\underline{c}(s)\|_1}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}} \\ &= \frac{\|f_{\underline{c}(s)}\|_{[a(s),a(s)+l(s)]}}{\|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}} > s,\end{aligned}$$

so that

$$\frac{\|f_{\hat{\underline{c}}(s)}\|_{[a(s),a(s)+l(s)]}}{s} \geq \|f_{\hat{\underline{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}.$$

By taking $s \to \infty$, we get

$$0 = \|f_{\mathbf{v}}\|_{I_\delta}.$$

This is a contradiction to Lemma 3.7.

Case 1.2. *$l(s)$ is unbounded in s.* In this case, we will achieve a contradiction by proving that:

$$\lim_{s\to\infty} \hat{\underline{c}}(s) = (v_1^-, ..., v_n^-, 0, ..., 0),$$

where $(v_1^-, ..., v_n^-) \neq 0$. This leads to a contradiction, because

$$f_{(v_1^-,...,v_n^-,0,...,0)} \in \mathscr{F} \setminus \{0\} \text{ and } \lim_{t\to\infty} f_{(v_1^-,...,v_n^-,0,...,0)}(t) = 0,$$

is contrary to our assumption on $\mathscr{F}$.

Remark 3.11. This is the only place in the proof of Theorem 1.12 where we use the non-decaying function assumption for the closed cone $\mathscr{F}$, and it highlights the necessity of the non-contracting assumption for obtaining the (C,α)-good property.

We now proceed with our proof. Since $l(s)$ is an unbounded definable function, we have that $l(s)\to\infty$. For $s\geq 1$, we define

(3.9)

$$C^+(s):=\sum_{i=0}^{m}|c_i^+(s)|l(s)^{r_i},\ C^-(s):=\sum_{j=1}^{n}|c_j^-(s)|,\text{ and } C(s):=C^+(s)+C^-(s).$$

Since $s\mapsto\frac{(c_0^+(s)l(s)^{r_0},\dots,c_m^+(s)l(s)^{r_m})}{C(s)}$ is definable and bounded, we get that:

$$(w_0^+,\dots,w_m^+):=\lim_{s\to\infty}\frac{(c_0^+(s)l(s)^{r_0},\dots,c_m^+(s)l(s)^{r_m})}{C(s)}\in[0,1]^{m+1}. \tag{3.10}$$

Claim. There exists an interval $J\subset[0,1]$ of positive length such that

$$w_0^+t^{r_0}+\cdots+w_m^+t^{r_m}=0,\ \forall t\in J.$$

Here, r_i are as in (3.5). In particular, $(w_0^+,\dots,w_m^+)=0$.

Proof of Claim. We consider

$$\varphi_{\underline{c}(s)}(t):=\frac{f_{\underline{c}(s)}(a(s)+tl(s))}{C(s)},\ \forall t\in[0,1].$$

Recall that, by our choice function for the definable set (3.6),

$$s<\frac{\|f_{\underline{c}(s)}\|_{[a(s),a(s)+l(s)]}}{\|f_{\underline{c}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}}=\frac{\|\varphi_{\underline{c}(s)}\|_{[0,1]}}{\|\varphi_{\underline{c}(s)}\|_{I_\delta(s)}}, \tag{3.11}$$

where $I_\delta(s):=\left[\frac{\alpha(s)-a(s)}{l(s)},\frac{\alpha(s)-a(s)+\delta l(s)}{l(s)}\right]\subseteq[0,1]$ having length $|I_\delta(s)|=\delta$ for all $s\geq 1$. Since $\frac{a(s)}{l(s)}$ is a bounded definable function and $l(s)$ is an unbouded definable function,

$$\lim_{s\to\infty}\frac{a(s)}{l(s)}=x_0\in[0,\infty).$$

Also, since the end-points of $I_\delta(s)$ are bounded, they converge to end-points of a sub-interval I_δ of $[0,1]$ of length δ. We will prove that

$$\limsup_{s\to\infty}\|\varphi_{\underline{c}(s)}\|_{I_\delta(s)}=0, \tag{3.12}$$

and that given any $t_0\in(0,1)$,

$$\lim_{s\to\infty}\sup_{t\in[t_0,1]}|\varphi_{\underline{c}(s)}(t)-w_0^+(x_0+t)^{r_0}+\cdots+w_m^+(x_0+t)^{r_m}|=0. \tag{3.13}$$

We note that the claim readily follows from (3.12) and (3.13). To prove (3.12), it is sufficient to show that

$$\sup_{s\to\infty}\|\varphi_{\underline{c}(s)}\|_{[0,1]}<\infty,$$

see (3.11). For each $0 \le i \le m$, there exists a $\nu_i > 0$ such that for all $t \in [0,1]$, we have as $l \to \infty$,

$$\begin{aligned}\frac{F_i^+(a+tl)}{l^{r_i}} &= \frac{(a+lt)^{r_i}}{l^{r_i}} + O\left(\frac{(a+lt)^{r_i-\nu_i}}{l^{r_i}}\right)\\ &= \left(\frac{a}{l}+t\right)^{r_i} + O\left(l^{-\nu_i}\left(\frac{a}{l}+t\right)^{r_i-\nu_i}\right). \qquad (3.14)\end{aligned}$$

We can pick $S_0 \ge 1$ such that $l(s) \ge 1$ for all $s \ge S_0$ and that the following two statements hold:

(1) $\sup_{t\in[0,1]}\left(\frac{a(s)}{l(s)}+t\right)^{r_i}$ *is uniformly bounded in* $s \ge S_0$ *and* $0 \le i \le m$.
Indeed, because $r_i \ge 0$ and $\lim_{s\to\infty}\frac{a(s)}{l(s)} = x_0$.

(2) $\sup_{t\in[0,1]} l(s)^{-\nu_i}\left(\frac{a(s)}{l(s)}+t\right)^{r_i-\nu_i}$ *is uniformly bounded in* $s \ge S_0$ *and* $0 \le i \le m$.
This is clear if $r_i - \nu_i \ge 0$. Now suppose that $r_i - \nu_i < 0$. Then for all $t \in [0,1]$,

$$\begin{aligned} l(s)^{-\nu_i}\left(\frac{a(s)}{l(s)}+t\right)^{r_i-\nu_i} &\le l(s)^{-\nu_i}\left(\frac{a(s)}{l(s)}\right)^{r_i-\nu_i}\\ &= a(s)^{-\nu_i}\left(\frac{a(s)}{l(s)}\right)^{r_i} = O(1),\end{aligned}$$

since $\nu_i > 0$, $a(s) \ge T_0 \ge 1$, $r_i > 0$, and $\frac{a(s)}{l(s)} = O(s^\kappa)$ with $\kappa \le 0$.

By (3.14) and (2), as $s \to \infty$, uniformly in $t \in [0,1]$,

$$f^+_{\underline{c}(s)}(a(s)+tl(s)) = \sum_{i=0}^{m}(c_i^+(s)l(s)^{r_i})\left(\frac{a(s)}{l(s)}+t\right)^{r_i} + O(C^+(s)). \qquad (3.15)$$

In addition, we observe that as $s \to \infty$:

$$\sup_{t\in[0,1]}\left|f^-_{\underline{c}(s)}(a(s)+tl(s))\right| = O(C^-(s)). \qquad (3.16)$$

Here, C^+, C^- are given by (3.9). Now:

$$\begin{aligned}|\varphi_{\underline{c}(s)}(t)| &= \frac{|f_{\underline{c}(s)}(a(s)+tl(s))|}{C(s)}\\ &= \frac{|f^+_{\underline{c}(s)}(a(s)+tl(s)) + f^-_{\underline{c}(s)}(a(s)+tl(s))|}{C(s)} = O(1), \text{ as } s\to\infty,\end{aligned}$$

uniformly in $t \in [0,1]$. This proves (3.12).

Next, we prove (3.13). We have $\lim_{s\to\infty} l(s) = \infty$ and there exists $\kappa' > 0$ such that $0 < a(s)/l(s) = x_0 + O(l(s)^{-\kappa'})$. Fix $t_0 \in (0,1)$. Then $t_0 \le |a(s)/l(s)+t| \le x_0+2$ for all $t \in [t_0,1]$, for all large s. So, for any $0 \le i \le m$,

by (3.14), as $s \to \infty$,

$$\sup_{t\in[t_0,1]} \left| \frac{F_i^+(a(s)+tl(s))}{l(s)^{r_i}} - (x_0+t)^{r_i} \right| = O(l(s)^{-\kappa'}) + O(l(s)^{-\nu_i}).$$

Let $\nu = \min\{\kappa', \nu_0, \dots, \nu_m\} > 0$. Then, as $s \to \infty$, uniformly in $t \in [t_0, 1]$,

$$f^+_{\underline{c}(s)}(a(s)+tl(s)) = \sum_{i=0}^{m} (c_i^+(s) l(s)^{r_i})(x_0+t)^{r_i} + O(C^+(s) l(s)^{-\nu}). \tag{3.17}$$

Also,

$$\sup_{t\in[t_0,1]} \left| f^-_{\underline{c}(s)}(a(s)+tl(s)) \right| = O(C^-(s) l(s)^{-\kappa_n}), \tag{3.18}$$

where κ_n is as in (3.5). By (3.17) and (3.18), for $t \in [t_0, 1]$, we have

$$\begin{aligned} &|\varphi_{\underline{c}(s)}(t) - (w_0^+(x_0+t)^{r_0} + \cdots + w_m^+(x_0+t)^{r_m})| \\ &\leq \sum_{i=0}^{m} \left| \frac{c_i^+(s) l(s)^{r_i}}{C(s)} - w_i^+ \right| (x_0+t)^{r_i} + O(l(s)^{-\kappa_n} + l(s)^{-\nu}), \end{aligned}$$

and the last expression decays to 0 uniformly for $t \in [t_0, 1]$, as $s \to \infty$. □

Using our claim and by recalling (3.10), we get that

$$\begin{aligned} &\lim_{s\to\infty} \frac{(c_1^-(s), ..., c_n^-(s), c_0^+(s) l(s)^{r_0}, ..., c_m^+(s) l(s)^{r_m})}{C(s)} \\ =&\lim_{s\to\infty} \frac{(c_1^-(s), ..., c_n^-(s), c_0^+(s) l(s)^{r_0}, ..., c_m^+(s) l(s)^{r_m})}{\|(c_1^-(s), ..., c_n^-(s), c_0^+(s) l(s)^{r_0}, ..., c_m^+(s) l(s)^{r_m})\|_1} \\ =&(w_1^-, ..., w_n^-, w_0^+, ..., w_m^+) = (w_1^-, ..., w_n^-, 0, ..., 0). \end{aligned}$$

Therefore,

$$\|(w_1^-, ..., w_n^-)\|_1 = 1. \tag{3.19}$$

We now again consider $\hat{\underline{c}}(s)$, as in (3.8). We have

$$\begin{aligned} \mathbf{v} = (v_1^-, ..., v_n^-, v_0^+, ..., v_m^+) &:= \lim_{s\to\infty} \hat{\underline{c}}(s) \\ &= \lim_{s\to\infty} \frac{(c_1^-(s), ..., c_n^-(s), c_0^+(s), ..., c_m^+(s))}{\|(c_1^-(s), ..., c_n^-(s), c_0^+(s), ..., c_m^+(s))\|_1}, \end{aligned}$$

where $\|\mathbf{v}\|_1 = 1$ and $\mathbf{v} \in \widetilde{\mathscr{F}} \setminus \{0\}$. By (3.19), and since $l(s) \to \infty$ and $r_i \geq 0$:

$$\begin{aligned} 1 &= \|(w_1^-, ..., w_n^-)\|_1 \\ &= \lim_{s\to\infty} \frac{\|(c_1^-(s), ..., c_n^-(s))\|_1}{\|(c_1^-(s), ..., c_n^-(s), c_0^+(s) l(s)^{r_0}, ..., c_m^+(s) l(s)^{r_m})\|_1} \\ &\leq \lim_{s\to\infty} \frac{\|(c_1^-(s), ..., c_n^-(s))\|_1}{\|(c_1^-(s), ..., c_n^-(s), c_0^+(s), ..., c_m^+(s))\|_1} \\ &= \|(v_1^-, ..., v_n^-)\|_1 \leq 1. \end{aligned}$$

Therefore, $\|(v_1^-, ..., v_n^-)\|_1 = 1$. As a consequence, we get $v_0^+ = \cdots = v_m^+ = 0$.

Case 2: $\kappa > 0$. Here, we consider two cases corresponding to $a(s)$ being bounded or $a(s) \to \infty$.

Case 2.1. *$a(s)$ is bounded.* We have that $\lim_{s\to\infty} a(s) = x_0 \geq T_0$. Since $\lim_{s\to\infty} \frac{a(s)}{l(s)} = \infty$, it follows that $\lim_{s\to\infty} l(s) = 0$. We again consider $\hat{\underline{c}}(s)$, as in (3.8), and recall $\lim_{s\to\infty} \hat{\underline{c}}(s) = \mathbf{v}$ with $\mathbf{v} \in \tilde{\mathscr{F}}$ such that $\|\mathbf{v}\|_1 = 1$.

By our choice of T_0, the basis functions F_i^-, F_j^+ are $(N+1)$-times continuously differentiable, where $N := n+m+1$. For $t \geq T_0$, let $W(t)$ be the Wronskian matrix $W(F_{-n}(t), \ldots, F_m(t))$, whose k-th row is $(F_{-n}^{(k)}(t), \ldots, F_m^{(k)}(t))$, for $k = 0, \ldots, N$. The Taylor polynomial of order N of the function $h \mapsto f_{\hat{\underline{c}}(s)}(a(s)+h)$ centered at $h = 0$ is expressed by:

$$Q_{N,a(s)}(h) = \hat{\underline{c}}(s) W(a(s))^\top \cdot (h^0, \ldots, h^N), \tag{3.20}$$

where $\cdot$ denotes the dot product on $\mathbb{R}^{N+1}$. By Taylor's theorem (mean value form for the remainder), for all $h \in [0, l(s)]$, there exists a $\xi \in a(s) + [0, l(s)]$ such that

$$f_{\hat{\underline{c}}(s)}(a(s)+h) - Q_{N,a(s)}(h) = \frac{f_{\hat{\underline{c}}(s)}^{(N+1)}(\xi)}{(N+1)!} h^{N+1}.$$

Since $a(s)$, $\hat{\underline{c}}(s)$, and $l(s)$ are bounded, we obtain that $|f_{\hat{\underline{c}}(s)}^{(N+1)}(\xi)|$ is uniformly bounded.

Denote $I(s) := [0, l(s)]$, and let $I_\delta(s) := [0, \delta l(s)]$. Thus, we conclude:

$$\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I(s)} = \|Q_{N,a(s)}\|_{I(s)} + O(l(s)^{N+1}),$$

and similarly,

$$\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)} = \|Q_{N,a(s)}\|_{I_\delta(s)} + O(l(s)^{N+1}). \tag{3.21}$$

Using Proposition 3.5,

$$\begin{aligned}
\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I(s)} =& \|Q_{N,a(s)}\|_{I(s)} + O(l(s)^{N+1}) \\
\leq& M_N(\delta) \|Q_{N,a(s)}\|_{I_\delta(s)} + O(l(s)^{N+1}) \\
\leq& M_N(\delta) \|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)} + O(l(s)^{N+1}) \\
=& M_N(\delta) \|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)} \left[1 + O\left(\frac{l(s)^{N+1}}{\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)}}\right)\right].
\end{aligned}$$

Thus,

$$\frac{\|f_{\underline{c}(s)}\|_{a(s)+I(s)}}{\|f_{\underline{c}(s)}\|_{a(s)+I_\delta(s)}} \ll 1 + O\left(\frac{l(s)^{N+1}}{\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)}}\right).$$

We will now show that

$$\lim_{s\to\infty} \frac{l(s)^{N+1}}{\|Q_{N,a(s)}\|_{I_\delta(s)}} = 0. \tag{3.22}$$

Before proceeding with the proof, note that by combining (3.22) with (3.21), we obtain that $\frac{l(s)^{N+1}}{\|f_{\hat{\underline{c}}(s)}\|_{a(s)+I_\delta(s)}}$ is bounded for all large s. This is a contradiction, since we are assuming that $s \leq \frac{\|f_{\underline{c}(s)}\|_{I(s)}}{\|f_{\underline{c}(s)}\|_{I_\delta(s)}}$ for all $s \geq 1$.

We proceed to prove (3.22). Observe that for a polynomial $q(x) = c_0 + c_1 x + ... + c_k x^k$ of degree bounded by k and an interval $J \subseteq [0,1]$, it follows by Proposition 3.5,

$$\|q\|_J \geq (k+1)^{-1} k^{-k} \|q\|_{[0,1]} |J|^k \gg \|(c_0, c_1, ..., c_k)\|_1 |J|^k. \tag{3.23}$$

Here, we have also used that $\|q\|_{[0,1]}$ and $\|\underline{c}\|_1$ are two norms on the $(k+1)$-dimensional space of polynomials of degree at most k, and we note that the implied constant depends only on k. By (3.20), $Q_{N,a(s)}$ is a polynomial of degree bounded by N whose coefficients are given by $\hat{\underline{c}}(s) W(a(s))^\top$. We have that $W(a(s))^\top \to W(x_0)^\top$ as $s \to \infty$, and we recall Lemma 3.8 which implies that $W(x_0)^\top$ is non-singular. Therefore,

$$\|\hat{\underline{c}}(s) W(a(s))^\top\| \gg \|\hat{\underline{c}}(s)\|_1 \|(W(a(s))^\top)^{-1}\|^{-1} = \|(W(x_0)^\top)^{-1}\|^{-1} > 0$$

for all $s \gg 1$. Then, by (3.23) applied to $J = [0, \delta l(s)]$, we get

$$\|Q_{N,a(s)}\|_{I_\delta(s)} \gg \|(W(x_0)^\top)^{-1}\|^{-1} \delta^N l(s)^N. \tag{3.24}$$

Since δ is fixed, and $l(s) \to 0$, we get (3.22).

Case 2.2. *$a(s)$ is unbounded.* Here $a(s) \asymp s^\theta$, for $\theta > 0$. We recall that for each $\mathrm{n} \in \mathbb{N}$, there is a T_n such that all basis functions F_i^-, F_j^+ will be continuously differentiable n times in the ray $[T_\mathrm{n}, \infty)$ for all i, j (see Theorem B.5). By Lemma 3.6, it holds that

$$\left| \frac{d^p}{dt^p} F_i^-(x) \right| \ll_p x^{-\kappa_i - p}, \text{ and } \left| \frac{d^q}{dt^q} F_j^+(x) \right| \ll_q x^{r_j - q},$$

where κ_i, r_j are as in (3.5), p, q are non-negative integers, and the implied constants depend on p, q. Then, for all sufficiently large integers n such that $r_i - (\mathrm{n}+1) < 0$ for all $0 \leq i \leq m$, it holds that

$$\sup_{t \in [a(s), a(s)+l(s)]} \left| \frac{d^{\mathrm{n}+1}}{dt^{\mathrm{n}+1}} f_{\hat{\underline{c}}(s)}(t) \right| = O_\mathrm{n}(a(s)^{r_m - (\mathrm{n}+1)}),$$

where the implied constant depends on n. By Taylor's theorem, for the Taylor polynomial $P_{\mathrm{n},a(s)}(t)$ of $f_{\hat{\underline{c}}(s)}(t)$ of degree n centered around $a(s)$, we have

$$\begin{aligned} \sup_{t \in [a(s), a(s)+l(s)]} |f_{\hat{\underline{c}}(s)}(t) - P_{\mathrm{n},a(s)}(t)| &= O_\mathrm{n}(a(s)^{r_m - (\mathrm{n}+1)} l(s)^{\mathrm{n}+1}) \\ &= O_\mathrm{n}\left(a(s)^{r_m} \left(\frac{l(s)}{a(s)} \right)^{\mathrm{n}+1} \right) \\ &= O_\mathrm{n}\left(s^{r_m \theta} s^{-(\mathrm{n}+1)\kappa} \right), \end{aligned} \tag{3.25}$$

where the last line follows by the assumption that $a(s)/l(s) \sim s^{\kappa}$ with $\kappa > 0$. Due to Lemma 3.7, the function $\Psi(s) := \|f_{\underline{\hat{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]}$ is positive. Since Ψ is definable in a polynomially bounded o-minimal structure, there exists an $\eta > 0$ such that

$$\|f_{\underline{\hat{c}}(s)}\|_{[\alpha(s),\alpha(s)+\delta l(s)]} \gg s^{-\eta}, \text{ as } s \to \infty. \tag{3.26}$$

We fix n large enough such that $\nu := r_m\theta - (\mathrm{n}+1)\kappa < -\eta$. We denote $J(s) = [a(s), a(s)+l(s)]$ and we let $J_\delta(s) = [\alpha(s), \alpha(s)+\delta l(s)]$. Then:

$$\begin{aligned}
\|f_{\underline{\hat{c}}(s)}\|_{J(s)} &\underbrace{\leq}_{(3.25)} \|P_{\mathrm{n},a(s)}\|_{J(s)} + O(s^{\nu}) \\
&\underbrace{\leq}_{\text{Prop. 3.5}} M_{\mathrm{n}}(\delta)\|P_{\mathrm{n},a(s)}\|_{J_\delta(s)} + O(s^{\nu}) \\
&\underbrace{\leq}_{(3.25)} M_{\mathrm{n}}(\delta)\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)} + O(s^{\nu}) \\
&= M_{\mathrm{n}}(\delta)\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)} \left[1 + O\left(\frac{s^{\nu}}{\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)}}\right)\right] \\
&\underbrace{\ll}_{(3.26)} M_{\mathrm{n}}(\delta)\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)} \left[1 + O\left(\frac{s^{\nu}}{s^{-\eta}}\right)\right]
\end{aligned}$$

Namely, $\frac{\|f_{\underline{\hat{c}}(s)}\|_{J(s)}}{\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)}}$ is bounded for all large s, contradicting the assumption $s < \frac{\|f_{\underline{\hat{c}}(s)}\|_{J(s)}}{\|f_{\underline{\hat{c}}(s)}\|_{J_\delta(s)}}$. □

3.3. **Relative time near varieties.** The following proposition is an analog of [DM93, Proposition 4.2], and it is key for the linearization technique.

Proposition 3.12. *Let $\psi : [0,\infty) \to \mathrm{GL}(m,\mathbb{R})$ be a continuous, unbounded curve definable in $\mathscr{S}_{\mathrm{pbd}}$ such that $\lim_{t\to\infty} \psi(t)v \neq 0, \forall v \in \mathbb{R}^m \setminus \{0\}$. Fix a nonzero polynomial $Q : \mathbb{R}^m \to \mathbb{R}$, and denote*

$$\mathcal{A} := \{x \in \mathbb{R}^m : Q(x) = 0\}.$$

Then, there exists a $T_0 > 0$ such that the following holds: for an $\epsilon > 0$ and a compact subset $\mathrm{K}_1 \subseteq \mathcal{A}$, there exists a compact subset K_2 with

$$\mathrm{K}_1 \subseteq \mathrm{K}_2 \subseteq \mathcal{A}$$

such that for every compact neighborhood Φ of K_2 in $\mathbb{R}^m$, there exists a compact neighborhood Ψ of K_1 in $\mathbb{R}^m$, with $\Psi \subseteq \mathring{\Phi}$, where $\mathring{\Phi}$ denotes the interior of Φ, such that:

$$|\{t \in [a,b] : \psi(t)v \in \Psi\}| \leq \epsilon(b-a), \tag{3.27}$$

for all $v \in \mathbb{R}^m$ and $[a,b] \subseteq [T_0,\infty)$, for which:

- $\psi(t)v \in \Phi,\ \forall t \in [a,b]$, *and*
- $\psi(b)v \in \Phi \setminus \mathring{\Phi}$.

Remark 3.13. Compared to [DM93, Proposition 4.2], the statement our proposition is different in the assumption that the curve $\psi(t)v$ remains in Φ throughout the whole interval $[a,b]$, and exits $\mathring{\Phi}$ at $t=b$. This is a stronger requirement than the one in [DM93, Proposition 4.2] which only asks for $\psi(t_0)v \notin \mathring{\Phi}$ for some $t_0 \in [a,b]$.

Remark 3.14. Notice that for any affine variety $\mathcal{A} \subseteq \mathbb{R}^m$ there exists a polynomial $Q : \mathbb{R}^m \to \mathbb{R}$ such that $\mathcal{A}$ is the zero set of Q. Indeed, by the Hilbert basis theorem, $\mathcal{A}$ is the zero set of finitely many polynomials $f_1, ..., f_r$, and we let $Q := f_1^2 + \cdots + f_r^2$.

Our proof of Proposition 3.12 builds on the following versions of the modified (C,α)-good property.

Proposition 3.15. *Let $\psi : [0,\infty) \to \mathrm{GL}(m,\mathbb{R})$ be a continuous curve definable in $\mathscr{S}_{\mathrm{pbd}}$, and let $Q : \mathbb{R}^m \to \mathbb{R}$ be a polynomial. Then there exist constants $\nu, C, \alpha, T_0 > 0$ such that for any $v \in \mathbb{R}^m$ exactly one of the following properties hold:*

(1) $Q(\psi(t)v) = 0, \forall t \geq T_0$;
(2) $\Theta_v(t) := t^\nu Q(\psi(t)v)$ is (C,α)-good on $[T_0,\infty)$.

Proof. Consider the vector space

$$\mathcal{V} := \mathrm{Span}_{\mathbb{R}}\{Q(\psi(t)v),\ t \geq 0 : v \in \mathbb{R}^m\}.$$

It is straightforward to verify that $\mathcal{V}$ is finite-dimensional, and that all functions in $\mathcal{V}$ are definable. In view of Remark 3.10, there is a $T_0 \geq 0$ such that if $f \in \mathcal{V}$ is eventually zero, then $f(t) = 0, \forall t \in [T_0,\infty)$. We consider $\mathscr{F}$ to be the space of functions of $\mathcal{V}$ restricted to $[T_0,\infty)$. By Lemma 3.9, there is a basis $\{f_0, f_1, ..., f_n\}$ for $\mathscr{F}$ such that $\deg(f_0) < \deg(f_1) < ... < \deg(f_n)$. Let $\nu > 0$ such that $\deg(t^\nu f_0) \geq 0$. Then, by Theorem 1.12, we pick $T_1 > 0$ such that any nonzero function

$$f \in t^\nu \otimes \mathscr{F} = \mathrm{Span}_{\mathbb{R}}\{t^\nu Q(\psi(t)v),\ t \geq T_1 : v \in \mathbb{R}^m\}$$

is (C,α)-good. □

We now note the following corollary.

Corollary 3.16. *Let $\psi : [0,\infty) \to \mathrm{GL}(m,\mathbb{R})$ be a continuous curve, definable in $\mathscr{S}_{\mathrm{pbd}}$. Fix a polynomial $Q : \mathbb{R}^m \to \mathbb{R}$. Then there exist $T_0, C, \alpha > 0$ such that for any $v \in \mathbb{R}^m$ and any interval $[x,y] \subseteq [T_0,\infty)$, if*

$$\sup_{t\in[x,y]} |Q(\psi(t)v)| = |Q(\psi(y)v)|,$$

then

$$|\{t \in [x,y] : |Q(\psi(t)v)| \leq \eta\}| \leq C\left(\frac{\eta}{|Q(\psi(y)v)|}\right)^\alpha (y-x), \tag{3.28}$$

for all $\eta > 0$; we may say that $t \mapsto |Q(\psi(t)v)|$ is right-max-(C,α)*-good in $[T_0,\infty)$.*

Proof. Fix $\nu, C, \alpha, T_0 > 0$ as in Proposition 3.15. Let $[x, y] \subseteq [T_0, \infty)$ and let $v \in \mathbb{R}^m$ be a vector such that

$$|Q(\psi(y)v)| = \delta > 0,$$

and such that

$$|Q(\psi(t)v)| \leq \delta, \ \forall t \in [x, y].$$

Then,

$$\sup_{t\in[x,y]} |t^\nu Q(\psi(t)v)| = y^\nu \delta.$$

Now, by the (C, α)-good property of $|t^\nu Q(\psi(t)v)|$, we get:

$$\begin{aligned} |\{t \in [x, y] : |Q(\psi(t)v)| \leq \eta\}| &= |\{t \in [x, y] : |t^\nu Q(\psi(t)v)| \leq t^\nu \eta\}| \\ &\leq |\{t \in [x, y] : |t^\nu Q(\psi(t)v)| \leq y^\nu \eta\}| \\ &\leq C \left(\frac{y^\nu \eta}{y^\nu \delta}\right)^\alpha (y - x) = C \left(\frac{\eta}{\delta}\right)^\alpha (y - x). \end{aligned} \tag{3.29}$$

□

Proof for Proposition 3.12. Let $\psi : [0, \infty) \to \mathrm{GL}(m, \mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous unbounded curve such that $\lim_{t\to\infty} \psi(t)v \neq 0, \forall v \neq 0$. Pick $T_0, C, \alpha > 0$ such that the (C, α)-good property holds for the family

$$\mathscr{F}_{\mathrm{norm}} := \{\|\psi(t)v\|, \ t \geq T_0 : v \in \mathbb{R}^m \setminus \{0\}\},$$

see Proposition 1.13, and such that the right-max-(C, α)-property holds for the functions in

$$\mathscr{F}_{\mathrm{variety}} := \{|Q(\psi(t)v)|, \ t \geq T_0 : v \in \mathbb{R}^m\},$$

which satisfy the requirements of Corollary 3.16.
Let $\epsilon > 0$. Given a compact set $\mathrm{K}_1 \subseteq \mathcal{A}$, let $R_1 > 0$ be such that $\mathrm{K}_1 \subseteq B_{R_1}(0)$. We choose $R_2 > R_1$ is such that $C(R_1/R_2)^\alpha \leq \epsilon$, and we denote $\mathrm{K}_2 := \overline{\mathrm{B}_{R_2}(0)} \cap \mathcal{A}$; see (3.32) for the use of this choice. We define for $\epsilon' > 0$:

$$\mathcal{A}^{\epsilon'} := \{x \in \mathbb{R}^m : |Q(x)| < \epsilon'\}.$$

Let Φ be a compact neighborhood of K_2. Then, we can pick an $\epsilon_1 > 0$ such that

$$B_{R_2}(0) \cap \mathcal{A}^{\epsilon_1} \subseteq \mathring{\Phi}. \tag{3.30}$$

We denote $\epsilon_2 := \frac{R_1}{R_2}\epsilon_1$, and we note that $K_1 \subseteq B_{R_1}(0) \cap \mathcal{A}^{\epsilon_2}$ and we observe that $C(\epsilon_2/\epsilon_1)^\alpha \leq \epsilon$; see (3.33) for the use of this choice. We consider

$$\Psi := \overline{B_{R_1}(0) \cap \mathcal{A}^{\epsilon_2}}. \tag{3.31}$$

Let $v \in \mathbb{R}^m$ and an interval $[a, b] \subseteq [T_0, \infty)$ such that:

- $\psi(t)v \in \Phi, \ \forall t \in [a, b]$, and
- $\psi(b)v \in \Phi \setminus \mathring{\Phi}$.

Then by (3.30), $\|\psi(b)v\| \geq R_2$ or $|Q(\psi(b)v)| \geq \epsilon_1$.

Suppose that $\|\psi(b)v\| \geq R_2$. By (3.31) and by Proposition 1.13,

$$\begin{aligned} |\{t \in [a,b] | \psi(t)v \in \Psi\}| \leq & |\{t \in [a,b] | \|\psi(t)v\| \leq R_1\}| \\ \leq & C\left(\frac{R_1}{R_2}\right)^{\alpha}(b-a) \leq \epsilon(b-a). \end{aligned} \tag{3.32}$$

Suppose now that $|Q(\psi(b)v)| \geq \epsilon_1$. Since $t \mapsto |Q(\psi(t)v)|$ is continuous and definable, and since a definable map takes a particular value only finitely many times, there is a decomposition

$$\{t \in [a,b] : |Q(\psi(t)v)| \leq \epsilon_1\} = [a_1, b_1] \sqcup \cdots \sqcup [a_k, b_k],$$

for some $k \in \mathbb{N}$, such that for all $1 \leq i \leq k$:

$$|Q(\psi(b_i)v)| = \epsilon_1 \text{ and } |Q(\psi(t)v)| \leq \epsilon_1, \forall t \in [a_i, b_i].$$

Since $\Psi \subseteq \mathcal{A}^{\epsilon_2}$, we have

$$\{t \in [a,b] : \psi(t)v \in \Psi\} \subseteq \{t \in [a,b] : |Q(\psi(t)v)| \leq \epsilon_2\},$$

and since $\epsilon_2 < \epsilon_1$, we have

$$\{t \in [a,b] : |Q(\psi(t)v)| \leq \epsilon_2\} = \bigsqcup_{i=1}^{k} \{t \in [a_i, b_i] : |Q(\psi(t)v)| \leq \epsilon_2\}.$$

We conclude by (3.28) of Corollary 3.16 that

$$|\{t \in [a_i, b_i] : |Q(\psi(t)v)| \leq \epsilon_2\}| \leq C\left(\frac{\epsilon_2}{\epsilon_1}\right)^{\alpha}(b_i - a_i) \leq \epsilon(b_i - a_i), \tag{3.33}$$

for each $1 \leq i \leq k$. This proves that

$$|\{t \in [a,b] | \psi(t)v \in \Psi\}| \leq \epsilon(b-a).$$

Thus, (3.27) holds in all cases. □

4. Non-escape of mass

Let $X^* := X \cup \{\infty\}$ be the one-point compactification of

$$X := \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}).$$

In what follows, $\mathscr{S}_{\mathrm{pbd}}$ is a fixed polynomially bounded o-minimal structure. Let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous and unbounded curve. For $x \in X$ and $T > 0$ let $\mu_{T,\varphi,x}$ be the measure defined in (1.1). By the Banach-Alaoglu theorem, any weak-star limit μ as $T \to \infty$ of the measures $\mu_{T,\varphi,x}$ is a probability measure on X^*. We now state and prove the non-divergence property in the setup of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$. Later in the section, we extend the result to the case of finite volume quotient spaces of Lie groups. In the general case, when the Lie group is of higher rank, the non-divergence result builds on the non-divergence in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ by using Margulis' arithmeticity theorem see [Zim84, page 3].

Proposition 4.1. *Let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous, unbounded, non-contracting curve. Then, there exists $T_0 > 0$ such that given a compact set $F \subseteq \mathrm{SL}(n,\mathbb{R})$ and $\epsilon > 0$, there exists a compact set $K \subseteq \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ such that for any $g \in F$ and $T \geq T_0$, we have*

$$|\{t \in [T_0, T] : \varphi(t)g\mathrm{SL}(n,\mathbb{Z})/\mathrm{SL}(n,\mathbb{Z}) \in K\}| \geq (1-\epsilon)(T - T_0).$$

We need the following preliminaries for the proof. In the discussion that follows, we will refer to rank-k discrete subgroups of $\mathbb{R}^n$ as *k-lattices*. For a k-lattice $\Lambda \subseteq \mathbb{Z}^n$, let $\|\Lambda\|$ be the volume of a fundamental domain $F \subseteq \mathrm{Span}_{\mathbb{R}}\{\Lambda\}$ of Λ with respect to the usual measure on the subspace $\mathrm{Span}_{\mathbb{R}}\{\Lambda\} \subseteq \mathbb{R}^n$ (obtained by restricting the usual Euclidean inner product). Recall that if $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ forms a $\mathbb{Z}$-basis for Λ, then

$$\|\Lambda\| = \|\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_k\|.$$

The norm above is defined by choosing the inner product for which the pure wedges of k-positively oriented tuples of the canonical basis vectors $e_i, i = 1, ..., n$ are orthonormal. Consider the representation of $\mathrm{SL}(n,\mathbb{R})$ on $\bigwedge^k \mathbb{R}^n$ defined by

$$g \cdot (\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_k) := g\mathbf{v}_1 \wedge g\mathbf{v}_2 \wedge \cdots \wedge g\mathbf{v}_k, \quad \forall g \in \mathrm{SL}(n,\mathbb{R}). \tag{4.1}$$

For a fixed $k < n$, the action of $\mathrm{SL}(n,\mathbb{R})$ is transitive on the space of k-lattices of rank $k < n$. We denote $\mathbb{Z}^k := \mathrm{Span}_{\mathbb{Z}}\{e_1, ..., e_k\}$, and we observe that for a unimodular n-lattice $L = g\mathbb{Z}^n \leq \mathbb{R}^n$, where $g \in \mathrm{SL}(n,\mathbb{R})$, the collection of primitive k-sublattices of L is given by

$$g\gamma\mathbb{Z}^k, \gamma \in \mathrm{SL}(n,\mathbb{Z}).$$

The following powerful theorem on quantitative non-divergence due to Kleinbock [Kle10] will be needed.

Theorem 4.2. *Let $B \subseteq \mathbb{R}$ be an interval, let $C \in \mathbb{R}, \alpha > 0$, $0 < \rho < 1$ and let $h : B \to \mathrm{SL}(n,\mathbb{R})$ be a continuous map. Assume that for every $\gamma \in \mathrm{SL}(n,\mathbb{Z})$, and for every $1 \leq k < n$ both of the following conditions are satisfied*

(1) the function $x \to \|h(x)\gamma \cdot e_1 \wedge \cdots \wedge e_k\|$ is (C,α)-good on B, and
(2) $\sup_{x \in B} \|h(x)\gamma \cdot e_1 \wedge \cdots \wedge e_k\| \geq \rho^k$.

Then, for every positive number ρ such that $\epsilon < \rho$, we have

$$|\{x \in B : \lambda_1(h(x)\mathbb{Z}^n) \leq \epsilon\}| \leq Cn2^n \left(\frac{\epsilon}{\rho}\right)^{\alpha} |B|, \tag{4.2}$$

where $\lambda_1(\cdot)$ is the function that outputs the length of the shortest nonzero vector of an Euclidean lattice.

Proof of Proposition 4.1. We identify $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ with the space of unimodular lattices

$$\mathcal{L}_n := \{L := \mathrm{Span}_{\mathbb{Z}}\{v_1, ..., v_n\} : \det(v_{i,j}) = 1\}.$$

Consider

$$\mathcal{B}_\delta := \{L \in \mathcal{L}_n : (L \setminus \{0\}) \cap B_\delta(0) \neq \emptyset\}.$$

Here $B_\delta(0) \subseteq \mathbb{R}^n$ is the ball of radius δ centered at the origin. By Mahler's criterion, $\mathcal{L}_n \setminus \mathcal{B}_\delta$ is compact for all $\delta > 0$ and thus $\mathcal{B}_\delta$ is a neighborhood of ∞. Fix $g\mathbb{Z}^n \in \mathcal{L}_n$ for $g \in \mathrm{SL}(n,\mathbb{R})$, and consider the measures

$$\mu_T(f) = \frac{1}{T}\int_0^T f(\varphi(t)g\mathbb{Z}^n)dt,\ \forall f \in C_c(\mathcal{L}_n). \tag{4.3}$$

Let μ be a weak-* limit of the measures μ_T as $T \to \infty$. In order to show that $\mu(\infty) = 0$, it is enough to prove that for every $\epsilon > 0$, there is δ such that

$$\limsup_{T\to\infty} \frac{|\{t \in [0,T] : \lambda_1(\varphi(t)g\mathbb{Z}^n) \le \delta\}|}{T} \le \epsilon. \tag{4.4}$$

This will be concluded by Theorem 4.2 as follows. We first verify the conditions. For $k \in \{1,2,..,n-1\}$, consider the representation (4.1). By Proposition 1.13, there exists $T_0, C, \alpha > 0$ such that for any fixed $k \in \{1,2,..,n-1\}$, and $\gamma \in \mathrm{SL}(n,\mathbb{Z})$ it holds that

$$\Theta(t) = \|\varphi(t)g\gamma \cdot (e_1 \wedge \cdots \wedge e_k)\| \tag{4.5}$$

is (C,α)-good in $[T_0,\infty)$. Let $F \subseteq \mathrm{SL}(n,\mathbb{R})$ be a given compact subset. Then, the number ρ defined by

$$\rho := (\inf\{\|\varphi(T_0)g\gamma \cdot (e_1 \wedge \cdots \wedge e_k)\| : g \in F,\ \gamma \in \mathrm{SL}(n,\mathbb{Z})\})^{\frac{1}{k}}$$

is positive. Thus, by Theorem 4.2, for any $g \in F$,

$$|\{t \in [T_0,T] : \lambda_1(\varphi(t)g\mathbb{Z}^n) \le \delta\}| \le C\left(\frac{\delta}{\rho}\right)^\alpha (T - T_0). \tag{4.6}$$

□

4.1. **Non-divergence for homogeneous space of Lie groups.** For the following statement, we recall that for a Lie group G, G_u denotes the Lie-subgroup generated by the one-parameter unipotent subgroups of G.

Proposition 4.3. *Let G be a Lie subgroup of $\mathrm{SL}(n,\mathbb{R})$ and let Γ be a lattice in G. Suppose that $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ is a $\mathscr{S}_{\mathrm{pbd}}$-definable, non-contracting curve. Assume further that $\varphi([0,\infty)) \subseteq G_u$. Then, there exists a $T_0 > 0$ such that given $\epsilon > 0$ and a compact set $F \subseteq G$, there exists a compact set $K \subseteq G/\Gamma$ such that for all $g \in F$ and all $T > T_0$,*

$$|\{t \in [T_0,T] : \varphi(t)g\Gamma \in K\}| \ge (1-\epsilon)(T - T_0). \tag{4.7}$$

Remark 4.4. We note that the assumption that $\varphi([0,\infty)) \subseteq G_u$ is non restrictive for our purposes in view of the following. In Proposition 5.13, we show that the hull of a non-contracting curve, definable in a polynomially bounded o-minimal structure and taking values in a Lie subgroup $G \subseteq \mathrm{SL}(n,\mathbb{R})$, is contained in G_u. Thus, by multiplying the original curve with a correcting curve, the assumption $\varphi([0,\infty)) \subseteq G_u$ will be satisfied.

Proof. We follow the arguments as in [DM93, Theorem 6.1] to reduce the problem to the case when G is a semisimple group and Γ is an irreducible lattice in G. We note that if G/Γ is compact, there is nothing to prove. So, we now assume that G/Γ is not compact.

Let M be the smallest closed normal subgroup of G such that $\bar{G} = G/M$ is semisimple with the trivial center and no compact factors. Then $M/(M \cap \Gamma)$ is compact, see [Rag72, Theorem 8.24] and [Sha96, Proof of Theorem 2.2]. Let $q : G \to \bar{G}$ denote the quotient homomorphism. Then $q(\Gamma)$ is a lattice in $\bar{G}$, and the natural quotient map $\bar{q} : G/\Gamma \to \bar{G}/q(\Gamma)$ is proper. We note that if $\tilde{G}$ and $\tilde{M}$ denote the Zariski closures of G and M in $\mathrm{SL}(n, \mathbb{R})$, respectively, then $\bar{G} = G/M = \tilde{G}^0/\tilde{M}^0$. Therefore, $q|_{G_u} : G_u \to \bar{G}$ is a rational homomorphism of real algebraic groups. Thus $q \circ \varphi$ is a curve definable in $\mathscr{S}_{\mathrm{pbd}}$. By Lemma 2.17, $q \circ \varphi : [0, \infty) \to \bar{G}$ is a non-contracting. Since $\bar{q}$ is proper, proving the theorem for $\bar{G}$ in place of G is sufficient.

Therefore, we assume that G is semisimple with the trivial center and no compact factors. Then, by [Rag72, Theorem 5.22], there exist closed normal subgroups $G_1, \ldots, G_k$ of G for some k such that G equals the direct product $G_1 \times \cdots \times G_k$, such that $\Gamma_i := G_i \cap \Gamma$ is an irreducible lattice in G_i, for all i. Moreover, $\Gamma_1 \times \cdots \times \Gamma_k$ is a normal subgroup of finite index in Γ. Hence, G/Γ is finitely covered by $G_1/\Gamma_1 \times \cdots \times G_k/\Gamma_k$. Let $q_i : G \to G_i$ denote the natural factor map, and observe that $q_i \circ \varphi$ is a non-contracting curve in G_i definable in $\mathscr{S}_{\mathrm{pbd}}$ by Lemma 2.17. So, proving the theorem for G_i/Γ_i separately for each i will imply the theorem for G/Γ. Therefore, without loss of generality, we may assume that Γ is an irreducible lattice in G, which is semisimple with the trivial center and no compact factors.

First, suppose that the real rank of G is 1. It is straightforward to adapt the proof of [Dan84, Proposition 1.2] to conclude (4.7). For this purpose, it is enough to use the following property of the map φ in place of [Dan84, Lemmas 2.5 and 2.7]: Let $V = \wedge^d \mathfrak{g}$, where d is the dimension of the maximal unipotent subgroup of G and $\mathfrak{g}$ the Lie algebra of G. Now, G acts on V via $\wedge^d \mathrm{Ad}$. Then there exists $T > 0$, $C > 0$ and $\alpha > 0$ such that for any $g \in G$, the map $t \mapsto \|\varphi(t) g \cdot p\|$ has the (C, α)-good property on $[T_0, \infty)$ by Proposition 1.13.

Now we suppose that the real rank of G is at least 2. Then, by the arithmeticity theorem due to G. A. Margulis, Γ is arithmetic; that is, see [Zim84, page 3]: There exists an $m \in \mathbb{N}$, an algebraic semisimple group $H \subseteq \mathrm{SL}(m, \mathbb{R})$ defined over $\mathbb{Q}$ and a surjective algebraic homomorphism

$$\rho : H^0 \to G,$$

such that $\ker \rho$ is compact and $\rho(H^0(\mathbb{Z}))$ is commensurable with Γ, where $H^0(\mathbb{Z}) = H^0 \cap \mathrm{SL}(m, \mathbb{Z})$. Since G has no compact factors, the map ρ restricted to H_u is a finite cover of G. By Lemma 2.21, we can lift φ to a $\mathscr{S}_{\mathrm{pbd}}$-definable curve $\psi : [0, \infty) \to H_u \subseteq H^0 \subseteq \mathrm{SL}(m, \mathbb{R})$ which is non-contracting in H_u, such that $\rho \circ \psi = \varphi$. By Proposition 2.22, ψ is non-contracting in $\mathrm{SL}(m, \mathbb{R})$. Denote $Y := \mathrm{SL}(m, \mathbb{R})/\mathrm{SL}(m, \mathbb{Z})$. As in Proposition 4.1, let

$T_0 > 0$ be such that for any compact set $F_1 \subseteq \mathrm{SL}(m,\mathbb{R})$, there exists a compact set $K_1 \subseteq Y$ such that for any $T \geq T_0$ and $h \in F_1$

$$|\{t \in [T_0, T] : \psi(t)h\mathrm{SL}(m,\mathbb{Z}) \in K_1\}| \geq (1-\epsilon)(T-T_0). \tag{4.8}$$

Therefore, since $H^0/H^0(\mathbb{Z}) \hookrightarrow Y$ is a proper H^0-equivariant continuous injection, the conclusion of the proposition holds for H^0 in place of G, $H^0(\mathbb{Z})$ in place of Γ, and ψ in place of φ. And, since $\rho \circ \psi = \varphi$, the conclusion of the proposition holds for $\rho(H^0(\mathbb{Z}))$ in place of Γ, and hence it holds for Γ due to the commensurability. □

As an immediate consequence of Proposition 4.3 we obtain the following.

Theorem 4.5. *Let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous, unbounded, non-contracting curve. Let $G \subseteq \mathrm{SL}(n,\mathbb{R})$ be a Lie subgroup, Γ be a lattice in G, and suppose that $\varphi([0,\infty)) \subseteq G_u$. Then, any weak-* limit μ of $\mu_{T,\varphi,x}$ as $T \to \infty$ is a probability measure on G/Γ, for all $x \in G/\Gamma$.*

5. Unipotent invariance

We say that a measure μ on G/Γ is *invariant* under $g \in G$ if $g_*\mu = \mu$, where g_* denotes the pushforward by the left-translation map by g on G/Γ. We say that a measure is *invariant under a subgroup*, if it is invariant under every element of the subgroup.

For the rest of the section, $\mathscr{S}_{\mathrm{pbd}}$ is a fixed polynomially bounded o-minimal structure.

Proposition 5.1. *Let $G \subseteq \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup. Consider a continuous, unbounded, $\mathscr{S}_{\mathrm{pbd}}$-definable curve $\varphi : [0,\infty) \to G$ such that $\det(\varphi(t))$ is constant in t. Suppose further that for all $v \in \mathbb{R}^n \setminus \{0\}$, $\lim_{t\to\infty} \varphi(t)v \neq 0$. Then there exists a nontrivial one-parameter unipotent group $P \subseteq G$ such that if $\Gamma \subseteq G$ is a discrete subgroup, $g \in G$ and μ is weak-* limit of the measures*

$$\mu_{T,\varphi,g\Gamma}(f) := \frac{1}{T}\int_0^T f(\varphi(t)g\Gamma)dt,\ f \in C_c(G/\Gamma),$$

then μ is invariant under P.

The key idea behind the proof is that o-minimal curves exhibit a kind of "tangency at infinity." Specifically, one can construct definable *change-of-speed* functions $h(t)$ that diverge to infinity and such that

$$\lim_{t\to\infty} \varphi(h(t))\varphi(t)^{-1} = \rho,$$

where ρ is a nontrivial element. Under our assumptions, these limiting elements ρ generate a one-parameter unipotent subgroup, and moreover, it holds that $h'(t) = 1 + o(t^{-\nu})$. The latter derivative asymptotic ensures that any limiting measure of $\mu_{T,\varphi,g\Gamma}$ as $T \to \infty$ is invariant under the subgroup generated by ρ; see Lemma 5.6 below.

This idea was previously applied by Shah in [Sha94] in the context of averages along polynomial curves. In the polynomial case, one can construct

suitable change-of-speed maps using Taylor expansions. In the o-minimal setting, Peterzil and Steinhorn [PS99] showed that the collection of all possible limits of the form $\lim_{t\to\infty}\varphi(h(t))\varphi(t)^{-1}$, where h is a definable function diverging to ∞, forms a one-dimensional torsion-free group. We refer to this group as the *Peterzil-Steinhorn subgroup*, which we define in a simplified form in Definition 5.3 to suit our needs.

Poulios, in his thesis [Pou13], further studied Peterzil-Steinhorn groups and showed that the group associated with an unbounded curve definable in a polynomially bounded structure is either unipotent or $\mathbb{R}$-diagonalizable. Importantly, Poulios establishes a convenient criterion for when the Peterzil-Steinhorn group is unipotent.

We now recall the required notions and results from [Pou13]. For the following, for $r<1$, consider

$$h_{r,s}(t) := \left(t^{1-r}+(1-r)s\right)^{\frac{1}{1-r}} = t+st^r+o(t^r), \text{ where } s\in\mathbb{R}, \tag{5.1}$$

and let

$$h_{1,s}(t) := st, \text{ where } s>0. \tag{5.2}$$

Proposition 5.2. [Pou13, Theorem 3.2.4, Theorem 3.3.1, Corollary 3.4.3] *Let $G\subseteq \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup, and let $\varphi:[0,\infty)\to G$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, unbounded curve. Then, there exists a unique $r\leq 1$ such that*

$$M_\varphi := \lim_{t\to\infty} t^r\cdot\varphi'(t)\cdot\varphi(t)^{-1} \tag{5.3}$$

with $M_\varphi\in\mathfrak{g}\neq\{0\}$, where $\mathfrak{g}$ represents the Lie-algebra of G.
It holds that M_φ is nilpotent $\iff$ $r<1$, and M_φ is $\mathbb{R}$-diagonalizable $\iff$ $r=1$. Moreover:

(1) If $r<1$, consider $\rho(s):=\exp(sM_\varphi)$, $s\in\mathbb{R}$. Then:

$$\rho(s) := \lim_{t\to\infty}\varphi(h_{r,s}(t))\varphi(t)^{-1},\ s\in\mathbb{R}. \tag{5.4}$$

(2) If $r=1$, let $d(s):=\exp(\log(s)M_\varphi)$, then:

$$d(s) := \lim_{t\to\infty}\varphi(h_{1,s}(t))\varphi(t)^{-1},\ s>0.$$

Definition 5.3. For an unbounded curve $\varphi:[0,\infty)\to\mathrm{GL}(n,\mathbb{R})$ definable in a polynomially bounded o-minimal structure, we say that $r\leq 1$ is the *P.S. order* of φ, if r satisfies (5.3). We call the one-parameter subgroup P generated by $\exp(M_\varphi)$ as the *P.S. group* of φ.

Remark 5.4. In the notations of Proposition 5.2, if φ is a restriction of a one-parameter group to $[0,\infty)$, then $r=0$, $h_{r,s}(t)=t+s$, and $\rho(s)=\varphi(s)$ for all $s\in\mathbb{R}$; that is, φ is a restriction of a unipotent one-parameter subgroup.

5.1. **Unipotent P.S. groups and limiting unipotent invariance.** We now show that when the P.S. group is unipotent, any limiting measure of measures of the form (1.1) will be invariant under the P.S. group. This proof is presented in [Sha94] and [PS21], and we provide it here for completeness. The following elementary calculus lemma will be needed.

Lemma 5.5. *Let $h : (0,\infty) \to \mathbb{R}$ be a differentiable function such that $\lim_{t\to\infty} h'(t) = 1$. Then, for any bounded continuous function $F : [0,\infty) \to \mathbb{R}$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (F(h(t)) - F(t))dt = 0 \tag{5.5}$$

For the proof below, notice that for $r < 1$, it holds that $h'_{s,r}(t) = 1 + O(t^{r-1})$.

Lemma 5.6. *Let $G \subseteq \mathrm{GL}(n,\mathbb{R})$ be a closed subgroup and let $\Gamma \subseteq G$ be a discrete subgroup. Suppose that $\varphi : [0,\infty) \to G$ is a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous, unbounded curve such that its P.S. group is unipotent (namely, (5.3) holds for $r < 1$). Let $\rho(s) = \exp(sM_\varphi)$, as in (5.4). Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (f(\rho(s)\varphi(t)x) - f(\varphi(t)x))dt, \tag{5.6}$$

for all $f \in C_c(G/\Gamma)$, $x \in G/\Gamma$ and $s \in \mathbb{R}$.

Proof. For $f \in C_c(G/\Gamma)$ and an $\epsilon > 0$, by uniform continuity of f, there is an T_ϵ such for all $t \geq T_\epsilon$:

$$\begin{aligned}|f(\rho(s)\varphi(t)x) - f(\varphi(h_{r,s}(t)x)| &= |f(\rho(s)\varphi(t)x) - f([\varphi(h_{r,s}(t))\varphi(t)^{-1}]\varphi(t)x)| \\ &\leq \epsilon/2,\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{T}\int_0^T (f(\rho(s)\varphi(t)x) - f(\varphi(t)x))\,dt \\ &= O\left(\frac{T_\epsilon}{T}\right) + O(\epsilon) + \frac{1}{T}\int_{T_\epsilon}^T (f(\varphi(h_{r,s}(t)x) - f(\varphi(t)x))\,dt.\end{aligned}$$

The last term converges to zero by Lemma 5.5. □

If a curve is contained in a unipotent group, as in [PS21], then the P.S. group is automatically unipotent. In an overview, to treat our more general setup, we will distinguish between two type of curves to those that are *essentially diagonal* (see the definition below), and those that are or not. We will show that if a curve is not essentially diagonal, then its P.S. group is unipotent.

Definition 5.7. We call a curve $\{\varphi(t)\}_{t\geq 0} \subseteq \mathrm{GL}(n,\mathbb{R})$ *essentially diagonal* if there exists a decomposition

$$\varphi(t) = \sigma(t)b(t)C, \tag{5.7}$$

where $b(t) \in \mathrm{GL}(n,\mathbb{R})$ is a diagonal matrix for all large t, $C \in \mathrm{GL}(n,\mathbb{R})$ is a constant matrix, and $\sigma(t) \in \mathrm{SL}(n,\mathbb{R})$ is convergent as $t \to \infty$.

The following is our key observation.

Proposition 5.8. *Let $\varphi : [0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, unbounded, continuous curve. Suppose that $\det(\varphi(t))$ is constant for all large enough t. Then, the P.S. group of φ is unipotent if and only if the curve φ is not essentially diagonal.*

We will prove the above Proposition 5.8 in the subsection below. Before we proceed, we consider the following lemma which proves Proposition 5.1 by assuming Proposition 5.8 in combination with Lemma 5.6.

Lemma 5.9. *Let $\varphi : [0,\infty) \to \mathrm{GL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, unbounded, continuous curve. Suppose that the determinant $\det(\varphi(t))$ is constant in t and suppose that $\lim_{t\to\infty} \varphi(t)v \neq 0$, for all $v \in \mathbb{R}^n \setminus \{0\}$. Then, φ is not essentially diagonal and in particular, the P.S. group of φ is unipotent.*

Proof. Suppose for contradiction that the P.S. group is not unipotent. Then, by Proposition 5.8, φ is essentially diagonal. Namely $\varphi(t) = \sigma(t)b(t)C$, where C, $\sigma(t) \in \mathrm{SL}(n,\mathbb{R})$ are bounded and $b(t)$ is a diagonal matrix with $\det(b(t)) = c_0$ for all t. Since we assume that φ is unbounded, it follows that $b(t)$ is unbounded, which in turn implies that there is an index $1 \leq i \leq n$ such that the corresponding entry function $b_{ii}(t)$ on the diagonal of $b(t)$ decays to zero. In particular, for $v := C^{-1}e_i$ it holds that $\lim_{t\to\infty} \varphi(t)v = 0$, which is a contradiction. □

5.1.1. *Proving Proposition 5.8.* We will call the (i,j) index of an upper-triangular matrix $b \in \mathrm{SL}(n,\mathbb{R})$ of the form

$$b = \begin{bmatrix}
f_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & f_{i,i} & 0 & \cdots & 0 & \boldsymbol{f_{i,j}} & \cdots & f_{i,n} \\
\vdots & & \vdots & \ddots & \ddots & & & \vdots & & \vdots \\
\vdots & & \vdots & & \ddots & \ddots & & \vdots & & \vdots \\
\vdots & & \vdots & & & \ddots & \ddots & \vdots & & \vdots \\
\vdots & & \vdots & & & & 0 & f_{j,j} & \cdots & f_{j,n} \\
\vdots & & \vdots & & & & & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & f_{n,n}
\end{bmatrix}, \tag{5.8}$$

where $f_{i,j} \neq 0$, the *first nonzero off-diagonal entry*. More precisely, (i,j) is the first nonzero entry among the off-diagonal entries according to the following lexicographic order on $\mathbb{N}^2$:

$$(i,j) \prec (k,l) \iff i < k \text{ or } (i = k \text{ and } j < l). \tag{5.9}$$

Let $B \leq \mathrm{SL}(n,\mathbb{R})$ be the subgroup of upper-triangular matrices. Let $b : [0,\infty) \to B$ be a definable curve. Recall that each definable function $f(t)$

is either zero for all large enough t or $|f(t)| > 0$ for all large t. Thus, for all large enough t, there exists a unique first nonzero off-diagonal entry in $b(t)$, or $b(t)$ is diagonal for all large t. We will refer to this entry as *the first nonzero off-diagonal entry* of the curve $b(t)$.

Lemma 5.10. *Let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable curve. Then there exists a definable curve $b : [0,\infty) \to B$ and a constant matrix $C \in \mathrm{SL}(n,\mathbb{R})$ such that*

$$\varphi(t) = \sigma(t)b(t)C,$$

where $\sigma(t) \in \mathrm{SL}(n,\mathbb{R})$ converges as $t \to \infty$. Furthermore, for sufficiently large t, the curve $b(t)$ satisfies one of the following:

- *$b(t)$ is diagonal, or*
- *the first nonzero off-diagonal entry $f_{i,j}$ of $b(t)$ satisfies:*
  (1) $\deg f_{i,i} \neq \deg f_{i,j}$, *and*
  (2) $\deg f_{i,j} > \deg f_{j,j}$.

Proof. Using the KAN decomposition, we first write $\varphi(t) \in \mathrm{SL}(n,\mathbb{R})$, as $\varphi(t) = k(t)p(t)$, where $k(t) \in \mathrm{SO}(n,\mathbb{R})$ and $p(t) \in B$ an upper-triangular matrix. In view of the Gram-Schimdt process, $k(t)$ and $p(t)$ are definable. Since $k(t)$ is bounded and definable, $\lim_{t\to\infty} k(t)$ exists. For all t large, $p(t)$ is either diagonal or $p(t)$ takes the form of (5.8).
We employ the following algorithm to achieve the outcome described in the lemma. At each step, we perform either a column or a row operation on $p(t)$:

(1) All off-diagonal entries are eventually 0. Then we are done.
(2) We pick the first index (i,j) (with $i < j$) such that the (i,j)-th entry is not eventually 0, and proceed to the next step.
(3) Suppose that $\deg(f_{i,j}) = \deg(f_{i,i})$. Then there is an $c \in \mathbb{R}$ such that $\deg(f_{i,j} - cf_{i,i}) < \deg(f_{i,i})$. For this step, subtract from the j-th column the i-th column multiplied by c. This amounts to multiplying $p(t)$ by a constant unipotent matrix from the right. The obtained matrix is the same besides the i,j-th entry which is replaced with $f_{i,j} - cf_{i,i}$. There are two possibilities now:
   (a) $f_{i,i} - cf_{i,j}$ is eventually zero. If all off-diagonal entries are eventually 0, then we are done. Otherwise, the first eventually nonzero off-diagonal entry in the resulting matrix has a strictly larger index (according to the lexicographic order), and we go back to step 2.
   (b) $f_{i,i} - cf_{i,j}$ is not eventually zero: Then the first requirement of Lemma 5.10 is satisfied. We continue then with the following step.
(4) If $\deg(f_{i,j}) > \deg(f_{j,j})$, then the second requirement of Lemma 5.10 is satisfied, and we are done, otherwise we proceed to the next step.
(5) Now suppose $\deg(f_{i,j}) \leq \deg(f_{j,j})$. Then subtract from the i-th row the j-th row multiplied by $\frac{f_{i,j}}{f_{j,j}}$. This amounts to multiplying from the left by a unipotent matrix, which converges as $t \to \infty$. This is

so because

$$\deg\left(\frac{f_{i,j}}{f_{j,j}}\right) = \deg(f_{i,j}) - \deg(f_{j,j}) \leq 0.$$

Then, either all the off-diagonal entries in the resulting matrix are eventually 0, and we are done; or the first eventually nonzero off-diagonal entry in the resulting matrix has a strictly larger index (according to the lexicographic order), and we go back to step 2.

The algorithm ends with finitely many steps with either a diagonal matrix or a definable curve $b(t)$ satisfying the requirements of the lemma. □

Proof for Proposition 5.8. First, note that it is enough to prove the statement for $\varphi \subseteq \mathrm{SL}(n,\mathbb{R})$ since the P.S. group is unchanged if we multiply the curve by a constant invertible matrix from the right. In the notations of Lemma 5.10:

$$\varphi(t) = \sigma(t)b(t)C,$$

where $\lim_{t\to\infty}\sigma(t) = g \in \mathrm{SL}(n,\mathbb{R})$. Let $r \leq 1$ be such that (5.3) holds for $\{b(t)\}$. Let $h_{s,r}(t)$ be is as in (5.1)–(5.2). We note that

$$\lim_{t\to\infty}\varphi(h_{s,r}(t))\varphi(t)^{-1} = g\left[\lim_{t\to\infty} b(h_{s,r}(t))b(t)^{-1}\right]g^{-1}. \tag{5.10}$$

Therefore, by Proposition 5.2, the r corresponding to (5.3) for $\varphi(t)$ and $b(t)$ are the same. Now, $b(t)$ is either eventually diagonal or upper-triangular and satisfies the conditions of the Lemma 5.10 for the first (eventually) nonzero off-diagonal entry. If $b(t)$ is eventually diagonal, then for all $s > 0$, $\lim_{t\to\infty} b(h_{1,s}(t))b^{-1}(t)$ converges to a diagonal matrix. Namely, the P.S. group is diagonal in this case.

Suppose that $b(t)$ is not eventually diagonal. Let (i,j) be the first nonzero off-diagonal entry in $b(t)$ (for all large enough t). We observe that (i,j)-th entry in the matrix $b'(t)b(t)^{-1}$ is

$$\frac{-f'_{i,i}f_{i,j} + f'_{i,j}f_{i,i}}{f_{i,i}f_{j,j}} = \left(\frac{f_{i,j}}{f_{i,i}}\right)' \cdot \frac{f_{i,i}}{f_{j,j}}. \tag{5.11}$$

Since $r_{i,i} := \deg f_{i,i} \neq r_{i,j} := \deg f_{i,j}$, we have that

$$\deg\left(\frac{f_{i,j}}{f_{i,i}}\right)' = r_{i,j} - r_{i,i} - 1.$$

Thus,

$$\deg\left(\left(\frac{f_{i,j}}{f_{i,i}}\right)' \cdot \frac{f_{i,i}}{f_{j,j}}\right) = r_{i,j} - r_{j,j} - 1.$$

Recall by Lemma 5.10 that $r_{i,j} - r_{j,j} > 0$. Thus the (i,j)-th entry in $tb'(t)b(t)^{-1}$ is unbounded. Hence $r \neq 1$. □

5.2. **Unipotent invariance in quotients.** For our purposes, it will not suffice only to establish unipotent invariance for our limiting measures, but we will also require unipotent invariance in certain quotients. We will now describe more precisely our motivation for the results in this section and how they play a role in the proof of Theorem 1.7 in Section 6.

Consider a non-contracting curve $\varphi : [0,\infty) \to G := \mathbf{G}(\mathbb{R})$ definable in $\mathscr{S}_{\mathrm{pbd}}$, and consider the measures $\mu_{T,\varphi,x}$; see (1.1). By Theorem 4.5, there exists a subsequence $\mu_{T_i,\varphi,x}$ which converges to a probability measure μ, as $i \to \infty$. By Proposition 5.1, μ is invariant under a nontrivial unipotent one-parameter subgroup. Let W be the subgroup generated by all unipotent one-parameter subgroups that preserve μ. We employ Ratner's measure classification theorem for the W-invariant ergodic measures and the linearization technique to prove that $\varphi(t)$ is contained in a certain algebraic subgroup N, such that $W \trianglelefteq N$ and such that $H_\varphi \subseteq N$, where H_φ is the hull of φ. Our goal will be to prove that $H_\varphi \subseteq W$. To achieve this goal, we would like to show that if $\varphi(t)$ is not bounded modulo W, then the image of φ in N/W, which is definable, is, in fact, non-contracting, and hence its corresponding P.S. group in N/W is unipotent. The latter outcome implies that μ is invariant under a unipotent one-parameter subgroup of N, which is not contained in W. This contradicts our choice of W as the group generated by *all* unipotents preserving μ. Before proceeding with the relevant result, we have the following remark which emphasizes that the requirement that $H_\varphi \subseteq N$ is important.

Remark 5.11. Consider $\varphi(t) := \left[\begin{smallmatrix} t & t^2 \\ 0 & 1/t \end{smallmatrix}\right]$ for $t \geq 1$. Then φ is non-contracting in $\mathrm{SL}(2,\mathbb{R})$, and its P.S. group is $U := \{u(s) := \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right] : s \in \mathbb{R}\}$. Now consider the group $B \leq \mathrm{SL}(2,\mathbb{R})$ of upper triangular matrices. Then, B/U is naturally identified with the group $A \leq \mathrm{SL}(2,\mathbb{R})$ of diagonal matrices. In particular, $\varphi(t)$ modulo U is unbounded, but it is a diagonal curve. Due to the next result, this is explained by the fact that H_φ is not contained in B.

Proposition 5.12. *Let $\psi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be an unbounded, non-contracting curve definable in $\mathscr{S}_{\mathrm{pbd}}$. Suppose that $\psi([0,\infty)) \subseteq H_\psi$, where H_ψ is the hull of ψ. Let $N \subseteq \mathrm{SL}(n,\mathbb{R})$ be a Zariski closed subgroup such that $H_\psi \subseteq N$, and let $H \trianglelefteq N$ be a closed normal subgroup.*

Assume that the image of ψ in N/H_u is unbounded. Let $q : N \to N/H$ be the natural quotient map. Then, there exists $r < 1$ such that

$$\lim_{t\to\infty} q(\psi(h_{s,r}(t)))q(\psi(t))^{-1} = p(s),\ \forall s \in \mathbb{R}, \tag{5.12}$$

where $\{p(s) : s \in \mathbb{R}\}$ is a nontrivial one-parameter subgroup of N/H. Here, $h_{s,r}(t)$ is given by (5.1)*. Importantly, there exists a one-parameter unipotent subgroup $\{u(s) : s \in \mathbb{R}\} \subseteq N$ such that $p(s) = q(u(s))$ for all $s \in \mathbb{R}$.*

Proof of Proposition 5.12. Let $\tilde{H}_u := \mathrm{zcl}(H_u)$, where H_u is the subgroup generated by all one-parameter unipotent subgroups contained in H. Since $H \trianglelefteq N$, it follows that $H_u \trianglelefteq N$, and by Zariski density $\tilde{H}_u \trianglelefteq N$. Now, H_u

is of finite index in $\tilde{H}_u$ (see Remark 2.7), therefore, since the image of ψ is unbounded in N/H_u it follows that the image of ψ is unbounded in $N/\tilde{H}_u$.

We realize $N/\tilde{H}_u$ as a real algebraic subgroup of $\mathrm{SL}(m,\mathbb{R})$ for some m. For each $t \in [0,\infty)$, let $\bar{\psi}(t)$ denote the image of $\psi(t)$ in $\mathrm{SL}(m,\mathbb{R})$. We claim that $\bar{\psi}$ is non-contracting for $\mathrm{SL}(m,\mathbb{R})$. Indeed, because any finite-dimensional linear representation of $\mathrm{SL}(m,\mathbb{R})$ is an algebraic representation of $\tilde{H}_\psi \subseteq N$, and ψ is non-contracting for $\tilde{H}_\psi$, where $\tilde{H}_\psi$ is the Zariski closure of the hull of ψ, see Theorem 2.5, (3).

Therefore by Lemma 5.9, there exists $r < 1$ such that

$$\lim_{t\to\infty} \bar{\psi}(h_{s,r}(t))\bar{\psi}(t)^{-1} =: \bar{p}(s),\ \forall s \in \mathbb{R}, \tag{5.13}$$

where $s \mapsto \overline{p}(s)$ is a nontrivial one-parameter unipotent subgroup of $\mathrm{SL}(m,\mathbb{R})$ contained in $N/\tilde{H}_u$. Let $\vartheta : N \to N/H_u$ denote the quotient homomorphism. Since the quotient map $N/H_u \to N/\tilde{H}_u$ is a homomorphism with a finite kernel, for each $s \in \mathbb{R}$, the following limit exists:

$$\tilde{p}(s) := \lim_{t\to\infty} \vartheta(\psi(h_{s,r}(t)))\vartheta(\psi(t))^{-1}. \tag{5.14}$$

In view of Jordan decomposition, there exists a unipotent one-parameter subgroup $\{u(s) : s \in \mathbb{R}\} \subseteq N$ such that for all $s \in \mathbb{R}$, the image of $u(s)$ in $N/\tilde{H}_u$ is $\bar{p}(s)$, and hence $\tilde{p}(s) = \vartheta(u(s))$. Since $\{\bar{p}(s) : s \in \mathbb{R}\}$ is a unipotent group, we obtain that $\{u(s) : s \in \mathbb{R}\}$ is a nontrivial unipotent one-parameter subgroup of N not contained in H_u, and hence it is not contained in H. Thus, $\{q(u(s)) : s \in \mathbb{R}\}$ is a nontrivial subgroup of N/H.

Denote $\eta : N/H_u \to N/H$ the quotient homomorphism, and note that $\eta \circ \vartheta = q$. Then, for all $s \in \mathbb{R}$, $p(s) := \eta(\tilde{p}(s)) = q(u(s))$, and by (5.14),

$$p(s) = \lim_{t\to\infty} \eta \circ \vartheta(\psi(h_{s,r}(t))\psi(t)^{-1}) = \lim_{t\to\infty} q(\psi(h_{s,r}(t)))q(\psi(t))^{-1}.$$

□

5.3. **Inclusion of the hull in a Lie subgroup.**

Proposition 5.13. *Let $G \subseteq \mathrm{SL}(n,\mathbb{R})$ be a connected unimodular closed subgroup, and let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, non-contracting, continuous curve such that $\varphi([0,\infty)) \subseteq G$. Then, $H_\varphi \subseteq G_u$, and in particular, any correcting curve of φ takes values in G_u.*

The result does not follow from Theorem 2.5, because we do not assume that G is an observable subgroup of $\mathrm{SL}(n,\mathbb{R})$.

If G is not algebraic, the hull H_φ may be contained in the Zariski closure of G, but not in G. The unimodularity of G ensures that $H_\varphi \subseteq G$.

The unimodularity assumption is essential even in the algebraic subgroup case. For example, in Remark 5.11, the curve is non-contracting in $\mathrm{SL}(2,\mathbb{R})$ and its image is contained in the non-unimodular Borel subgroup B of $\mathrm{SL}(2,\mathbb{R})$. But its hull is not contained in B.

The following is an immediate corollary of Proposition 5.13.

Corollary 5.14. *Let $G_1 \subseteq G_2$ be real points of real algebraic groups. Let $\varphi : [0,\infty) \to G_2$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, non-contracting, continuous curve. Suppose that $\varphi([0,\infty)) \subseteq G_1$ and G_1 is unimodular. Then $\varphi : [0,\infty) \to G_1$ is non-contracting.*

The proof will be based on Proposition 5.12 and the following key lemma.

Lemma 5.15. *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(n,R)$. Let $\mathfrak{g}_u$ be the subalgebra generated by nilpotent matrices in $\mathfrak{g}$. Then for any $X \in \mathfrak{g}$,*

$$\mathrm{tr}(\mathrm{ad}X) = \mathrm{tr}(\mathrm{ad}X|_{\mathfrak{g}_u}),$$

where $\mathrm{tr}(A)$ denotes the trace of A.

So, for the connected Lie subgroup G corresponding to $\mathfrak{g}$, we have

$$\det(\mathrm{Ad}(g)|_{\mathfrak{g}_u}) = \det(\mathrm{Ad}(g)),\ \forall g \in G. \tag{5.15}$$

Proof. We will be using repeatedly the following fact – If T is a linear map on a vector space V preserving a subspace W, and $T_{V/W}$ is the factored map on V/W, then

$$\mathrm{tr}(T|_V) = \mathrm{tr}(T|_W) + \mathrm{tr}(T_{V/W}), \tag{5.16}$$

where $T|_W$ is the restriction of T to W.

Let $X \in \mathfrak{g}$. Then $\mathrm{ad}X$ acts trivially on $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Hence,

$$\mathrm{tr}(\mathrm{ad}X|_{\mathfrak{g}}) = \mathrm{tr}(\mathrm{ad}X|_{[\mathfrak{g},\mathfrak{g}]}). \tag{5.17}$$

Let $\tilde{\mathfrak{g}}$ denote the Lie algebra of the Zariski closure of the Lie group associated with $\mathfrak{g}$ in $\mathrm{GL}(n,\mathbb{R})$. By [Che51, Chapter II, Theorem 13],

$$[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}] = [\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g}.$$

Let $\mathfrak{u}$ denote the radical of the $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]$. Then $\mathfrak{u}$ consists of nilpotent matrices; we can derive this from [Rag72, (2.5) A structure theorem]. So, $\mathfrak{u} \subseteq \mathfrak{g}_u$.

Since $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]$ and $\mathfrak{u}$ are ideals in $\tilde{\mathfrak{g}}$ and $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]/\mathfrak{u}$ is semisimple, the trace of the derivation on it corresponding to $\mathrm{ad}X$ is 0. Therefore,

$$\mathrm{tr}(\mathrm{ad}X|_{[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]}) = \mathrm{tr}(\mathrm{ad}X|_{\mathfrak{u}}). \tag{5.18}$$

Let $\mathfrak{v}$ denote the radical of $\mathfrak{g}_u$. Then, $\mathfrak{u} \subseteq \mathfrak{v}$. Since $\mathfrak{g}_u$ is a Lie subalgebra generated by nilpotent matrices, $\mathfrak{v}$ consists of nilpotent matrices. Also, $\mathfrak{g}_u$, and hence $\mathfrak{v}$ are ideals of $\mathfrak{g}$. So, by definition of $\tilde{\mathfrak{g}}$, $\mathfrak{v}$ is an ideal of $\tilde{\mathfrak{g}}$. So, $\mathfrak{v} \subseteq \mathfrak{u}$. Hence $\mathfrak{v} = \mathfrak{u}$.

As above, since $\mathfrak{g}_u/\mathfrak{v}$ is semisimple, we conclude that

$$\mathrm{tr}(\mathrm{ad}X|_{\mathfrak{g}_u}) = \mathrm{tr}(\mathrm{ad}X|_{\mathfrak{v}}). \tag{5.19}$$

Now, (5.16) follows from (5.17)-(5.19). □

Proof of Proposition 5.13. Consider the exterior product of the Adjoint representation of $\mathrm{SL}(n,\mathbb{R})$ on $\wedge^{\dim \mathfrak{g}_u}\mathfrak{sl}(n,\mathbb{R})$, and fix $p \in \wedge^{\dim \mathfrak{g}_u}\mathfrak{g}_u \setminus \{0\}$. Let

$$\begin{aligned} N &:= \{g \in \mathrm{SL}(n,\mathbb{R}) : g \cdot p = p\} \\ &= \{g \in \mathrm{SL}(n,\mathbb{R}) : \mathrm{Ad}(g)\mathfrak{g}_u = \mathfrak{g}_u,\ \det(\mathrm{Ad}(g)|_{\mathfrak{g}_u}) = 1\} \end{aligned}$$

So, N normalizes G_u. Since G is unimodular, by (5.15), $G \subseteq N$.

We have $\varphi : [0,\infty) \to G$. Let $\beta : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a correcting curve for φ, and denote $\psi(t) := \beta(t)\varphi(t)$. Now N is observable in $\mathrm{SL}(n,\mathbb{R})$ (Definition 2.2), and $\varphi([0,\infty)) \subseteq G \subseteq N$. So $H_\varphi \subseteq N$ by the minimality of the hull, Theorem 2.5,(1). Now, by Remark 2.8, we have that $\psi([0,\infty)) \subseteq H_\psi = H_\varphi$.

Assume for contradiction that ψ is not bounded modulo G_u. Let $q : N \to N/G_u$ be the quotient map. Then, by Proposition 5.12, there exists a one-parameter unipotent subgroup of $\{u(s)\}_{s\in\mathbb{R}} \subseteq N$ such that

$$q(u(s)) = \lim_{t\to\infty} q(\psi(h_{s,r}(t)))q(\psi(t))^{-1}, \forall s \in \mathbb{R},$$

and $q \circ u$ is a nontrivial unipotent one-parameter subgroup of N/G_u. Let $\beta_\infty = \lim_{t\to\infty} \beta(t)$. Then, for all $s \in \mathbb{R}$,

$$q(\beta_\infty^{-1} u(s) \beta_\infty) = \lim_{t\to\infty} q(\varphi(h_{s,r}(t)))q(\varphi(t))^{-1},$$

and since $\varphi([0,\infty)) \subseteq G \subseteq N$, we get that $\{q(\beta_\infty^{-1}u(s)\beta_\infty) : s \in \mathbb{R}\}$ is a nontrivial one-parameter subgroup of G/G_u. It then follows that $\{\beta_\infty^{-1}u(s)\beta_\infty : s \in \mathbb{R}\}$ is a nontrivial one-parameter unipotent subgroup contained in G not contained in G_u, a contradiction. Finally, since φ is bounded modulo G_u, and since $\mathrm{zcl}(G_u)$ is observable, we get that $H_\varphi \subseteq G_u$ by the minimality of the hull, Theorem 2.5,(1). □

6. Avoidance of singular sets

Suppose that $\Gamma \subseteq \mathrm{SL}(n,\mathbb{R})$ is a discrete subgroup, and μ is a probability measure on $X := \mathrm{SL}(n,\mathbb{R})/\Gamma$ which is invariant under a nontrivial one-parameter unipotent subgroup of $\mathrm{SL}(n,\mathbb{R})$. Then, by the celebrated Ratner's theorem (see [Rat91a]), every W-ergodic component of μ is homogeneous.

The measure μ that we will consider is a weak-* limit of the measures $\mu_{T,\varphi,x}$ (see (1.1)), for a non-contracting, definable polynomially bounded curve φ. By Theorem 4.5 and Proposition 5.1, μ is a probability measure, and it is invariant under a nontrivial unipotent subgroup.

As an immediate consequence of Ratner's theorem (see Theorem 6.3) if μ is not $\mathrm{SL}(n,\mathbb{R})$-invariant, then it is strictly positive on a Ratner's singular set, which is the image of a certain strictly lower dimensional algebraic subvariety of $\mathrm{SL}(n,\mathbb{R})$ on $X = \mathrm{SL}(n,\mathbb{R})/\Gamma$. To analyze this situation, we will consider a finite-dimensional representation V of $\mathrm{SL}(n,\mathbb{R})$ and represent the singular set via an algebraic subvariety, say $\mathcal{A}$, in V. The trajectory $t \mapsto \varphi(t)x$ defining μ will be represented by countably many simultaneous trajectories $t \mapsto \varphi(t)\gamma \cdot p$ $(\forall \gamma \in \Gamma)$ for a well chosen $p \in V$ with $\Gamma \cdot p$ discrete. We will show that μ cannot be positive on the singular set unless for some $\gamma \in \Gamma$, $\varphi(t)\gamma \cdot p$ converges to a point of $\mathcal{A}$ as $t \to \infty$. This leads to group-theoretic constraints.

Studying the dynamics of $\varphi(t)x$-trajectories in thin neighborhoods of a singular set in X via simultaneously analyzing countably many trajectories

of φ in a thin neighborhood of an algebraic variety in a vector space is known as the *linearization technique*. Showing that an unbounded trajectory of φ avoids remaining too close to an algebraic variety for too long is called the *avoidance of singular sets* - this is achieved via the (C,α)-good property. The remainder of the section is devoted to employing these techniques in our situation.

6.1. **Description of Ratner's singular sets.**

Definition 6.1. Let $\mathcal{H}$ be the class of all closed connected subgroups H of $\mathrm{SL}(n,\mathbb{R})$ such that $H/H\cap\Gamma$ admits an H-invariant probability measure and the closed subgroup $H_u \subseteq H$ which is generated by all unipotent one-parameter subgroups of H acts ergodically on $H/H\cap\Gamma$ concerning the H-invariant probability measure.

Theorem 6.2 ([Rat91a, Theorem 1.1])**.** *The collection $\mathcal{H}$ is countable.*

We note that if $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, then $\mathrm{zcl}(H\cap\Gamma)$ is defined over $\mathbb{Q}$ and $H = \mathrm{zcl}(H\cap\Gamma)(\mathbb{R})^0$ [Sha91, (2.3) and (3.2)] for all $H\in\mathcal{H}$, so $\mathcal{H}$ is countable.

Let W be a subgroup generated by unipotent one-parameter subgroups of $\mathrm{SL}(n,\mathbb{R})$. Let $H\in\mathcal{H}$. Define

$$\begin{aligned} N(H,W) =& \{g\in \mathrm{SL}(n,\mathbb{R}) : W\subseteq gHg^{-1}\}, \text{ and}\\ S(H,W) =& \bigcup_{H'\in\mathcal{H}, H'\subsetneq H} N(H',W). \end{aligned}$$

Note that $N(W)N(H,W)N(H) = N(H,W)$, where $N(F)$ denotes the normalizer of F in $\mathrm{SL}(n,\mathbb{R})$. Also, for all $\gamma\in\Gamma$, we have $\gamma H\gamma^{-1}\in\mathcal{H}$,

$$N(H,W)\gamma = N(\gamma^{-1}H\gamma, W) \text{ and } S(H,W)\gamma = S(\gamma^{-1}H\gamma, W).$$

Let $\pi : \mathrm{SL}(n,\mathbb{R}) \to X = \mathrm{SL}(n,\mathbb{R})/\Gamma$ denote the quotient map. Consider the refined singular set

$$T_H(W) = \pi(N(H,W)\setminus S(H,W)) = \pi(N(H,W))\setminus\pi(S(H,W)). \tag{6.1}$$

for the second equality see [MS95, Lemma 2.4]. Also, $T_H(W) = T_{\gamma H\gamma^{-1}}(W)$ for all $\gamma\in\Gamma$.

Theorem 6.3. (Ratner [Rat91a]) [MS95, Theorem 2.2] *Let ν be a W-invariant Borel probability measure on X. For each $H\in\mathcal{H}$, let ν_H denote the restriction of ν on $T_H(W)$, which is W-invariant. Then ν decomposes as:*

$$\nu = \sum_{H\in\mathcal{H}^*} \nu_H,$$

where $\mathcal{H}^*\subseteq\mathcal{H}$ is a countable set of subgroups, each of which is a representative of a distinct Γ-conjugacy class.

6.2. **O-minimal curves in representations – a dichotomy theorem.** For $H \in \mathcal{H}$, consider

$$V_H := \bigwedge^{\dim(H)} \mathfrak{sl}(n,\mathbb{R}) \text{ and } p_H \in \bigwedge^{\dim(H)} \mathfrak{h} \setminus \{0\}, \tag{6.2}$$

where $\mathfrak{h}$ denotes the Lie algebra of H. Consider the action of $\mathrm{SL}(n,\mathbb{R})$ on V_H induced by its Adjoint representation its Lie algebra; that is, $g \cdot (\bigwedge_{i=1}^d x_i) := \bigwedge_{i=1}^d \mathrm{Ad}_g(x_i)$ and extended linearly. We denote by $\eta_H : \mathrm{SL}(n,\mathbb{R}) \to V_H$ the orbit map

$$\eta_H(g) = g \cdot p_H, \forall g \in \mathrm{SL}(n,\mathbb{R}). \tag{6.3}$$

The notation η_H is used only for denoting preimages under the orbit map. We note some basic observations.

Theorem 6.4. *Let $H \in \mathcal{H}$. Then the following hold:*

(1) [DM93, Proof of Proposition 3.2] *Let $\mathfrak{w}$ be the Lie algebra of W, and consider the linear subspace*

$$A_H := \{v \in V_H : v \wedge w = 0, \forall w \in \mathfrak{w}\}. \tag{6.4}$$

Then $\eta_H^{-1}(A_H) = N(H,W)$.

(2) [DM93, Theorem 3.4]

$$\Gamma \cdot p_H \text{ is closed, and hence discrete, in } V_H.$$

For example, if $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, then $\mathfrak{h}$ is defined over $\mathbb{Q}$, so we can pick $p_H \in \wedge^d \mathfrak{h}(\mathbb{Z}) \setminus \{0\}$, and hence, $\Gamma \cdot p_H \subseteq V_H(\mathbb{Z})$.

Next, we will provide a version of [MS95, Proposition 3.4] for definable curves. Our proof is based on the linearization technique developed by Dani and Margulis [DM93, Proof of Theorem 1]. The result can be considered an analogue of Theorem 4.2. In what follows, $\mathscr{S}_{\mathrm{pbd}}$ denotes a polynomially bounded o-minimal structure.

Theorem 6.5. *Let $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a $\mathscr{S}_{\mathrm{pbd}}$-definable, continuous, unbounded, non-contracting curve. Let $\beta : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ be a correcting curve of φ and let H_φ be the hull of φ such that $\beta(t)\varphi(t) \subseteq H_\varphi$ for all t, see Definition 2.6. Fix $H \in \mathcal{H}$. Let $\mathrm{K}_1 \subseteq A_H$ be a compact set. Then, given $x \in X$, one of the following holds*:

(1) There exists $w \in \pi^{-1}(x) \cdot p_H \cap \mathrm{K}_1$ such that

$$H_\varphi \cdot w = \{w\}. \tag{6.5}$$

(2) Given $0 < \epsilon < 1$, there exists a closed set $\mathcal{S} \subseteq \pi(S(H,W))$ such that given a compact set $K \subseteq X \setminus \mathcal{S}$, there exist a compact neighborhood Ψ of K_1 in V_H and a $T_x > 0$ such that for all $T > T_x$,

$$|\{t \in [0,T] : \beta(t)\varphi(t)x \in K \cap \pi(\eta_H^{-1}(\Psi))\}| \leq \epsilon T. \tag{6.6}$$

We now recall some observations and results that will be used in the proof. Let $N(H)$ be the normalizer of H in $\mathrm{SL}(n,\mathbb{R})$. Note that,

$$g \cdot p_H = \det(\mathrm{Ad}_g|_{\mathfrak{h}}) p_H,\ \forall g \in N(H). \tag{6.7}$$

Set $N_\Gamma := N(H) \cap \Gamma$. By [DM93, Lemma 3.1], $N_\Gamma \subseteq \{\pm p_H\}$. If $N_\Gamma = \{\pm p_H\}$, let $\tilde{V}_H := V_H/\{\pm 1\}$, the space V_H modulo the equivalence relation defined by identifying $v \in V_H$ with $-v$. If $N_\Gamma \cdot p_H = \{p_H\}$, let $\tilde{V}_H := V_H$.

For any $g \in \mathrm{SL}(n,\mathbb{R})$ and $y = \pi(g) \in X$, we treat $\pi^{-1}(y) \cdot \tilde{p}_H = g\Gamma \cdot \tilde{p}_H$ as the discrete *set of representatives* of y in $\tilde{V}_H$, where $\tilde{p}_H$ denotes the image of p_H in $\tilde{V}_H$. We need the following result on the *uniqueness of representatives* in thin neighborhoods.

Proposition 6.6. [MS95, Proposition 3.2] *Let D be a compact subset of A_H, and consider*

$$S(D) = \{g \in \eta_H^{-1}(D) : g\gamma \in \eta_H^{-1}(D) \text{ for some } \gamma \in \Gamma \setminus N_\Gamma\} \subseteq \mathrm{SL}(n,\mathbb{R}).$$

Then the following hold:

1. *$S(D) \subseteq S(H,W)$.*
2. *$\pi(S(D))$ is closed in X.*
3. *Suppose that $K \subseteq X \setminus \pi(S(D))$ is compact. Then, there is a compact neighborhood Φ of D in V_H such that any $y \in K$ has at most one representative in $\tilde{\Phi}$, where $\tilde{\Phi}$ is the image of Φ in $\tilde{V}_H$.*

   That is, for all $y \in \pi(\eta_H^{-1}(\Phi)) \cap K$, the set $\pi^{-1}(y).\tilde{p}_H \cap \tilde{\Phi}$ consists of a single element.

Proof of Theorem 6.5. We assume for simplicity that $N_\Gamma \cdot p_H = \{p_H\}$, namely $\tilde{V}_H = V_H$. The case for which $N_\Gamma \cdot p_H = \{\pm p_H\}$ can be treated in essentially the same way, and is left to the readers.

By choosing a basis for V_H, we identify $V_H \cong \mathbb{R}^m$ for some m. We will denote by $\psi(t) \in \mathrm{GL}(m,\mathbb{R})$ the action of $\beta(t)\varphi(t)$ on V_H. We will apply Proposition 3.12 for $\psi(t)$. Since A_H is a linear subspace, $A_H = \{Q = 0\}$ for a polynomial map $Q : V_H \to \mathbb{R}$, see Remark 3.14. Let $K_1 \subseteq A_H$ and let $0 < \epsilon < 1$ be given. Let $\mathrm{K}_2 \subseteq A_H$ be a compact set that is given by Proposition 3.12 with $\mathrm{K}_1 \subseteq \mathrm{K}_2$.

Choices of $\mathcal{S}$ and Φ for injective property. For applying Proposition 6.6, we use K_2 in place of D, and let $\mathcal{S} := \pi(S(\mathrm{K}_2))$. Let $K \subseteq X \setminus \mathcal{S}$ be a compact set, and let $\Phi \subseteq V_H$ be a compact neighborhood of K_2 which satisfies the outcome of Proposition 6.6,(3). As a consequence, for any $t \geq 0$, $w_1, w_2 \in \Gamma \cdot p_H$, and $g \in \mathrm{SL}(n,\mathbb{R})$, if $g\Gamma \in K$ and $g \cdot w_i \in \Phi$ for $i = 1, 2$, then $w_1 = w_2$; let us call this the *injective property of K in Φ*.

Choice of Ψ using (C,α)-good property. We take $T_0' > 0$ to satisfy the outcomes of Propositions 1.13 and 3.12 for ψ. By Proposition 3.12, we choose a compact neighborhood Ψ of K_1 in V_H such that $\Psi \subseteq \mathring{\Phi}$ and for any $v \in V$

and any closed interval $[a,b] \subseteq [T_0', \infty)$, if $\psi([a,b])v \subseteq \Phi$ and $\psi(b)v \notin \mathring{\Phi}$, then

$$|\{s \in [a,b] : \psi(s)v \in \Psi\}| \leq \epsilon(b-a). \tag{6.8}$$

It may be noted that in view of (3.30) and (3.31), there exist $R_2 > 0$ and $\epsilon_1 > 0$ such that

$$\Psi \subseteq \{v \in V_H : \|v\| < R_2 \text{ and } |Q(v)| < \epsilon_1\} \subseteq \mathring{\Phi}. \tag{6.9}$$

Modifying Ψ for dichotomy. Let $x \in X$ be given. Since $\pi^{-1}(x) \cdot p_H \cap \Psi$ is finite, by replacing Ψ with a smaller compact neighborhood of K_1, we may assume that $\pi^{-1}(x) \cdot p_H \cap \Psi = \pi^{-1}(x) \cdot p_H \cap K_1$.

By Theorem 2.5,(3), the following two mutually exclusive cases exhaust all possibilities.

Case 1 – *There exists $w \in \pi^{-1}(x) \cdot p_H \cap \Psi$ such that $H_\varphi \cdot w = \{w\}$.*
Case 2 – *For all $w \in \pi^{-1}(x) \cdot p_H \cap \Psi$, $\lim_{s\to\infty} \psi(s)w = \infty$.*

If Case 1 holds, then by our choice of Ψ, the first conclusion of the theorem holds. Therefore, to complete the proof, we assume that Case 2 holds.

Consider

$$J = \{t \geq T_0' : \beta(t)\varphi(t)x \in \pi(\eta_H^{-1}(\Psi)) \cap K\}. \tag{6.10}$$

Then J is closed. Because $\pi(\eta_H^{-1}(\Psi))$ is closed in X. To show this, suppose $g_i \to g$ in $\mathrm{SL}(n,\mathbb{R})$ and $\gamma_i \in \Gamma$ are such that $g_i\gamma_i \cdot p_H \in \Psi$. Since Ψ is compact and $\Gamma \cdot p_H$ is discrete, we can pick $\gamma \in \Gamma$ such that $\gamma_i \cdot p_H = \gamma \cdot p_H$ for infinitely many i's. Therefore $g\gamma \cdot p_H \in \Psi$.

Then, to prove (6.6), it is enough to show that there exists $T_x \geq T_0'$ depending on x such that

$$|J \cap [T_0', T]| \leq 2\epsilon T, \ \forall\, T \geq T_x. \tag{6.11}$$

Without loss of generality, we may assume that J is unbounded.

Choice of a representative of x in for every $t \in J$. For every $t \in J$, there exists a representative of x, say $w \in \pi^{-1}(x) \cdot p_H$, such that $\psi(t)w \in \Psi \subseteq \Phi$. Since $\beta(t)\varphi(t)x \in K$, by the injective property of K in Φ, w is unique, and we denote it by w_t.

Let $t \in J$. We note that

$$\lim_{s\to\infty} \psi(s)w_t = \infty,$$

because otherwise w_t is fixed by H_φ due to (3) of Theorem 2.5, and since $\psi(t) \subseteq H_\phi$, we have $w_t \in \Psi \cap \pi^{-1}(x) \cdot p_H$, contradicting the Case 2.

Therefore, $\{s \geq T_0' : \psi(s)w_t \in \mathring{\Phi}\}$ is a bounded open subset of $[T_0', \infty)$ containing t. Let $U(t)$ be its connected component containing t, and denote

$$t^+ := \sup U(t) < \infty.$$

Then

$$\psi([t,t^+))w_t \subseteq \mathring{\Phi} \text{ and } \psi(t^+)w_t \in \Phi \setminus \mathring{\Phi}. \tag{6.12}$$

Bound on $|J \cap [t, t^+]|$. For any $s \in J \cap [t, t^+]$, we have $\beta(s)\varphi(s)x \in K$, $\beta(s)\varphi(s) \cdot w_s \in \Psi \subseteq \mathring{\Phi}$, and $\beta(s)\varphi(s) \cdot w_t \in \Phi$, and hence by the injectivity of K in Φ, $w_s = w_t$. Then,

$$\begin{aligned} J \cap [t, t^+] &= \{s \in [t, t^+] \cap J : \psi(s)w_s \in \Psi\} \\ &= \{s \in [t, t^+] \cap J : \psi(s)w_t \in \Psi\}. \end{aligned} \tag{6.13}$$

By (6.8), (6.12), and (6.13),

$$|J \cap [t, t^+]| \leq \big|\{s \in [t, t^+] : \psi(s)w_t \in \Psi\}\big| \leq \epsilon(t^+ - t). \tag{6.14}$$

Choosing a sequence $t_i \in J$. For each $i \in \mathbb{N}$, we define $t_i \in J$ inductively as follows: Let $t_1 = \inf J \in J$. Let $i \in \mathbb{N}$ be such that t_i is defined. Then we let $t_{i+1} = \inf J \cap (t_i^+, \infty)$. Since J is closed, $t_i \in J$.

By our construction, $t_i < t_i^+ < t_{i+1}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$. Since

$$J \cap [T_0', t_i^+] = \cup_{j=1}^{i} J \cap [t_j, t_j^+),$$

by (6.14),

$$|J \cap [T_0', t_i^+]| \leq \epsilon(t_i^+ - T_0'). \tag{6.15}$$

We will bound $|J \cap [t_{i+1}, T]|$ using Proposition 1.13. We need the following:

Claim. As $i \to \infty$, $\|w_{t_i}\| \to \infty$ and $t_i \to \infty$.

For $w \in \pi^{-1}(x) \cdot p_H$, the maps $s \mapsto \|\psi(s)w\|$ and $s \mapsto Q(\psi(s)w)$ are definable and for $s \geq T_0'$. Let

$$S_w := \{s \geq T_0' : \|\psi(s)w\| = R_2 \text{ or } |Q(\psi(s)w)| = \epsilon_1\}.$$

Since S_w is a definable subset of $\mathbb{R}$, it has only finitely many connected components, see Theorem B.6. For any $i \in \mathbb{N}$, if $w = w_{t_i}$, then (t_i, t_i^+) contains at least one connected component of S_w; indeed, since $\psi(t_i)w_{t_i} \in \Psi$ and $\psi(t_i^+)w_{t_i} \notin \mathring{\Phi}$, by (6.9), $t_i, t_i^+ \notin S_w$ and $(t_i, t_i^+) \cap S_w \neq \emptyset$. Therefore, since $\{t_i\}_{i=1}^{\infty}$ is an infinite sequence, the set $\{i \in \mathbb{N} : w = w_{t_i}\}$ is finite.

So, $\{w_{t_i} : i \in \mathbb{N}\}$ is infinite. Since $\pi^{-1}(x) \cdot p_H$ is discrete in V_H, we get $\lim_{i\to\infty} \|w_{t_i}\| = \infty$. Since $\psi(t_i)w_{t_i} \in \Psi$ for all i and the map $s \mapsto \|\psi(s)^{-1}\|$ is continuous on $[T_0', \infty)$, we conclude that $t_i \to \infty$. This proves the claim.

By Proposition 1.13, and recalling (1.6), given $\epsilon > 0$ we can pick $M_0 > 0$ such for any $w \in V_H$, if $\|\psi(T_0')w\| \geq M_0$, then

$$\big|\{s \in [T_0', T] : \psi(s)w \in \Psi\}\big| \leq \epsilon(T - T_0'). \tag{6.16}$$

Due to the above claim, let $i_x \in \mathbb{N}$ be such that $\|w_{t_i}\| \geq \|\psi(T_0')^{-1}\| \cdot M_0$ for all $i \geq i_x$. Set $T_x = t_{i_x}$. Let $T \geq T_x$. Since $t_j \to \infty$ as $j \to \infty$, $i := \max\{j : t_j \leq T\} < \infty$. Then $i \geq i_x$, so $\|\psi(T_0')w_{t_i}\| \geq M_0$. By (6.16),

$$|J \cap [t_i, T]| \leq |\{s \in [t_i, T] : \psi(s)w_{t_i} \in \Psi\}| \leq \epsilon(T - T_0'). \tag{6.17}$$

By combining (6.15) and (6.17), we obtain (6.11). □

6.3. **Proof of Theorem 1.7.** Let $G := \mathbf{G}(\mathbb{R})$. Suppose that $\mathcal{G} \subseteq G$ is a connected, closed Lie subgroup with a lattice $\Gamma \subseteq \mathcal{G}$. Let $\varphi : [0, \infty) \to G$ be a $\mathscr{S}_{\text{pbd}}$-definable, continuous, unbounded, non-contracting curve for G such that $\varphi([0, \infty)) \subseteq \mathcal{G}$. Let H_φ be the hull of φ in G and let β be a definable correcting curve in G, so that $\beta\varphi \subseteq H_\varphi$, see Definition 2.6. Then, by Proposition 5.13, $H_\varphi \subseteq \mathcal{G}$ and $\beta([0, \infty)) \subseteq \mathcal{G}$.

We choose an injective algebraic map $\iota : G \hookrightarrow \mathrm{SL}(n, \mathbb{R})$ for some $n \geq 2$. Note that $\iota(\Gamma)$ is a discrete subgroup of $\mathrm{SL}(n, \mathbb{R})$. Importantly, $\iota \circ \varphi$ is non-contracting in $\mathrm{SL}(n, \mathbb{R})$ by Lemma 2.17, and $\iota(H_\varphi) = H_{\iota\circ\varphi}$ by Lemma 2.10. By abuse of notations, in the following we replace φ with $\iota \circ \varphi$, replace Γ with $\iota(\Gamma)$ and replace G with $\iota(G)$.

Let $x_0 \in \mathcal{G}/\Gamma \hookrightarrow \mathrm{SL}(n, \mathbb{R})/\Gamma$, and fix a limiting measure μ as $T \to \infty$ of $\mu_{T,\beta\varphi,x_0}$ (defined in (1.1)), with respect to the weak-$*$ topology on the space of probability measures on $\mathrm{SL}(n, \mathbb{R})/\Gamma$. We fix a sequence $\{T_i\}$ such that $T_i \to \infty$ and

$$\text{weak*-}\lim_{i\to\infty} \mu_{T_i,\beta\varphi,x_0} = \mu. \tag{6.18}$$

By Theorem 4.5, μ is a probability measure.

Let W be the subgroup of $\mathrm{SL}(n, \mathbb{R})$ generated by all unipotent one-parameter subgroups under which the limiting measure μ is invariant. By Proposition 5.1, $\dim(W) > 0$. Using Theorem 6.3, there exists $H \in \mathcal{H}$ such that

$$\mu(\pi(N(H, W))) > 0 \text{ and } \mu(\pi(S(H, W))) = 0. \tag{6.19}$$

Here $H \in \mathcal{H}$ is of smallest dimension such that $\mu(\pi(N(H, W))) > 0$.

In view of (6.7), we define

$$\begin{aligned} N^1(H) := \eta_H^{-1}(p_H) &= \{g \in \mathrm{SL}(n, \mathbb{R}) : g \cdot p_H = p_H\} \\ &= \{g \in N(H) : \det((\mathrm{Ad}g)|_{\mathfrak{h}}) = 1\}, \end{aligned} \tag{6.20}$$

where p_H and η_H are as in (6.3). Since $H \cap \Gamma$ is a lattice in H, H is unimodular (see [Rag72, Remark 1.9]). So, $H \subseteq N^1(H)$.

The remainder of the proof proceeds as follows. In Lemma 6.7, we show that there exists an element $g_0 \in \mathrm{SL}(n, \mathbb{R})$ such that $g_0\Gamma = x_0$ and

$$\{\beta(t)\varphi(t)g_0\}_{t\geq 0} \subseteq g_0 N^1(H).$$

We first note that $\Gamma N^1(H) = \eta_H^{-1}(\Gamma \cdot p_H)$ is closed, as $\Gamma \cdot p_H$ is discrete in V_H (see Theorem 6.4, (2)). Consequently, the quotient $N^1(H)\Gamma/\Gamma \subseteq X$ is also closed.

By (6.18), it follows that μ is supported on $g_0 N^1(H)x_0$. One may view $g_0 N^1(H)\Gamma/\Gamma$ as foliated by orbits of $g_0 H g_0^{-1}$, and we will show that μ is invariant under $g_0 H g_0^{-1}$. The key step is, to prove that μ is actually supported on the single orbit of $g_0 H g_0^{-1}$ passing through x_0, a fact established in Corollary 6.9. This will then immediately imply our main result, Theorem 1.7, as it shows that

$$\overline{H_\varphi x_0} = g_0 H g_0^{-1} x_0,$$

and that $\beta\varphi$ is equidistributed on this orbit.

Lemma 6.7. *Fix $H \in \mathcal{H}$ such that (6.19) holds. Then, there exists $g_0 \in \mathrm{SL}(n,\mathbb{R})$ such that $x_0 = \pi(g_0) := g_0\Gamma$ and*

$$\left\{g_0^{-1}\beta(t)\varphi(t)g_0\right\}_{t\geq 0} \subseteq N^1(H). \tag{6.21}$$

Importantly,

$$\nu := (g_0^{-1})_*\mu, \tag{6.22}$$

is supported on $\pi(N^1(H))$, ν is H-invariant, and the largest group generated by unipotent one-parameter subgroups preserving ν is H_u.

Proof. In view of (6.1), due to (6.19), there is a compact subset $C \subseteq N(H,W) \setminus S(H,W)$ such that $\mu(\pi(C)) = \alpha$ for some $\alpha > 0$, and $\pi(C) \cap \pi(S(H,W)) = \varnothing$. Consider the compact subset $\mathrm{K}_1 \subseteq A_H$ given by $\mathrm{K}_1 := C \cdot p_H$. We will now apply Theorem 6.5 to $x_0 \in X$.

We show that outcome (2) of Theorem 6.5 cannot hold. Let $\epsilon := \frac{\alpha}{2}$. Denote by $\mathcal{S} \subseteq \pi(S(H,W))$ the closed subset given by outcome (2) of Theorem 6.5. Let $K \subseteq X \setminus \mathcal{S}$ be an open neighborhood of $\pi(C)$, which exists as $\pi(C)$ is compact and $\mathcal{S}$ is closed. Also, fix an arbitrary compact neighborhood U of K_1 in V_H. Then, $K \cap \pi(\eta_H^{-1}(U))$ is an open neighborhood of $\pi(C)$. By (6.18), for all i large enough,

$$|\{t \in [0,T_i] : \beta(t)\varphi(t)x_0 \in K \cap \pi(\eta_H^{-1}(U))\}| \geq \epsilon T_i,$$

which contradicts outcome (2) of Theorem 6.5.

Therefore, the first outcome (1) of Theorem 6.5 must hold. That is, we can pick $g_0 \in \pi^{-1}(x_0)$ such that $g_0 \cdot p_H \in \mathrm{K}_1 \subseteq A_H$ and such that:

$$\beta(t)\varphi(t)g_0 \cdot p_H = g_0 \cdot p_H, \text{ for } t \geq 0. \tag{6.23}$$

Namely, $\beta(t)\varphi(t)g_0 \in g_0N^1(H)$ for all $t \geq 0$. Since $\pi(N^1(H))$ is closed, we obtain that μ is supported on a subset of $\pi(g_0N^1(H))$. In fact, we claim that $\mu(\pi(g_0N^1(H))) = 1$

To see this, observe that since $g_0 \cdot p_H \in A_H$, then $g_0 \in N(H,W)$ by Theorem 6.4,(1). Moreover, $g_0N^1(H) \subseteq N(H,W)$, since $N(H,W)$ is right-$N^1(H)$ invariant.

In particular, we conclude that $\mu(\pi(g_0N^1(H) \setminus S(H,W))) = 1$ (see (6.19)). Therefore, by [MS95, Lemma 2.4], almost every W-ergodic component of μ is invariant under $gHg^{-1} = g_0Hg_0^{-1}$ for some $g \in g_0N^1(H) \setminus S(H,W)$. Therefore μ is $g_0Hg_0^{-1}$-invariant. Then, $\nu = (g_0^{-1})_*\mu$ is H-invariant, and especially, H_u preserves ν.

Finally, we claim that $H_u = g_0^{-1}Wg_0 \subseteq H$. To see this equality, recall that $g_0 \in N(H,W)$, so that $g_0^{-1}Wg_0 \subseteq H$, and recall that W is chosen to be the subgroup of $\mathrm{SL}(n,\mathbb{R})$ generated by all unipotent one-parameter subgroups preserving μ. □

We will use the following notations: $\psi(t) := g_0^{-1}\beta(t)\varphi(t)g_0$, for $t \geq 0$, $N := N^1(H)$, $\Delta := N^1(H) \cap \Gamma$ and $q : N \to N/H$ denotes the natural quotient map.

Consider $\bar{N} := q(N)$, $\bar{\Delta} := q(\Delta)$. Since $H\Delta = \Delta H$ is closed, we have that $\bar{\Delta} \leq \bar{N}$ is discrete (since ΔH is closed, then $q(\Delta) = \Delta H/H \cong \Delta/\Delta \cap H$). We let $\bar{q} : N/\Delta \to \bar{N}/\bar{\Delta}$ be the natural map. Notice that the fibers of $\bar{q}$ are H-orbits. Namely,

$$\bar{q}^{-1}(\bar{n}\bar{\Delta}) = nH\Delta/\Delta = Hn\Delta/\Delta.$$

Since ν is a probability measure on $N/\Delta \cong \pi(N^1(H))$, it follows that $\bar{\nu} := \bar{q}_*\nu$ is a probability measure on $\bar{N}/\bar{\Delta}$.

Lemma 6.8. *Suppose that $\psi(t)$ is unbounded in N/H_u. Then there exists a unipotent one-parameter subgroup $U \leq N$ such that U is not contained in H, the measure $\bar{\nu}$ is invariant under $q(U)$, and $q(U)$ is a nontrivial one-parameter subgroup.*

Moreover, ν is U-invariant.

Proof. We note that the measure push-forward map q_* is continuous, and in particular,

$$\bar{\nu}(f) = \lim_{i\to\infty} \frac{1}{T_i} \int_0^{T_i} f(q(\psi(t))\bar{\Delta})dt, \forall f \in C_c(\bar{N}/\bar{\Delta}).$$

Thus, we may conclude the statement by applying Proposition 5.12, and Lemma 5.5 (see the proof of Lemma 5.6. The only difference here is that N/H is not a subgroup of $\mathrm{GL}(n,\mathbb{R})$).

By the ergodic decomposition for the H-action on N/Δ (see e.g. [EW11, Theorem 8.20]), for all $f \in C_c(N/\Delta)$,

(6.24)
$$\int_{N/\Delta} f(n\Delta)d\nu(n\Delta) = \int_{\bar{N}/\bar{\Delta}} \int_{H\Delta/\Delta} f(nh\Delta)d\mu_{H\Delta/\Delta}(h\Delta)d\bar{\nu}(q(n)\bar{\Delta}),$$

where $\mu_{H\Delta/\Delta}$ denotes the H-invariant probability measure on $H\Delta/\Delta$.

Therefore, since $\bar{\nu}$ is $q(U)$-invariant, ν is U-invariant. □

Corollary 6.9. *It holds that $\psi(t) \in H_u$ for all $t \geq 0$, and ν is the H-invariant probability on $H\Delta/\Delta \cong H\Gamma/\Gamma$. Moreover, H is the smallest subgroup such that $g_0^{-1}H_\varphi g_0 \subseteq H$ and that the orbit $H\Delta/\Delta$ is closed.*

Proof. We first show that $\psi(t)$ is bounded in N/H_u. Suppose $\psi(t)$ is unbounded in N/H_u. Then by Lemma 6.8, ν is invariant by the group generated by U and H, which strictly includes H_u. This is a contradiction, because H_u is the maximal subgroup generated by unipotent one-parameter subgroups preserving ν.

Now, since the image of $\psi(t) = g_0^{-1}\beta(t)\varphi(t)g_0$ is boundned in $\mathrm{SL}(n,\mathbb{R})/H_u$, we get that the image of $\beta(t)\varphi(t)$ is bounded in $\mathrm{SL}(n,\mathbb{R})/g_0H_ug_0^{-1}$. Thus, since $\mathrm{zcl}(H_u)$ is observable (see Lemma 2.3), by minimality of the hull (see Theorem 2.5), $H_\varphi \subseteq g_0H_ug_0^{-1}$, and in particular $\beta(t)\varphi(t) \in g_0H_ug_0^{-1}$ for all t. Namely, $\psi(t) \in g_0^{-1}H_\varphi g_0 \subseteq H_u$ for all $t \geq 0$. In particular, $\bar{\nu}$ is the Dirac measure at the identity coset. Therefore $\nu = \mu_{H\Delta/\Delta}$ by (6.24). In particular

$\{\psi(t)\Delta\}_{t\geq 0}$ is dense in $H\Delta/\Delta$. In particular $g_0^{-1}H_\varphi g_0\Delta$ is dense in $H\Delta/\Delta$, which finishes the proof.

□

Theorem 1.7 follows directly from the above corollary.

7. Curves with a largest possible hull

The main goal of this section is to prove Proposition 1.9 on the equidistribution of the curve (1.5). As an overview, the key idea is that the curve (1.5) will be strongly non-contracting, in the sense that it takes nonzero wedges to ∞, Lemma 7.2. The following result, of independent interest, implies the hull of (1.5) is semi-simple and acts irreducibly on $\mathbb{R}^n$. Finally, a key classification of [SY24, Theorem A.7] will allow to prove that the hull equals to the full group $\mathrm{SL}(n,\mathbb{R})$.

Lemma 7.1. *Suppose that $\varphi : [0,\infty) \to \mathrm{SL}(n,\mathbb{R})$ is a continuous curve definable in an o-minimal structure such that φ has the following divergence property:*

$$\lim_{t\to\infty} \varphi(t)\cdot(v_1\wedge\cdots\wedge v_k) = \infty, \tag{7.1}$$

for all $1\leq k\leq n-1$ and all linearly independent vectors $v_1,\dots,v_k\in\mathbb{R}^n$. Then, the hull H_φ of φ is a connected semi-simple with no compact factors, and the action of H_φ on $\mathbb{R}^n$ is irreducible.

Proof. Since $\mathrm{zcl}(H_\varphi)$ is observable, we have a finite-dimensional rational representation V of $\mathrm{SL}(n,\mathbb{R})$ and $v\in V$ such that $\mathrm{zcl}(H_\varphi)$ is the isotropy group of v. Note that $\varphi(t)\cdot v, t\geq 0$ is bounded since φ is bounded in $\mathrm{SL}(n,\mathbb{R})/H_\varphi$. Since we assume (7.1), it follows by Lemma 2.1 that the orbit $\mathrm{SL}(n,\mathbb{R})\cdot v$ is Zariski closed. Then, Matsushima criterion [Mat60] implies that $\mathrm{zcl}(H_\varphi)$ is reductive. Since H_φ is connected and generated by one-parameter unipotent subgroups, it follows that H_φ is a connected semi-simple group with no compact factors. Finally, we prove that H_φ acts on $\mathbb{R}^n$ irreducibly. Suppose for contradiction that $\mathbb{R}^n = V_1\oplus V_2$ where both V_1 and V_2 are nontrivial subspaces invariant under H_φ. Let $v_1,\dots,v_k$ be a basis for V_1. Here $1\leq k<n$ because V_2 is not trivial. Since semi-simple groups have no nontrivial characters, we obtain that $H_\varphi\cdot(v_1\wedge\cdots\wedge v_k) = v_1\wedge\cdots\wedge v_k$. Since $\varphi([0,\infty))$ is bounded in $\mathrm{SL}(n,\mathbb{R})/H_\varphi$, we get $\varphi(t)\cdot(v_1\wedge\cdots\wedge v_k)$ is bounded as $t\to\infty$. This contradicts (7.1). □

Let φ be the curve given by (1.5) satisfying the conditions (1), (2), and (3) as stated in Proposition 1.9. We now verify that φ satisfies a divergence property, which is stronger than (7.1).

Lemma 7.2. *For any $1\leq k\leq n$, and $0\neq v\in\bigwedge^k\mathbb{R}^{n+1}$, $\lim_{t\to\infty}\varphi(t)\cdot v\to\infty$ as $t\to\infty$.*

Proof. We denote by $e_0, e_1, \ldots, e_n$ the canonical basis vectors of $\mathbb{R}^{n+1}$, where e_i denotes the vectors whose coordinates equal 0 besides the $(i+1)$th position in which 1 appears. Fix an integer $1 \leq k \leq n$ and let $\mathcal{I}$ be the collection of all subsets of $\{0, \ldots, n\}$ of cardinality k. Let φ be the curve given by (1.5). For any $I := \{i_1, \ldots, i_k\} \in \mathcal{I}$, where $i_1 < i_2 < \cdots < i_k$, let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. There are two cases for $\varphi(t) \cdot e_I$:

If $i_1 = 0$, then

$$\begin{aligned}\varphi(t) \cdot e_I =& f_0(t)e_0 \wedge [f_{i_2}(t)e_0 + h_{i_2}(t)^{-1}e_{i_2}] \wedge \cdots \wedge [f_{i_k}(t)e_0 + h_{i_k}(t)^{-1}e_{i_k}] \\ =& f_0(t) \left(\prod_{l=2}^{k} h_{i_l}(t)^{-1} \right) e_I; \end{aligned} \tag{7.2}$$

and if $i_1 \geq 1$, then

$$\begin{aligned}\varphi(t) \cdot e_I =& [f_{i_1}(t)e_0 + h_{i_1}(t)^{-1}e_{i_1}] \wedge \cdots \wedge [f_{i_k}(t)e_0 + h_{i_k}(t)^{-1}e_{i_k}] \\ =& \sum_{l=1}^{k} (-1)^{l-1} f_{i_l}(t) \left(\prod_{j \in \{1,\ldots,k\} \setminus \{l\}} h_{i_j}(t)^{-1} \right) e_{\{0\} \cup I \setminus \{i_l\}} \\ &+ \prod_{j \in \{1,\ldots,k\}} h_{i_j}(t)^{-1} e_I. \end{aligned} \tag{7.3}$$

Let $\mathbf{v} \in \bigwedge^k \mathbb{R}^{n+1} \setminus \{0\}$. Then $\mathbf{v} = \sum_{I \in \mathcal{I}} c_I e_I$, where $c_I \in \mathbb{R}$. Then,

$$\varphi(t) \cdot \mathbf{v} = \sum_{I \in \mathcal{I}} C_I(t) e_I, \tag{7.4}$$

where $C_I(t) \in \mathbb{R}$. The coefficients of our interest are $C_I(t)$ such that $0 \in I$. For any $1 \leq i_2 < \cdots < i_k \leq n$, and $J = \{i_2, \ldots, i_k\}$,

$$C_{\{0\} \cup J}(t) = \left(\prod_{j \in J} h_j(t)^{-1} \right) \sum_{l \in \{0,1,\ldots,k\} \setminus J} f_l(t) c_{J \cup \{l\}} (-1)^{j_l}, \tag{7.5}$$

where $j_l = \min\{j \in \{2, \ldots, k\} : i_j > l\}$.

Let $\mathcal{I}_{\mathbf{v}} = \{I \in \mathcal{I} : c_I \neq 0\} \neq \varnothing$, as $\mathbf{v} \neq 0$. Let $i_1 = \min \cup \mathcal{I}_{\mathbf{v}}$. We pick $I \subseteq \mathcal{I}_{\mathbf{v}}$ such that $i_i \in I$. Let $J = I \setminus \{i_1\}$. Then, $c_{J \cup \{l\}} = 0$ for every $l < i_1$. Therefore, by (7.5)

$$C_{\{0\} \cup J}(t) = \left(\prod_{j \in J} h_j(t) \right)^{-1} \left(f_{i_1} c_I + \sum_{l \in \{i_1+1,\ldots,k\} \setminus J} f_l(t) c_{J \cup \{l\}} (-1)^{j_l} \right). \tag{7.6}$$

By condition (1) of Proposition 1.9, $\sum_{i=1}^{n} \deg h_i = n$ and $\deg h_1 \geq \cdots \geq \deg h_n$. Therefore

$$\sum_{i=1}^{j} \deg h_i \geq j \tag{7.7}$$

for each $1 \leq j \leq n$. Since $1 \leq k \leq n$ and $\deg h_n > 0$, we get

$$\deg h_{i_2} + \cdots + \deg h_{i_k} \leq \deg h_1 + \cdots + \deg h_{k-1} \leq n - \deg h_n < n, \tag{7.8}$$

and if $i_1 \geq 1$, then by (7.7),

$$\deg h_{i_2} + \cdots + \deg h_{i_k} \leq n - (\deg h_1 + \cdots + \deg h_{i_1}) \leq n - i_1. \tag{7.9}$$

Since $c_I \neq 0$, by our assumptions (2) and (3) of Proposition 1.9,

$$\deg \left(c_I f_{i_1} + \sum_{l \in \{i_1+1,\ldots,k\} \setminus J} (-1)^{j_l} c_{J \cup \{l\}} f_l(t) \right) \begin{array}{ll} \geq n & \text{if } i_1 = 0 \\ > n - i_1 & \text{if } i_1 \geq 1. \end{array} \tag{7.10}$$

By combining (7.6), (7.8), (7.9) and (7.10), we conclude that $\deg C_{\{0\}\cup J}(t) > 0$. Hence, the conclusion of the lemma follows from (7.4). □

Lemma 7.3. *Let $\{\rho(s) : s \in \mathbb{R}\}$ be the P.S. group of φ (Definition 5.3). Then, there exists $v = (v_1, \ldots, v_n) \in \mathbb{R}^n \setminus \{0\}$, such that*

$$\rho(s) := \left[\begin{smallmatrix} 1 & sv \\ 0 & I_n \end{smallmatrix}\right] \in \mathrm{SL}(n+1, \mathbb{R}), \forall s \in \mathbb{R}. \tag{7.11}$$

Proof. Due to Lemma 7.2, by Lemma 5.9, $\rho(s)$ is unipotent. Moreover, there exists $r < 1$ such that for any $s \in \mathbb{R}$,

$$\varphi(h_{r,s}(t))\varphi(t)^{-1} \to \rho(s), \tag{7.12}$$

where $h_{r,s}$ is as in (5.1). We note that $\varphi(s)$ is contained in the subgroup F of $\mathrm{SL}(n+1, \mathbb{R})$ whose nonzero entries are only on the diagonal and in the top row. Therefore, by (7.12), $\rho(s)$ is a one-parameter unipotent subgroup of F. Hence, $\rho(s)$ is of the form given by (7.11). □

Proof of Proposition 1.9. Let $\{e_0, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^{n+1}$. Let $(v_1, \ldots, v_n) \in \mathbb{R}^n$ be as in Lemma 7.3. Let W_1 denote the orthogonal complement of $v = \sum_{i=1}^n v_i e_i$ in the span of $\{e_1, \ldots, e_n\}$. Then the fixed points space of $\rho(s)$ in $\mathbb{R}^{n+1}$ is $W := \mathbb{R}e_0 + W_1$.

Now let β denote the correcting definable curve such that $\beta(t)\varphi(t) \subseteq H_\varphi$. Since $\beta(t) \to \beta(\infty) \in \mathrm{SL}(n+1, \mathbb{R})$ as $t \to \infty$, by (7.12), we conclude that $\rho_1(s) := \beta(\infty)\rho(s)\beta(\infty)^{-1} \in H_\varphi$ for all $s \in \mathbb{R}$. By Lemma 7.2 and Lemma 7.1, H_φ is a semisimple group acting irreducibly on $\mathbb{R}^{n+1}$. Therefore, by Jacobson-Morosov theorem, there exists a homomorphism $\eta : \mathrm{SL}(2, \mathbb{R}) \to H_\varphi$ such that $\eta(u(s)) = \rho_1(s)$, where $u(s) := \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{R})$. As noted earlier, the fixed points space of ρ_1 on $\mathbb{R}^{n+1}$ is $\beta(\infty)W$, and it has dimension n. Expressing the representation of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^{n+1}$ via η as a direct sum of irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, and noting that the dimension of the $u(s)$-fixed subspace is n, we conclude that $\mathbb{R}^{n+1} = V_1 \oplus V_2$, where V_1 is isomorphic to the standard representation of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$, and $\mathrm{SL}(2, \mathbb{R})$ acts trivially on V_2. Therefore, by [SY24, Theorem A.7], either $H_\varphi = \mathrm{SL}(n+1, \mathbb{R})$ or $n + 1 = 2d$ for some $d \geq 2$ and H_φ is conjugate to the standard symplectic group $\mathrm{SP}(\mathbb{R}^{2d})$.

Consider the case when $H_\varphi \neq \mathrm{SL}(n+1,\mathbb{R})$. Then, there exists a basis $w_1, w_1', w_2, w_2', \ldots, w_d, w_d'$ of $\mathbb{R}^{2d}$ such that H_φ stabilizes

$$\mathbf{v} = w_1 \wedge w_1' + w_2 \wedge w_2' + \cdots + w_n \wedge w_n' \in \wedge^2 \mathbb{R}^{2d}.$$

Now, by Lemma 7.2, $\varphi(t)$ cannot be bounded in $\mathrm{SL}(2d,\mathbb{R})/H_\varphi$, contradictions the assumption that H_φ is the hull of φ. Therefore, $H_\varphi = \mathrm{SL}(n+1,\mathbb{R})$. This completes the proof of Proposition 1.9. □

Proof of Corollary 1.10. If G is a linear Lie group, the result follows immediately from Proposition 1.9 and Theorem 1.7. If G is not a linear Lie group, for our proof to work, it is sufficient that after the composition with the Adjoint representation of G on its Lie algebra $\mathfrak{g}$, the curve is definable and non-contracting in $\mathrm{GL}(\mathfrak{g})$. This property is valid in our situation. □

Appendix A. Kempf's Lemma for real algebraic groups

We note that [Kem78, Lemma 1.1(b)] is stated for reductive algebraic groups over an arbitrary field. We need the same result for any algebraic group defined over $\mathbb{R}$. Here, we verify that the proof given by Kempf is valid for real algebraic groups G using the most naive language.

For finite-dimensional real vector spaces V and W, we say that $P : V \to W$ is a *polynomial map* if, after choosing bases for V and W, the map P in those coordinates is a multi-variable polynomial in each coordinate. We say that the map is G-equvairant if $g \cdot P(x) = P(g \cdot x)$ for all $g \in G$ and $x \in V$.

Lemma A.1 (Kempf)**.** *Let G be the set of $\mathbb{R}$-points of an algebraic group defined over $\mathbb{R}$. Let V be a rational representation of G. Suppose that for $v \in V$, it holds that $S := \mathrm{zcl}(G \cdot v) \setminus G \cdot v$ is non-empty. Then there exists a rational representation W and an equivariant polynomial map $P : V \to W$ such that $P(S) = \{0\}$ and $P(v) \neq 0$.*

Proof. First, note that since $\mathrm{zcl}(G \cdot v)$ and $G \cdot v$ are G-invariant, we get that S is G-invariant. Let $\mathbb{R}[V]$ be the coordinate ring on the vector space V and consider $I \subseteq \mathbb{R}[V]$ the polynomial ideal of the polynomials vanishing on S. For $d \in \mathbb{N}$, let $I(d) \subseteq I$ be the subset of polynomials in I with degree bounded by d. We may choose d such that the ideal generated by $I(d)$ is I. Notice that $I(d)$ is a finite-dimensional vector space. Consider the right action of G on $\mathbb{R}[V]$ defined by

$$(f.g)(x) := f(g \cdot x),\ g \in G,\ x \in V,\ f \in \mathbb{R}[V]. \tag{A.1}$$

The action preserves the degree of a polynomial. So, it preserves $I(d)$ because S is G-invariant. Let $I(d)^*$ be the space of linear functionals on $I(d)$. Using the representation (A.1), we get a left action on $I(d)^*$ defined by

$$g \cdot l(f) := l(f \cdot g),\ g \in G,\ l \in I(d)^*,\ f \in I(d). \tag{A.2}$$

Now consider the map $P : V \to I(d)^*$ defined by

$$P(x) := \mathrm{ev}_x, \tag{A.3}$$

where ev_x is the functional mapping $f \in I(d)$ to $f(x)$. It is straightforward to verify that P is an equivariant polynomial map. Finally, we claim that $P(v) = \mathrm{ev}_v$ is a nonzero functional. By the definition of S and P, $P(x) = 0$ for all $x \in S$. It remains to show that $P(v) \neq 0$. For that, recall the closed orbit lemma [Bor91, Section 1.8], which states that S is Zariski closed. This implies that there exists a $p \in I(d)$ such that $p(S) = 0$ and $p(v) \neq 0$. Since $P(v)(p) = p(v)$, $P(v) \neq 0$. □

Appendix B. Basic notions in the theory of o-minimal structures

In order to keep our paper self contained, we will now recall the basic definitions and results on o-minimal structures on the field of real numbers. We refer to [Dri98] and [DM96] for more details.

We begin with the notion of a structure *on the ordered field of real numbers*, and later introduce the stronger notion of an o-minimal structure.

Definition B.1. *A structure $\mathscr{S}$ on the ordered field of real numbers* is a sequence of families of sets

$$\mathscr{S}_d \subseteq \{\text{subsets of } \mathbb{R}^d\},\ d \in \mathbb{N},$$

such that the following requirements are satisfied:

(1) For each $n \in \mathbb{N}$, $\mathscr{S}_n$ is a Boolean algebra of sets, and $\mathbb{R}^n \in \mathscr{S}_n$.
(2) For each $n \in \mathbb{N}$, $\mathscr{S}_n$ contains the diagonals $\{(x_1, \dots, x_n) : x_i = x_j\}$, for all $1 \leq i < j \leq n$.
(3) For each $n \in \mathbb{N}$, if $A \in \mathscr{S}_n$, then $A \times \mathbb{R} \in \mathscr{S}_{n+1}$ and $\mathbb{R} \times A \in \mathscr{S}_{n+1}$.
(4) For each $n \in \mathbb{N}$, if $A \in \mathscr{S}_{n+1}$, then $\pi(A) \in \mathscr{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection to the first n coordinates.
(5) $\mathscr{S}_3$ contains the graph of addition $\{(x, y, x + y) : x, y \in \mathbb{R}\}$ and the graph of multiplication $\{(x, y, xy) : x, y \in \mathbb{R}\}$.

Given a structure $\mathscr{S}$, we say that $A \subseteq \mathbb{R}^d$ is definable if A belongs to $\mathscr{S}_d$. We say that a function $f : B \to \mathbb{R}^m$ is definable, for $B \subseteq \mathbb{R}^d$, if its graph $\{(x, f(x) : x \in B\}$ belongs to $\mathscr{S}_{d+m}$.

We note the following basic properties, which are proved directly from Definition B.1

Lemma B.2. *Let $\mathscr{S}$ be a structure on the field of real numbers. Then*:

(1) The restriction of a definable function to a definable set is definable function on that set.
(2) If $f : \mathbb{R}^d \to \mathbb{R}$ is definable, and $A \subseteq \mathbb{R}^d$, is definable, then the subsets:

$$\{\mathbf{x} \in A \mid f(\mathbf{x}) = 0\},\ \{\mathbf{x} \in A \mid f(\mathbf{x}) > 0\}$$

are definable.
(3) If $f : A \to B$ and $g : B \to C$ are definable, then the composition $g \circ f$ is also definable.

(4) If $f, g : A \to \mathbb{R}$ are definable, then $f+g$, $f-g$, $f \cdot g$ are definable, and f/g on the domain $\{x \in A : g(x) \neq 0\}$ is definable.

The notion of a structures belongs to model theory, and it is most natural to describe definable sets using first order logic formulae, see [DM96, Appendix A] for a concise introduction. The advantage of the definition above is that it allows one to define structures without resorting to notions of model theory.

The proof of the statement below demonstrates the technique of proving the definability of functions using first-order logic formulae. The claim is useful for us since the type of the function considered there is used in the proof of Theorem 3.2.

Lemma B.3. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded definable function, and let $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be definable functions such that $a(s) < b(s)$, $\forall s$. Then the function $\Psi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Psi(s) := \sup \{|f(s,t)| : t \in [a(s), b(s)]\}, \ \forall s \in \mathbb{R},$$

is definable.

Proof. To prove the Ψ is definable, we need to verify that the graph of Ψ, $\Gamma(\Psi) := \{(s, \Psi(s)) : s \in \mathbb{R}\}$ is definable. Consider the definable set:

$$A_1 := \{(s, y, t) : a(s) \leq t \leq b(s) \text{ and } |f(s,t)| > y\},$$

and consider the projection $\pi_{3,2} : \mathbb{R}^3 \to \mathbb{R}^2$ to the first coordinates. Then $B_1 := (\pi_{3,2}(A_1))^c$ is definable, and it can be alternatively described by:

$$B_1 := \{(s, y) : (\forall t \in [a(s), b(s)]) \ |f(s,t)| \leq y\}.$$

In a similar way (while slightly complicated), the set:

$$B_2 := \{(s, y) : (\forall \epsilon > 0)(\exists t \in [a(s), b(s)]) \ y - \epsilon \leq |f(s,t)|\}$$

is definable. The graph of Ψ is then $B_1 \cap B_2$. □

B.1. **O-minimal structures.** The notion of o-minimal structures was introduced in [Dri86], as a generalization of the real semi-algebraic geometry.

Definition B.4. (*o-minimal structures*) A structure $\mathscr{S}$ expanding the real field is called o-minimal if $\mathscr{S}_1$ consists of all possible sets which are finite unions of points and intervals.

The semi-algebraic sets are definable in every o-minimal structure. Here, the semi-algebraic sets are boolean combinations of sets of the form $\{x \in \mathbb{R}^d : P(x) = 0\}$, $\{x \in \mathbb{R}^d : P(x) > 0\}$, where $P \in \mathbb{R}[X_1, \ldots, X_d]$. By the Tarski-Seidenberg theorem, the semi-algebraic sets form an o-minimal structure.

We collect the following important results, see [Dri98] for detailed proofs. For the following, fix an o-minimal structure.

Theorem B.5. [Dri98, Ch. 3, Thm. 1.2 and Ch. 7, Prop. 2.5] *Suppose that $f : (a, b) \to \mathbb{R}$ is definable, where $-\infty \leq a < b \leq \infty$. Then, there exist $a_0, a_1, ..., a_{k+1}$ such that $a =: a_0 < a_1 < \cdots < a_k < a_{k+1} := b$ and such that in each sub-interval $I = (a_i, a_{i+1})$, $f|_I$ is differentiable and $f|_I$ is either strictly monotonic, or constant.*

A direct consequence of the monotonicity theorem is that a definable function $f : \mathbb{R} \to \mathbb{R}$ either converges as $t \to \infty$ or diverges to ∞ or to $-\infty$.

Theorem B.6. [Dri98, Ch.3, Cor. 3.7]. *If $A \subseteq \mathbb{R}^n$ is definable, then A has finitely many connected components. Moreover, the following uniformity holds: For any definable $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that for any $x \in \mathbb{R}^m$, the set $A_x := \{y \in \mathbb{R}^n : (x, y) \in A\}$ has at most N connected components.*

Theorem B.7. [Dri98, Ch. 3, Prop. 1.2]. *Suppose that $A \subseteq \mathbb{R}^{m+n}$ is definable. Let $\pi_{m+n,m} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the projection to the first m coordintates. Then, there is a definable function $f : \pi_{m+n,m}(A) \to \mathbb{R}^n$ such that the graph of f is contained in A.*

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