Math. Ann.
DOI 10.1007/s00208-010-0561-z

Mathematische Annalen

Khinchin's theorem for approximation by integral points on quadratic varieties

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Received: 22 April 2008 / Revised: 8 July 2010

Abstract We prove an analogue of the Khinchin's theorem for the Diophantine approximation by integer vectors lying on a quadratic variety. The proof is based on the study of a dynamical system on a homogeneous space of the orthogonal group. We show that in this system, generic trajectories of a certain geodesic flow visit a family of shrinking subsets infinitely often.

Mathematics Subject Classification (2000) 11J83 · 22E40

1 Introduction

Let us consider the following question in the theory of Diophantine approximation: given a vector $v \in \mathbb{R}^d$, is it possible to approximate v by a sequence of rational vectors $\frac{x}{y}$ that come from integer points lying on the quadratic surface $x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1$?

A. Gorodnik's research was supported in part by NSF grant 0654413 and RCUK Fellowship.
N. Shah's research was supported in part by Swarnajayanti Fellowship.

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Published online: 24 September 2010

Namely, we are interested in the integral solutions $(x, y) \in \mathbb{Z}^{d+1}$ of

$$\left\| v - \frac{x}{y} \right\| < \epsilon, \quad x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1. \tag{1}$$

It is easy to see that if (1) has infinitely many integral solutions for every $\epsilon > 0$, then the vector v has to lie on the surface $\mathcal{Q} = \{v_1^2 \pm \cdots \pm v_d^2 = 1\}$. On the other hand, it follows, for instance, from the results in [10] (see [10, Corollary 1.7, Sect. 2.1]) that for every $v \in \mathcal{Q}$ and $\epsilon > 0$, (1) has infinitely many solutions. In this paper, we consider a more delicate question about the order of approximation in (1). For a given function $\psi : (0, \infty) \to (0, \infty)$, we study whether there are infinitely many integral solutions $(x, y) \in \mathbb{Z}^{d+1}$ of

$$\left\| v - \frac{x}{y} \right\| < \psi(|y|), \quad x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1; \tag{2}$$

and show that the answer is determined by integrability of the function $t^{d-2}\psi(t)^{d-1}$. This result is analogous to the Khinchin's theorem, which we now recall. A vector $v \in \mathbb{R}^d$ is called *ψ-approximable* if the inequality

$$\left\| v - \frac{x}{y} \right\| < \psi(|y|)$$

has infinitely many integral solutions $(x, y) \in \mathbb{Z}^{d+1}$. The Khinchin's theorem determines the size of the set of ψ-approximable vectors:

Theorem 1.1 (Khinchin) *Let $\psi : (0, \infty) \to (0, \infty)$ be a non-increasing function. Then the following statements hold:*

(i) *If $\int_1^\infty t^d \psi(t)^d dt = \infty$, then almost every vector in $\mathbb{R}^d$ is ψ-approximable.*
(ii) *If $\int_1^\infty t^d \psi(t)^d dt < \infty$, then almost every vector in $\mathbb{R}^d$ is not ψ-approximable.*

A function $\psi : (0, \infty) \to (0, \infty)$ is called *quasi-conformal* if there exists $c > 0$ such that

$$\psi(ht) \le c\,\psi(t) \quad \text{for all } h \in [1/2, 2] \text{ and } t > 0.$$

It was proved by Sullivan [18, Sect. 3] that the Khinchin's theorem also holds for measurable quasi-conformal functions (which are not necessarily non-increasing).

Note that question (2) is about approximation by radial projections of integral points lying on the surface $x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1$ to the plane $\{y = 1\}$, and when ψ is quasi-conformal, it does not depend on a choice of the radial projection. Hence, it is natural to restate question (2) in a more general geometric fashion.

Let X be an algebraic variety in the Euclidean space $\mathbb{R}^{d+1}$. We denote by $\pi : \mathbb{R}^{d+1} \backslash \{(0, \ldots, 0)\} \to S^d$ the radial projection on the unit sphere S^d. We say that a vector $v \in S^d$ is *(X, ψ)-approximable* if the inequality

$$\|\pi(x) - v\| < \psi(\|x\|).$$

has infinitely many solutions $x \in X(\mathbb{Z}) := X \cap \mathbb{Z}^{d+1}$.

Now for quasi-conformal functions ψ, problem (2) can be restated as a question about (X, ψ)-approximable vectors, where $X = \{x_1^2 \pm \cdots \pm x_d^2 - y^2 = 1\}$, and the Khinchin's theorem is about $(\mathbb{R}^{d+1}, \psi)$-approximable vectors.

We define the *boundary* ∂X of a variety X to consist of the points $\lim_{n\to\infty} \pi(x_n)$ with $x_n \in X \backslash \{(0, \ldots, 0)\}$, $\|x_n\| \to \infty$. Note that if $\psi(t) \to 0$ as $t \to \infty$, then the set of (X, ψ)-approximable vectors is always contained in ∂X.

1.1 Quadratic varieties

Let X be a nonsingular rational quadratic; that is, $X = \{w \in \mathbb{R}^{d+1} : Q(w) = m\}$ for some $m \in \mathbb{Q} \smallsetminus \{0\}$, where Q is a rational nondegenerate indefinite quadratic form. In this case,

$$\partial X = \{x \in \mathbb{R}^{d+1} : Q(x) = 0\} \cap S^d.$$

We assume that $d \geq 3$ and $X(\mathbb{Z}) \neq \emptyset$.

Let $G = \mathrm{O}(Q)$ be the orthogonal group. By Witt's theorem G acts transitively on X. The variety X supports a G-invariant measure, which we denote by vol. We also consider the smooth action of G on S^d by $g \cdot \pi(w) = \pi(gw)$ for all $g \in G$ and $w \in \mathbb{R}^{d+1}$. Under this action, ∂X is homogeneous space of G admitting a unique G-semi-invariant probability measure μ_∞.

Theorem 1.2 *Let the notation be as above and $\psi : (0, \infty) \to (0, \infty)$ be a measurable quasi-conformal function. Then the following statements hold:*

- (i) *If $\int_1^\infty t^{d-2}\psi(t)^{d-1}\, dt = \infty$, then μ_∞-almost every $v \in \partial X$ is (X, ψ)-approximable.*
- (ii) *If $\int_1^\infty t^{d-2}\psi(t)^{d-1}\, dt < \infty$, then μ_∞-almost every $v \in \partial X$ is not (X, ψ)-approximable.*

1.1.1 Remarks

- (a) It is important to note that Theorem 1.2 holds only almost everywhere. There are examples of not (X, ψ)-approximable vectors under assumption (i) and examples of (X, ψ)-approximable vectors under assumption (ii) (see Sect. 5).
- (b) Since the function ψ is quasi-conformal, it is clear that the claim of Theorem 1.2 does not depend on a choice of the norm. Hence, we assume that $\|\cdot\|$ is the standard Euclidean norm.
- (c) In this paper, we consider the surface $\{Q = m\}$ with $m \neq 0$. The structure of the singular surface $\{Q = 0\}$ is different. It seems that the integral/rational points on the later surface can studied using the methods from [6].

For $v \in \partial X$, we define the *cusp* $C(v, \psi)$ at v:

$$C(v, \psi) = \{x \in X : \|\pi(x) - v\| < \psi(\|x\|)\}. \tag{3}$$

Using this notation, Theorem 1.2 can be restated as follows:

Theorem 1.3 *For μ_∞-almost every $v \in \partial X$,*

$$\mathrm{vol}(C(v, \psi)) = \infty \iff \#(X(\mathbb{Z}) \cap C(v, \psi)) = \infty.$$

It follows from Theorem 2.1 below that Theorem 1.3 is equivalent to Theorem 1.2, and the condition $\mathrm{vol}(C(v, \psi)) = \infty$ is independent of $v \in \partial X$.

1.2 Shrinking targets

We will prove Theorem 1.2 using a shrinking target property for a flow on a suitable homogeneous space. Let G be a noncompact real algebraic group and H an algebraic subgroup which is the set of fixed points of an involution σ. We fix a nontrivial one parameter subgroup $\{a_t\}$ of G such that $\sigma(a_t) = a_{-t}$. Let Z denote the centralizer of $\{a_t\}$ in G and U^- the contracting horospherical subgroup of a_t, i.e.,

$$U^- = \{g \in G : a_t g a_t^{-1} \to e \text{ as } t \to \infty\}.$$

Given a lattice Γ in G, we consider the flow a_t on the space G/Γ. We are interested in visits of generic trajectories of a_t to shrinking boxes of the form $\Psi_t B\Gamma \subset G/\Gamma$ where $\Psi_t \subset U^-$ and $B \subset ZH$. The following is our main result:

Theorem 1.4 *Given a bounded measurable subset of $B \subset ZH$ of positive measure, there exists a neighborhood $\mathcal{O}$ of identity in U^- such that for any sequence of measurable subsets $\Psi_n \subset \mathcal{O}$, an increasing sequence $t_n \to \infty$ of real numbers, and $y_0 \in G/\Gamma$, the following statements hold:*

(i) *If* $\inf_{n\in\mathbb{N}}(t_{n+1} - t_n) > 0$ *and* $\sum_{n=1}^{\infty} \mathrm{vol}_{U^-}(\Psi_n) = \infty$, *then*

$$\mathrm{vol}_{G/\Gamma}(\{z \in G/\Gamma : \#(\{n \in \mathbb{N} : a_{t_n}^{-1} z \in \Psi_n B y_0\}) = \infty\}) > 0.$$

(ii) *If* $\sum_{n=1}^{\infty} \mathrm{vol}_{U^-}(\Psi_n) < \infty$, *then*

$$\mathrm{vol}_{G/\Gamma}(\{z \in G/\Gamma : \#(\{n \in \mathbb{N} : a_{t_n}^{-1} z \in \Psi_n B y_0\}) = \infty\}) = 0.$$

The problem of shrinking targets, that is, the problem about visits of trajectories to a family shrinking subsets, has been an active topic of research over the past decades (see [1,3,4,9,12,13,15,16,18]). It seems that in the context of partially hyperbolic systems, there are two main approaches to this question. One is based on (strong) mixing properties of the flow, and the other uses geometric properties of the space such as negative curvature. As usual, the crucial step is to show that the sets $\{a_{t_n}\Psi_n B\Gamma\}$ are quasi-independent (see Proposition 4.1 and Theorem 3.1). Our proof of quasi-independence is quite different from the previous works and is based on the simple observation that the sets $a_{t_n}(\Psi_n B)a_{t_n}^{-1}$ and $a_{t_m}(\Psi_m B)a_{t_m}^{-1}$ are "transversal" if $|n - m|$ is sufficiently large (see the proof of Proposition 3.2). It is interesting to note that similar transversality argument was also used in a very different context in [2].

As a consequence of Theorem 1.4, we obtain

Corollary 1.5 *Suppose that $\Psi_{n+1} \subset \Psi_n$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \mathrm{vol}_{U^-}(\Psi_n) = \infty$, and the action of $T = a_{t_0}$ on G/Γ is ergodic for some $t_0 > 0$. Then for any $y_0 \in G/\Gamma$ and for almost every $z \in G/\Gamma$,*

$$\#(\{n \in \mathbb{N} : T^{-n} z \in \Psi_n B y_0\}) = \infty.$$

1.3 Organization of the paper

In Sect. 2, we introduce a convenient coordinate system to describe the cusps $C(v, \psi)$ and show that Theorems 1.2 and 1.3 are equivalent. Section 3 contains the proof of quasi-independence which is crucial for Theorem 1.4. In Sect. 4, we prove the theorems from the introduction, and in Sect. 5, we give some examples to justify the need for almost everywhere condition in the statement of Theorem 1.2.

2 Description of the cusps

In this section, we use notation from Sect. 1.1, and we will prove

Theorem 2.1 *Let $\psi : (0, \infty) \to (0, \infty)$ be a measurable quasi-conformal function. Then for any $v \in \partial X$,*

$$\mathrm{vol}(C(v, \psi)) = \infty \quad \iff \quad \int_1^\infty t^{d-2} \psi(t)^{d-1}\, dt = \int_0^\infty (e^t \psi(e^t))^{d-1}\, dt = \infty.$$

Recall that we are assuming that $\|\cdot\|$ is the standard Euclidean norm. In this case, $\pi(x) = x/\|x\|$.

We choose a basis $\{f_1, \ldots, f_{d+1}\}$ of $\mathbb{R}^{d+1}$ such that

$$Q(x_1 f_1 + \cdots + x_{d+1} f_{d+1}) = 2x_1 x_{d+1} + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_d^2, \quad (x_i \in \mathbb{R}).$$

As we have noted before, the variety $X = \{Q = m\}$, where $m \in \mathbb{Q} \setminus \{0\}$, is a homogeneous space of $G = \mathrm{O}(Q) \simeq \mathrm{O}(p, q)$.

Lemma 2.2 *Given a compact set $\mathcal{K} \subset G$, there exists a constant $\kappa > 1$ such that for any $v \in \partial X$ and $g \in \mathcal{K}$,*

$$g(C(v, \psi)) \subset C(\pi(gv), \kappa\psi).$$

Proof Let $x \in C(v, \psi)$. Then

$$\|\pi(x) - v\| < \psi(\|x\|).$$

We need to prove that $gx \in C(\pi(gv), \kappa\psi)$ for some $\kappa > 0$, i.e.,

$$\|\pi(gx) - \pi(gv)\| < \kappa\psi(\|gx\|). \tag{4}$$

We have

$$\|\pi(gx) - \pi(gv)\| = \left\| \frac{g\pi(x)}{\|g\pi(x)\|} - \frac{gv}{\|gv\|} \right\|,$$

and using the inequality

$$\left\| \frac{w_1}{\|w_1\|} - \frac{w_2}{\|w_2\|} \right\| \leq \frac{2}{\|w_2\|}\|w_1 - w_2\|,$$

we deduce that for some $\kappa_1 = \kappa_1(\mathcal{K}) > 0$,

$$\begin{aligned} \|\pi(gx) - \pi(gv)\| &\leq \frac{2}{\|gv\|}\|g\pi(x) - gv\| \\ &\leq \frac{2\|g\|}{\|gv\|}\|\pi(x) - v\| < \kappa_1\psi(\|x\|). \end{aligned}$$

Since ψ is quasi-conformal, there exists $\kappa_2 = \kappa_2(\mathcal{K}) > 0$ such that

$$\psi(\|x\|) \leq \kappa_2\psi(\|gx\|).$$

This implies (4). □

Lemma 2.3 *Let $\phi : \mathbb{R} \to (0, \infty)$ be a measurable function such that for some $c > 1$,*

$$c^{-1}\phi(x) \leq \phi(x + h) \leq c\phi(x), \quad x \in \mathbb{R},\ h \in [-1, 1]. \tag{5}$$

(i) *If $\int_0^\infty \phi(t)\,dt < \infty$, then $\phi(t) \to 0$ as $t \to \infty$.*
(ii) *If $\int_0^\infty \phi(t)\,dt = \infty$, then there exists a function $\phi_1 : (0, \infty) \to (0, 1]$ such that ϕ_1 satisfies (5) with a possibly different constant $c > 0$, $\phi_1 \leq \phi$, $\phi_1(t) \to 0$ as $t \to \infty$, $\frac{\phi(t)}{\phi_1(t)} \to \infty$ as $t \to \infty$, and $\int_0^\infty \phi_1(t)\,dt = \infty$.*

We note that the function $\phi(t) := e^t\psi(e^t)$ satisfies the condition (5) for some $c > 1$.

Proof Suppose there exists $\delta > 0$ and a sequence $t_i \to \infty$ such that $\phi(t_i) \geq \delta$. We may assume that $t_{i+1} - t_i \geq 2$. By (5), $\phi(t) \geq c^{-1}\delta$ for all $t \in [t_i - 1, t_i + 1]$. Therefore $\int_0^\infty \phi(t)\,dt = \infty$. In particular, this proves (i).

Now we prove (ii). Let $\phi_2(t) = \min(\phi(t), 1)$. Then it follows from (5) that $\int_0^\infty \phi_2(t)\,dt = \infty$, and ϕ_2 satisfies (5) with a different constant $c > 0$. Let $T_0 = 0$ and $T_i > 0$ be such that $\int_0^{T_i} \phi_2(t)\,dt = i$ (it follows from (5) that the function $T \mapsto \int_0^T \phi_2(t)dt$ is continuous). Then $T_i - T_{i-1} \geq 1$ for all $i \in \mathbb{N}$. We define

$\rho(t) = 1/i$ for $t \in (T_{i-1}, T_i]$. Then $(1/2)\rho(t) \le \rho(t+h) \le 2\rho(t)$ for all $h \in [-1, 1]$ and $t > 1$, and it is easy to check that $\phi_1(t) := \rho(t)\phi_2(t)$ satisfies the conditions of (ii). $\square$

For $v \in \mathbb{R}^{d+1}$, we write $v = v_1 + v_2$, where $v_1 \in \mathbb{R} f_1$ and $v_2 \in \operatorname{span}\{f_2, \dots, f_{d+1}\}$. Let $p(v) := \|v_1\|$ and $\bar{f}_1 = \pi(f_1) \in \partial X$. For $T > 1$, we define

$$D_T(\bar{f}_1, \psi) = \{x \in X : \|x/p(x) - \bar{f}_1\| \le \psi(p(x));\ \|p(x)\| \ge T\},$$
$$C_T(\bar{f}_1, \psi) = \{x \in X : \|x/\|x\| - \bar{f}_1\| \le \psi(\|x\|);\ \|x\| \ge T\}.$$

Lemma 2.4 *Let $\psi : (0, \infty) \to (0, \infty)$ be quasi-conformal function such that $\psi(t) \to 0$ as $t \to \infty$. Then there exist $c > 1$ and $T_0 > 1$, depending on ψ, such that for all $T \ge T_0$,*

$$D_T(\bar{f}_1, \psi) \subset C_{T/2}(\bar{f}_1, c\psi), \tag{6}$$
$$C_T(\bar{f}_1, \psi) \subset D_{T/2}(\bar{f}_1, c\psi). \tag{7}$$

Proof Let $c > 1$ be such that

$$\psi(ht) \le (c/2)\psi(t) \quad \text{for all } h \in [1/2, 2].$$

For $x = x_1 + x_2 \in D_T(\bar{f}_1, \psi)$, we have $\|x/\|x_1\| - \bar{f}_1\| \le \psi(\|x_1\|)$. Since $x_1/\|x_1\| = \bar{f}_1$, we have

$$\|x_2\| \le \|x_1\|\psi(\|x_1\|),$$

and

$$\|x/\|x\| - x/\|x_1\|\| \le |\|x_1\| - \|x\||/\|x_1\| \le \|x_2\|/\|x_1\| \le \psi(\|x_1\|).$$

Therefore

$$\|x/\|x\| - \bar{f}_1\| \le \|x/\|x\| - x/\|x_1\|\| + \|x/\|x_1\| - \bar{f}_1\| \le 2\psi(\|x_1\|). \tag{8}$$

We choose $T_0 > 0$ such that $\psi(T) < 1/2$ for all $T \ge T_0$. Then $\|x_2\|/\|x_1\| < 1/2$. Therefore, $\|x_1\|/\|x\| \in (1/2, 2)$ and $\psi(\|x_1\|) \le (c/2)\psi(\|x\|)$. By (8), $x \in C_{T/2}(\bar{f}_1, \psi)$ and (6) follows. Similarly, one proves (7). $\square$

Proof of Theorem 2.1 Applying Lemma 2.3 to $\phi(t) = (e^t\psi(e^t))^{d-1}$, we may assume that $e^t\psi(e^t) \to 0$ as $t \to \infty$.

Since the action of g on X preserves the G-invariant measure on X, we have that $\operatorname{vol}(g\,C(v, \psi)) = \operatorname{vol}(C(v, \psi))$. Therefore due to Lemma 2.2, in order to prove Theorem 2.1, it is sufficient to prove it for a chosen $v \in \partial X$. Here we choose $v = \pi(f_1) = \bar{f}_1$.

Due to Lemma 2.4, it is enough to prove that for sufficiently large T,

$$\mathrm{vol}(D_T(\bar{f}_1, \psi)) = \infty \iff \int_0^\infty (e^t \psi(e^t))^{d-1} dt = \infty. \tag{9}$$

Let $w_0 = f_1 + (m/2) f_{d+1} \in X$. In what follows we will write the matrices of the linear transformation on $\mathbb{R}^{d+1}$ with respect to the basis $\{f_1, \ldots, f_{d+1}\}$. Let

$$a_t = \mathrm{diag}(e^t, 1, \ldots, 1, e^{-t}). \tag{10}$$

For $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2) \in \mathbb{R}^{d-1}$, where $\boldsymbol{s}_1 = (s_2, \ldots, s_p) \in \mathbb{R}^{p-1}$ and $\boldsymbol{s}_2 = (s_{p+1}, \ldots, s_d) \in \mathbb{R}^{d-p}$, let

$$u(\boldsymbol{s}) = \begin{pmatrix} 1 & & & \\ {}^t\boldsymbol{s}_1 & I_{p-1} & & \\ {}^t\boldsymbol{s}_2 & & I_{d-p} & \\ \frac{1}{2}(-\|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2) & -\boldsymbol{s}_1 & \boldsymbol{s}_2 & 1 \end{pmatrix}. \tag{11}$$

Then

$$a_t u(\boldsymbol{s}) w_0 = e^t f_1 + (s_2 f_2 + \cdots + s_d f_d) + (e^{-t}(m - \|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2)/2) f_{d+1}. \tag{12}$$

We observe that every $x \in X$ such that $p(x) \neq 0$ can be written in the form $x = a_t u(\boldsymbol{s}) w_0$. This implies that the set $D_T(\bar{f}_1, \psi)$ consists of $x = a_t u(\boldsymbol{s}) w_0$ such that

$$\frac{\|(s_2 f_2 + \cdots + s_d f_d) + (e^{-t}(m - \|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2)/2) f_{d+1}\|}{\|f_1\| e^t} \leq \psi(e^t \|f_1\|) \tag{13}$$

and $\|f_1\| e^t \geq T$. There exists a constant $c_2 > 1$ such that for any $(s_2, \ldots, s_{d+1}) \in \mathbb{R}^d$, we have

$$\sup(\|(s_2, \ldots, s_d)\|, |s_{d+1}|) / \|f_2 s_2 + \cdots + s_d f_d + s_{d+1} f_{d+1}\| \in (c_2^{-1}, c_2).$$

Since ψ is quasi-conformal, there exists $c_3 > 1$ such that

$$\psi(e^t \|f_1\|)/\psi(e^t) \in (c_3^{-1}, c_3), \quad \forall t > 0.$$

Let

$$\mathcal{U}_t(\psi) = \left\{ \boldsymbol{s} \in \mathbb{R}^{d-1} : \begin{array}{l} |m - \|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2|/2 \leq e^{2t}\psi(e^t), \\ \|\boldsymbol{s}\| \leq e^t \psi(e^t) \end{array} \right\}. \tag{14}$$

Then by (13), there exists $c > 1$ such that

$$\bigcup_{t\geq t_0} a_t u(\mathcal{U}_t(c^{-1}\psi))w_0 \subset D_T(\bar{f}_1, \psi) \subset \bigcup_{t\geq t_0} a_t u(\mathcal{U}_t(c\psi))w_0 \tag{15}$$

where $t_0 = \log(T/\|f_1\|)$. Since the volume form on X with respect to the $(t, \boldsymbol{s})$-coordinates is given by $dtd\boldsymbol{s}$, we have

$$\mathrm{vol}_X\left(\bigcup_{t\geq t_0} a_t u(\mathcal{U}_t(\psi))w_0\right) = \infty \iff \int_{t_0}^{\infty}\int_{\boldsymbol{s}\in\mathcal{U}_t(\psi)} dtd\boldsymbol{s} = \infty.$$

Hence to prove (9), it suffices to prove that

$$\int_0^\infty (e^t\psi(e^t))^{d-1}\, dt = \infty \iff \int_0^\infty \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_t(\psi))\, dt = \infty. \tag{16}$$

Let

$$\tilde{\mathcal{U}}_t(\psi) = \{\boldsymbol{s} \in \mathbb{R}^{d-1} : \|\boldsymbol{s}\| \leq e^t\psi(e^t)\}. \tag{17}$$

Let $t_1 > 0$ be such that $\psi(e^t) \leq 1$ for all $t \geq t_1$, and let

$$\mathcal{T} := \{t > 0 : e^{2t}\psi(e^t) \leq |m|\}. \tag{18}$$

For $t > t_1$, $t \notin \mathcal{T}$, and $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2) \in \tilde{\mathcal{U}}_t(\psi)$,

$$|m - \|\boldsymbol{s}_1\|^2 + \|\boldsymbol{s}_2\|^2| \leq |m| + \|\boldsymbol{s}\|^2 \leq |m| + e^{2t}\psi(e^t)^2 \leq 2e^{2t}\psi(e^t). \tag{19}$$

Hence, for such t,

$$\mathcal{U}_t(\psi) = \tilde{\mathcal{U}}_t(\psi) \quad \text{and} \quad \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_t(\psi)) = \omega_{d-1}(e^t\psi(e^t))^{d-1}.$$

where $\omega_{d-1} > 0$ is the volume of the unit ball in $\mathbb{R}^{d-1}$. On the other hand,

$$\begin{aligned}\int_{\mathcal{T}} \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_t(\psi))\, dt &\leq \int_{\mathcal{T}} \mathrm{vol}_{\mathbb{R}^{d-1}}(\tilde{\mathcal{U}}_t(\psi))\, dt \leq \omega_{d-1}\int_{\mathcal{T}} (e^t\psi(e^t))^{d-1}\, dt \\ &\leq \omega_{d-1}|m|^{d-1}\int_{\mathcal{T}} e^{-(d-1)t}\, dt < \infty. \end{aligned}\tag{20}$$

This implies (16) and completes the proof of Theorem 2.1. □

3 Volume estimate for intersections

Let G be a real algebraic group, and σ is an involution of G. Let $A = \{a_t\}$ be a one-parameter subgroup of G such that $\sigma(a_t) = a_{-t}$. We use notation:

$$H = \{g \in G : \sigma(h) = h\}, \quad U^+ = \{g \in G : a_{-t} g a_t \to_{t\to\infty} e\},$$
$$Z = Z_G(A), \quad U^- = \{g \in G : a_t g a_{-t} \to_{t\to\infty} e\}.$$

Note that $\sigma(U^-) = U^+$, $\sigma(Z) = Z$, and $\sigma(U^+) = U^-$.

We fix a (right) invariant Riemannian metric on G. For a subgroup S of G, we set $S_\epsilon = \{s \in S : d(s, e) < \epsilon\}$. (It will be convenient to use that $S_\epsilon^{-1} = S_\epsilon$ and $S_{\epsilon_1} S_{\epsilon_2} \subset S_{\epsilon_1+\epsilon_2}$.)

The following theorem is the main result of this section:

Theorem 3.1 *There exist constants $r_0, t_0 > 0$ such that for any measurable subsets $\Psi_i \subset U^-_{r_0}$ and $g \in G$, setting $D_i := \Psi_i Z_{r_0} H_{r_0} g\Gamma$, $i = 1, 2$, we have*

$$\mathrm{vol}_{G/\Gamma}(D_1 \cap a_t D_2) \le C\, \mathrm{vol}_{U^-}(\Psi_1)\, \mathrm{vol}_{U^-}(\Psi_2), \quad \forall t \ge t_0.$$

for some $C = C(g) > 0$.

The proof of Theorem 3.1 consists of two main steps: in Proposition 3.2, we estimate the volumes of the intersections of lifts of D_1 and $a_t D_2$ in G, and using Proposition 3.3 with Lemma 3.4, we estimate the number of lifts which intersect.

We start the proof by introducing convenient coordinate systems in G. Let $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{h}$, $\mathfrak{z}$, $\mathfrak{u}^+$, $\mathfrak{u}^-$ denote the corresponding Lie algebras. We have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

into $(+1)$- and (-1)-eigenspaces of σ. Since $\mathrm{Ad}(a_t)$ is skew-symmetric with respect to the form $\langle X, \sigma(X)\rangle$, $X \in \mathfrak{g}$, it follows that $\mathrm{Ad}(a_t)$ is diagonalizable and we have the decomposition:

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{z} \oplus \mathfrak{u}^+.$$

Hence, the product map $U^- \times Z \times U^+ \to G$ is a diffeomorphism in a neighborhood of e, and there exist $r_0 > 0$ and analytic maps $\boldsymbol{u}^- : G_{r_0} \to U^-$, $\boldsymbol{z} : G_{r_0} \to Z$, $\boldsymbol{u}^+ : G_{r_0} \to U^+$ such that every element $g \in G_{r_0}$ can be uniquely written as

$$g = \boldsymbol{u}^-(g)\boldsymbol{z}(g)\boldsymbol{u}^+(g). \tag{21}$$

For every $x \in \mathfrak{u}^+$, we have $x = -\sigma(x) + (x + \sigma(x))$ where $\sigma(x) \in \mathfrak{u}^-$ and $x + \sigma(x) \in \mathfrak{h}$. Hence, we also have the decomposition:

$$\mathfrak{g} = \mathfrak{u}^- \oplus (\mathfrak{z} + \mathfrak{h}) \tag{22}$$

and the decomposition:

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a}) \oplus \mathfrak{h}.$$

Let $B = \exp(\mathfrak{z}_{\mathfrak{q}}(a))$. It follows that there exist $r_0 > 0$ and analytic maps $\boldsymbol{v} : G_{r_0} \to U^-$, $\boldsymbol{b} : G_{r_0} \to B$, $\boldsymbol{h} : G_{r_0} \to U^+$ such that every element $g \in G_{r_0}$ can be uniquely written as

$$g = \boldsymbol{v}(g)\boldsymbol{b}(g)\boldsymbol{h}(g). \tag{23}$$

Proposition 3.2 *There exist $r_0, t_0, c > 0$ such that for measurable subsets $\Psi_1, \Psi_2 \subset U^-_{r_0}$, $t > t_0$, and $g \in G$, we have*

$$\mathrm{vol}_G(\Psi_1 Z_{r_0} H_{r_0} g \cap a_t \Psi_2 Z_{r_0} H_{r_0}) \le c \, \mathrm{vol}_{U^-}(\Psi_1)\, \mathrm{vol}_{U^-}(\alpha_t(\Psi_2))$$

where $\alpha_t(g) = a_t g a_{-t}$.

Proof Let $Y := \Psi_1 Z_{r_0} H_{r_0} g \cap a_t \Psi_2 Z_{r_0} H_{r_0}$ and $y \in Y$. Let $\zeta_i \in \Psi_i Z_{r_0} H_{r_0}$, $i = 1, 2$, be such that $y = \zeta_1 g = a_t \zeta_2$. Then

$$Y = (\Psi_1 Z_{r_0} H_{r_0} \zeta_1^{-1} \cap \alpha_t(\Psi_2 Z_{r_0} H_{r_0} \zeta_2^{-1}))y.$$

We express $\zeta_i = u_i z_i h_i$, where $u_i \in \Psi_i$, $z_i \in Z_{r_0}$ and $h_i \in H_{r_0}$. Then $H_{r_0} h_i^{-1} \subset H_{2r_0}$ and

$$\begin{aligned} Y &\subset (\Psi_1 Z_{r_0} H_{2r_0} (u_1 z_1)^{-1} \cap \alpha_t(\Psi_2 Z_{r_0} H_{2r_0} (u_2 z_2)^{-1}))y \\ &\subset (\Psi_1 Z_{r_0} H_{2r_0} \zeta \cap \alpha_t(\Psi_2 Z_{r_0} H_{2r_0}))y_1. \end{aligned}$$

where $\zeta = (u_1 z_1)^{-1} \alpha_t(u_2 z_2)$ and $y_1 = \alpha_t(u_2 z_2)^{-1} y$. Since the measure is right translation invariant, it remains to show that for the set

$$X := \Psi_1 Z_{r_0} H_{2r_0} \zeta \cap \alpha_t(\Psi_2 Z_{r_0} H_{2r_0}), \tag{24}$$

we have

$$\mathrm{vol}_G(X) \le c \; \mathrm{vol}_{U^-}(\Psi_1)\, \mathrm{vol}_{U^-}(\alpha_t(\Psi_2)). \tag{25}$$

Note that $\alpha_t|_{U^- Z}$ is Lipschitz (uniformly on t). Hence, there exists $l > 0$ such that $\zeta \in G_{l r_0}$.

For small $r > 0$, we have a well-defined map

$$\boldsymbol{p} = \boldsymbol{u}^- \times \boldsymbol{z} \times \boldsymbol{u}^+ : G_r \to U^- \times Z \times U^+$$

which is a diffeomorphism onto its image. For $\zeta, g \in G$ close to identity, we also have the map $\boldsymbol{p}_\zeta(g) = \boldsymbol{p}(g\zeta^{-1})$. Note that

$$\boldsymbol{p}_\zeta(g) \to \boldsymbol{p}(g) \quad \text{and} \quad D(\boldsymbol{p}_\zeta)_g \to D(\boldsymbol{p})_g \quad \text{as } \zeta \to e, \tag{26}$$

uniformly for g in a neighborhood of identity. Given $g = u^- z u^+ \in U^- Z U^+$ with components close to identity, we write

$$\begin{aligned} g = u^- z u^+ &= u^- z \boldsymbol{v}(u^+) \boldsymbol{b}(u^+) \boldsymbol{h}(u^+) \\ &= \psi(u^-, z, u^+) \cdot \eta(u^-, z, u^+) \cdot \boldsymbol{h}(u^+) \end{aligned}$$

where

$$\begin{aligned} \psi(u^-, z, u^+) &:= u^- z \boldsymbol{v}(u^+) z^{-1} \in U^-, \\ \eta(u^-, z, u^+) &:= z \boldsymbol{b}(u^+) \in Z. \end{aligned}$$

We claim that if $u_1 z_1 h_1 = u_2 z_2 h_2 \in U^- Z H$ with components close to identity, then $u_1 = u_2$. Indeed, we have

$$z_1^{-1}(u_2^{-1} u_1) z_1 = (z_1^{-1} z_2)(h_2 h_1^{-1}) \in U^- \cap ZH.$$

Since the map $u \mapsto z_1^{-1} u z_1$, $u \in U^-$, is Lipschitz in a neighborhood of identity, and U^- is transversal to ZH at identity (see (22)), it follows that $u_1 = u_2$. In particular, we note that

$$g \in \Psi_1 Z_{r_0} H_{2r_0} \zeta \implies \psi(\boldsymbol{p}_\zeta(g)) \in \Psi_1. \tag{27}$$

For $g = u^- z u^+ \in U^- Z U^+$ with components close to identity, we write

$$\begin{aligned} g = u^- z u^+ &= u^- z \alpha_t(\alpha_{-t}(u^+)) \\ &= u^- z \alpha_t(\boldsymbol{v}(\alpha_{-t}(u^+)) \boldsymbol{b}(\alpha_{-t}(u^+)) \boldsymbol{h}(\alpha_{-t}(u^+))) \\ &= \phi_t(u^-, z, u^+) \cdot z \boldsymbol{b}(\alpha_{-t}(u^+)) \cdot \alpha_t(\boldsymbol{h}(\alpha_{-t}(v))), \\ &\in U^- \cdot Z \cdot \alpha_t(H), \end{aligned}$$

where

$$\phi_t(u^-, z, u^+) := u^- z \alpha_t(\boldsymbol{v}(\alpha_{-t}(u^+))) z^{-1} \in U^-.$$

By the argument as above, we have

$$g \in \alpha_t(\Psi_2) Z_{r_0} \alpha_t(H_{2r_0}) \implies \phi_t(\boldsymbol{p}(g)) \in \alpha_t(\Psi_2). \tag{28}$$

As $t \to \infty$, the map $u^+ \mapsto \alpha_t(\boldsymbol{v}(\alpha_{-t}(u^+)))$ converges in C^1-topology to the constant function e. Therefore, setting $\phi(u^-, z, u^+) = u^-$, we have

$$\phi_t(\boldsymbol{p}(g)) \to \phi(\boldsymbol{p}(g)) \quad \text{and} \quad D(\phi_t)_{\boldsymbol{p}(g)} \to D(\phi)_{\boldsymbol{p}(g)} \quad \text{as } t \to \infty, \tag{29}$$

uniformly on g in a neighborhood of identity.

For small $r > 0$, consider the maps $\Phi_{\zeta,t}, \Phi : G_r \to U^- \times Z \times U^-$ defined by

$$\Phi_{\zeta,t}(g) = (\psi(\boldsymbol{p}_\zeta(g)), \eta(\boldsymbol{p}_\zeta(g)), \phi_t(\boldsymbol{p}(g))),$$
$$\Phi(g) = (\psi(\boldsymbol{p}(g)), \eta(\boldsymbol{p}(g)), \phi(\boldsymbol{p}(g))).$$

By (26) and (29),

$$\Phi_{\zeta,t}(g) \to \Phi(g) \quad \text{and} \quad D(\Phi_{\zeta,t})_g \to D(\Phi)_g \quad \text{as } (\zeta, t) \to (e, \infty), \tag{30}$$

uniformly on g in a neighborhood of identity. We have

$$D(\Phi \circ \boldsymbol{p}^{-1})_{(e,e,e)} = \begin{pmatrix} \mathrm{id} & 0 & \left(\frac{\partial \boldsymbol{v}}{\partial u^+}\right)_e \\ 0 & \mathrm{id} & \left(\frac{\partial \boldsymbol{b}}{\partial u^+}\right)_e \\ \mathrm{id} & 0 & 0 \end{pmatrix}.$$

Since every $x \in \mathfrak{u}^+$ can be written as

$$x = -\sigma(x) + 0 + (x + \sigma(x)) \in \mathfrak{u}^- \oplus \mathfrak{b} \oplus \mathfrak{h},$$

it follows that $(\frac{\partial \boldsymbol{v}}{\partial u^+})_e = -\sigma$ and $|\det D(\Phi \circ \boldsymbol{p}^{-1})_{(e,e,e)}| = 1$. Hence, by (30), there exists $c_1 > 0$ such that for every $\zeta \in G$ close to e, sufficiently large t, and $g \in G$ in a neighborhood of e, we have

$$|\det D(\Phi_{\zeta,t})_g| > c_1. \tag{31}$$

By (24), (27) and (28),

$$\Phi_{\zeta,t}(X) \subset \Psi_1 \times Z_{\kappa r_0} \times \alpha_t(\Psi_2)$$

for some $\kappa > 0$. Now taking $r_0 > 0$ sufficiently small and $c = c_1^{-1}$, (25) follows from (31). This proves the proposition. □

Proposition 3.3 *There exist $r_0 > 0$ and $d > 0$ such that for every $t > 0$,*

$$\#(\Gamma \cap G_{r_0} a_t G_{r_0}) \le d \det(\mathrm{Ad}(a_t)|_{\mathfrak{u}^+}).$$

To prove this proposition, we use the following lemma:

Lemma 3.4 *There exist $\epsilon_0 > 0$ and $l > 1$ such that for every $r \in (0, \epsilon_0)$ and $t > 0$,*

$$G_r a_t G_r \subset U^-_{lr} Z_{lr} a_t U^+_{lr}.$$

Proof There exists $l_1 > 1$ such that for every small $r > 0$, $G_r \subset U^-_{l_1 r} Z_{l_1 r} U^+_{l_1 r}$ (see (21)). Since the map $u \mapsto a_t u a_t^{-1}$, $u \in U^-$, is Lipschitz on compact sets, there exists $\kappa > 0$ such that $a_t U^-_r a_t^{-1} \subset U^-_{\kappa r}$. Therefore,

$$G_r a_t G_r \subset G_r a_t U^-_{l_1 r} Z_{l_1 r} U^+_{l_1 r} \subset G_r U^-_{\kappa l_1 r} Z_{l_1 r} a_t U^+_{l_1 r} \subset G_{l_2 r} a_t U^+_{l_2 r}.$$

where $l_2 = 1 + (\kappa + 1)l_1$. Similarly,

$$G_r a_t U_r^+ \subset U_{l_1 r}^- Z_{l_1 r} U_{l_1 r}^+ a_t U_r^+ \subset U_{l_1 r}^- Z_{l_1 r} a_t U_{\kappa l_1 r}^+ U_r^+ \subset U_{l_1 r}^- Z_{l_1 r} a_t U_{(\kappa l_1 + 1) r}^+.$$

This implies the claim. □

Proof of Proposition 3.3 Let $\epsilon > 0$ be such that

$$G_\epsilon \gamma_1 \cap G_\epsilon \gamma_2 = \emptyset \quad \text{for } \gamma_1 \neq \gamma_2.$$

Then

$$\#(\Gamma \cap G_r a_t G_r) \leq \mathrm{vol}_G(G_\epsilon G_r a_t G_r) / \mathrm{vol}_G(G_\epsilon).$$

Hence, it follows from Lemma 3.4, that for some $c_1, r_1 > 0$

$$\#(\Gamma \cap G_r a_t G_r) \leq c_1 \ \mathrm{vol}_G(U_{r_1}^- Z_{r_1} a_t U_{r_1}^+).$$

Since the Haar measure in $U^- Z U^+$-coordinates is given by

$$\det(\mathrm{Ad}(z)|_{\mathfrak{u}^+}) du^- dz du^+,$$

the claim follows. □

Proof of Theorem 3.1 Let $r_0 > 0$ be sufficiently small. We have

$$\mathrm{vol}_{G/\Gamma}(D_1 \cap a_t D_2) \leq \sum_{\gamma \in \Gamma} \mathrm{vol}_G(\Psi_1 Z_{r_0} H_{r_0} g \gamma \cap a_t \Psi_2 Z_{r_0} H_{r_0} g). \tag{32}$$

If $\gamma \in \Gamma$ satisfies

$$\Psi_1 Z_{r_0} H_{r_0} g \gamma \cap a_t \Psi_2 Z_{r_0} H_{r_0} g \neq \emptyset,$$

then

$$g \gamma g^{-1} \in (\Psi_1 Z_{r_0} H_{r_0})^{-1} a_t \Psi_2 Z_{r_0} H_{r_0} \subset G_{3r_0} a_t G_{3r_0}.$$

Hence, by Proposition 3.3, the number of terms in the sum (32) is bounded by $d \det(\mathrm{Ad}(a_t)|_{\mathfrak{u}^+})$ for some $d = d(g) > 0$. Applying Proposition 3.2, we deduce from (32) that

$$\mathrm{vol}_{G/\Gamma}(D_1 \cap a_t D_2) \leq d \det(\mathrm{Ad}(a_t)|_{\mathfrak{u}^+}) \cdot c \ \mathrm{vol}_{U^-}(\Psi_1) \ \mathrm{vol}_{U^-}(\alpha_t(\Psi_2)).$$

This proves the theorem. □

In the rest of this section we compute the asymptotics for the number of lattice points in the boxes $U^-_{r_1} Z_{r_2} a_t U^+_{r_3}$ as $t \to \infty$. This result is of independent interest, but it is not needed in the proof of the main theorem. We will assume that the action of $\{a_t\}$ on G/Γ is mixing. Due to the Howe–Moore theorem [14] on vanishing of matrix coefficients, the mixing condition is satisfied if Γ is an irreducible lattice in G. For example, this irreducibility condition is satisfied if G is a connected noncompact simple Lie group.

Theorem 3.5 *For every $r_1, r_2, r_3 > 0$,*

$$\#(\Gamma \cap U^-_{r_1} Z_{r_2} a_t U^+_{r_3}) \sim_{t\to\infty} \frac{\mathrm{vol}_G(U^-_{r_1} Z_{r_2} a_t U^+_{r_3})}{\mathrm{vol}_{G/\Gamma}(G/\Gamma)} = \frac{\lambda(\boldsymbol{r}) \det(\mathrm{Ad}\, a_t|_{\mathfrak{u}^+})}{\mathrm{vol}_{G/\Gamma}(G/\Gamma)},$$

where

$$\lambda(\boldsymbol{r}) = \mathrm{vol}_{U^-}(U^-_{r_1})\, \mathrm{vol}_{U^+}(U^+_{r_3}) \int\limits_{Z_{r_2}} \rho(z)\, dz,$$

$\rho(z) = |\det(\mathrm{Ad}\, z|_{\mathfrak{u}^+})|$, *and dz denotes the Haar integral on Z associated to* vol_Z.

We will prove this theorem using mixing of the flow a_t (as in [5,7]) and the following lemma:

Lemma 3.6 *For every $r_0 > 0$, there exist $l, \epsilon_0 > 0$ such that if $u \in U^-_{r_0}$, $z \in Z_{r_0}$ and $v \in U^+_{r_0}$, then for any $g \in G_s$ with $s \in (0, \epsilon_0)$ and $t > 0$, we have*

$$g(uza_t v) = (uu_1)(z_1 z) a_t (v_1 v),$$

where $u_1 \in U^-_{ls}$, $z \in Z_{ls}$, and $v_1 \in U^+_{ls}$.

Proof We have

$$g(uza_t v) = u(g_1 a_t) z v$$

where $g_1 = u^{-1} g u \in G_{l_1 s}$ for some $l_1 = l_1(r_0) > 0$. By Lemma 3.4,

$$g_1 a_t = u_1 z_1 a_t v_1$$

where $u_1 \in U^-_{l_2 s}$, $z_1 \in Z_{l_2 s}$ and $v_1 \in U^+_{l_2 s}$ for some $l_2 = l_2(r_0) > 0$. Hence,

$$g(uza_t v) = (uu_1)(z_1 z) a_t (v_2 v)$$

where $v_2 = z^{-1} v_1 z \in U^+_{l_3 s}$ for some $l_3 = l_3(r_0) > 0$. This implies the claim. □

Proof of Theorem 3.5 Let du, dz and dv denote the Haar integrals on U^-, Z and U^+, respectively. A Haar measure on G is defined by

$$\int_G f\,d\mu = \int_{U^-}\int_Z\int_{U^+} f(uzv)\rho(z)\,du\,dz\,dv, \quad \forall f \in \mathrm{C_c}(G). \tag{33}$$

Now given $\boldsymbol{r} = (r_1, r_2, r_3)$, we put $E_t(\boldsymbol{r}) = U^-_{r_1} Z_{r_2} a_t U^+_{r_3}$. Then

$$\mu(E_t(\boldsymbol{r})) = \mathrm{vol}_{U^-}(U^-_{r_1}) \left(\int_{Z_{r_2}} \rho(z)\,dz \right) \rho(a_t)\,\mathrm{vol}_{U^+}(U^+_{r_3}) = \lambda(\boldsymbol{r})\rho(a_t). \tag{34}$$

Let $l, \epsilon_0 > 0$ be as in Lemma 3.6. We use parameters $s \in (0, \epsilon_0)$, $r_i^\pm = r_i \pm ls$, and $\boldsymbol{r}^\pm = (r_1^\pm, r_2^\pm, r_3^\pm)$. Then by Lemma 3.6, for every $t > 0$,

$$E_t(\boldsymbol{r}^-) \subset \bigcap_{g\in G_s} gE_t(\boldsymbol{r}) \subset E_t \subset \bigcup_{g\in G_s} gE_t(\boldsymbol{r}) \subset E_t(\boldsymbol{r}^+). \tag{35}$$

Let $\bar{\mu}$ denote the finite G-invariant measure on G/Γ associated to μ. By our assumption the action of $\{a_t\}_{t>0}$ by left translations on G/Γ is mixing. In other words, if we put $y_0 = e\Gamma$, then given any $\phi \in \mathrm{C_c}(G/\Gamma)$ and small $r > 0$,

$$\frac{1}{\mu(U_r^- Z_r U_{r_3}^+)} \int_{U_r^-}\int_{Z_r}\int_{U_{r_3}^+} \phi(a_t uzvy_0)\rho(z)\,du\,dz\,dv \to \frac{1}{\bar{\mu}(G/\Gamma)} \int_{G/\Gamma} \phi\,d\bar{\mu}$$

at $t \to \infty$. Since $a_t uza_t^{-1} \to z$ as $t \to \infty$, and since $\rho(z) \to e$ as $z \to e$, from the uniform continuity of ϕ, we deduce that

$$\lim_{t\to\infty} \frac{1}{\mathrm{vol}_{U^+}(U_{r_3}^+)} \int_{U_{r_3}^+} \phi(a_t vy_0)\,dv = \frac{1}{\bar{\mu}(G/\Gamma)} \int_{G/\Gamma} \phi\,d\bar{\mu}.$$

Hence, in view of (33) and (34),

$$\lim_{t\to\infty} \frac{1}{\rho(a_t)} \int_{E_t(\boldsymbol{r})} \phi(gy_0)\,d\mu(g) = \frac{\lambda(\boldsymbol{r})}{\bar{\mu}(G/\Gamma)} \int_{G/\Gamma} \phi\,d\bar{\mu}. \tag{36}$$

Now as in [5,7], we introduce functions on G/Γ:

$$F_t(gy_0) = \sum_{\gamma\in\Gamma} \chi_{E_t(\boldsymbol{r})}(g\gamma) \quad \text{and} \quad F_t^\pm(gy_0) = \sum_{\gamma\in\Gamma} \chi_{E_t(\boldsymbol{r}^\pm)}(g\gamma).$$

We note that $F_t(y_0) = \#(\Gamma \cap E_t(\boldsymbol{r}))$. Let $\phi \in C_c(G/\Gamma)$ with $\mathrm{supp}(\phi) \subset G_s y_0$ and $\int_{G/\Gamma} \phi \, d\bar{\mu} = 1$. Then by (35),

$$\int_{G/\Gamma} F_t^-(y)\phi(y)\, d\bar{\mu}(y) \le F_t(y_0) \le \int_{G/\Gamma} F_t^+(y)\phi(y)\, d\bar{\mu}(y). \tag{37}$$

Using (36), we obtain

$$\begin{aligned} &\frac{1}{\rho(a_t)} \int_{G/\Gamma} F_t^{\pm}(y)\phi(y)\, d\bar{\mu}(y) \\ &= \int_{G/\Gamma} \sum_{\gamma\in\Gamma} \chi_{E_t^{\pm}(\boldsymbol{r})}(g\gamma)\phi(gy_0)\, d\bar{\mu}(g\Gamma) \\ &= \frac{1}{\rho(a_t)} \int_G \chi_{E_t(\boldsymbol{r}^{\pm})}(g)\phi(gy_0) d\mu(g) \to \frac{\lambda(\boldsymbol{r}^{\pm})}{\bar{\mu}(G/\Gamma)} \quad \text{as } t \to \infty. \end{aligned} \tag{38}$$

Since $\lambda(\boldsymbol{r}^+)/\lambda(\boldsymbol{r}^-) \to 1$ as $s \to 0$, from (37) and (38), we conclude that

$$\#(\Gamma \cap E_t(r)) = F_t(y_0) \sim_{t\to\infty} \frac{\lambda(\boldsymbol{r})\rho(a_t)}{\bar{\mu}(G/\Gamma)}$$

as required. □

4 Proof of the main theorems

To prove Theorem 1.4, we use a converse of the Borel–Cantelli lemma. It is well-known that such a converse holds under some quasi-independence condition. We will use the following version (see [18, Sect. 1], and also [11, Lemma 2.3], [17, Lemma 5] for more general results):

Proposition 4.1 *Let (Y, μ) be a finite measure space. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of measurable subsets of Y such that $\sum_{n=1}^{\infty} \mu(F_n) = \infty$. Suppose there exist $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that*

$$\mu(F_n \cap F_m) \le C\mu(F_n)\mu(F_m), \quad \forall m, n \in \mathbb{N},\ |m - n| \ge n_0. \tag{39}$$

Let $F = \cap_{n\in\mathbb{N}}(\cup_{m\ge n} F_m)$. Then $\mu(F) > 0$.

Proof of Theorem 1.4 First we suppose that

$$\sum_{n=1}^{\infty} \mathrm{vol}_{U^-}(\Psi_n) = \infty \quad \text{and} \quad t_{n+1} - t_n \ge \delta_0 > 0, \quad \forall n \in \mathbb{N}.$$

Let $r_0 > 0$ be as in Theorem 3.1. Since B has positive measure, there exist $z \in Z$ and $h \in H$ such that for every $r_0 > 0$, the set $B \cap z Z_{r_0} H_{r_0} h$ has positive measure as well. Let $B_0 := z^{-1} B h^{-1} \cap Z_{r_0} H_{r_0}$. We consider the sets

$$F_n = a_{t_n} \Psi_n z B_0 h y_0 = z a_{t_n} (z^{-1} \Psi_n z) B_0 h y_0 \subset G/\Gamma.$$

Recall that we are assuming that the sets Ψ_n are contained in a small neighborhood of identity. Hence, taking r_0 sufficiently small, the sets $(z^{-1}\Psi_n z) B_0 h$ project injectively on G/Γ, and

$$\sum_{n=1}^{\infty} \mathrm{vol}_{G/\Gamma}(F_n) = \infty.$$

By Theorem 3.1, there exist $n_0 \in \mathbb{N}$ and $C_1, C_2 > 0$ such that

$$\mathrm{vol}_{G/\Gamma}(F_k \cap F_l) \le C_1 \,\mathrm{vol}_{U^-}(\Psi_k)\, \mathrm{vol}_{U^-}(\Psi_l) \le C_2 \,\mathrm{vol}_{G/\Gamma}\, \mathrm{vol}(F_k)\, \mathrm{vol}_{G/\Gamma}(F_l)$$

for all $k, l \in \mathbb{N}$ such that $|k - l| \ge n_0$. Let $F = \cap_{n \in \mathbb{N}} \cup_{m \ge n} F_m$. Then by Proposition 4.1 applied to $Y = G/\Gamma$, we conclude that $\mathrm{vol}_{G/\Gamma}(F) > 0$. Now for any $y \in F$, we have that

$$\#(\{n \in \mathbb{N} : y \in a_{t_n} \Psi_n B y_0\}) \ge \#(\{n \in \mathbb{N} : y \in F_n\}) = \infty. \tag{40}$$

This proves the first part of the theorem.

To prove the second part, we assume that $\sum_{n=1}^{\infty} \mathrm{vol}_G(\Psi_n) < \infty$. If we set $F_n = a_{t_n} \Psi_n B y_0$, then

$$\sum_{n=1}^{\infty} \mathrm{vol}_{G/\Gamma}(F_n) < \infty.$$

Therefore by the Borel–Cantelli Lemma for almost all $y \in G/\Gamma$, we have that $y \in F_n$ for only finitely many $n \in \mathbb{N}$. This proves the second part of the theorem. □

Proof of Corollary 1.5 If we put $t_n = n t_0$, the condition of the first part of Theorem 1.4 is satisfied; and let the notation be as in the proof of this part given as above. Since $\Psi_{n+1} \subset \Psi_n$ for all $n \in \mathbb{N}$, we have $F_{n+1} \subset T(F_n)$, and $F \subset T(F)$. Hence, since T is ergodic, the set F has full measure. Now the claim follows from (40). □

To prove Theorem 1.2, we will need the following:

Proposition 4.2 *The Γ-action on ∂X is ergodic with respect to μ_∞.*

Proof We denote by G^0 the connected component of identity of G. The space ∂X is not connected in general, but it consists of at most two connected components, which are mapped to each other by the transformation $x \mapsto -x$. Since this transformation is in Γ, it suffices to show that $G^0 \cap \Gamma$ acts ergodically on the connected components of

∂X. Each connected component can be identified with a homogeneous space G^0/P of G^0 where P is a closed noncompact algebraic subgroup of G^0. Note that G^0 is a connected simple Lie group unless the signature of the quadratic form is (2, 2), and in the later case, G^0 is semisimple and one can check that the projection of P to the nontrivial simple factors of G^0 are noncompact. Hence, by Mautner's lemma, the P-action on $G^0/(G^0 \cap \Gamma)$ is ergodic with respect to the G^0-invariant probability measure. Therefore $(G^0 \cap \Gamma)$-action on G^0/P is ergodic with respect to the G^0-semi-invariant probability measure (see, for example, [8]). $\square$

Now we begin the proof of Theorem 1.2. We use notation as in Sect. 2. In particular, we recall that $G = \mathrm{O}(Q)$, $A = \{a_t\}$ is the one-parameter subgroup defined in (10), and $w_0 = (f_1 + (m/2) f_2) \in X$. Let $H = \mathrm{Stab}_G(w_0)$. Note that H is the set of fixed points of the involution $\sigma(g) = s_0 g s_0$, where $s_0 \in \mathrm{O}(Q)$ is given by

$$s_0 : f_1 \mapsto -\frac{m}{2} f_{d+1}, \quad f_{d+1} \mapsto -\frac{2}{m} f_1, \quad f_i \mapsto f_i, \quad i = 2, \ldots, d.$$

Moreover, $\sigma(a_t) = a_{-t}$. Let $\Gamma = G(\mathbb{Z})$, which is a lattice in G by the Borel–Harish-Chandra theorem. Hence, we are in the setting of Theorem 1.4. Note that $U^- = \{u(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^{d-1}\}$, where $u(\mathbf{s})$ is defined in (11).

Let $x_0 \in X(\mathbb{Z})$ and $g_0 \in G$ be such that $w_0 = g_0 x_0$.

Proof of Theorem 1.2(i) Suppose that

$$\int_1^\infty t^{d-2} \psi(t)^{d-1}\, dt = \int_0^\infty (e^t \psi(e^t))^{d-1}\, dt = \infty.$$

Then by Lemma 2.3(ii), there exists a measurable quasi-conformal function $\psi_1 \le \psi$ such that

$$e^t \psi_1(e^t) \to_{t\to\infty} 0, \quad \frac{\psi(t)}{\psi_1(t)} \to_{t\to\infty} \infty, \quad \int_0^\infty (e^t \psi_1(e^t))^{d-1}\, dt = \infty.$$

In view of Lemma 2.4 and (15), there exist $c, t_2 > 0$ such that

$$\bigcup_{t > t_2} a_t u(\mathcal{U}_t(\psi_1)) w_0 \subset C(\bar{f}_1, c\psi_1). \tag{41}$$

(Recall that $\mathcal{U}_t(\psi_1) \subset \mathbb{R}^{d-1}$ is as defined in (14).) Since ψ_1 is quasi-conformal, we have

$$\sum_{k=1}^\infty (e^k \psi_1(e^k))^{d-1} = \infty,$$

and by the discrete version of (16),

$$\sum_{k=1}^{\infty} \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_k(\psi_1)) = \infty.$$

For $p > 0$, let $\mathcal{T}_p = \mathcal{T} + (-p, p)$, where $\mathcal{T}$ is defined in (18). By the same argument as in (20),

$$\sum_{k \in \mathcal{T}_p \cap \mathbb{N}} \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_k(\psi_1)) < \infty.$$

It follows that there exists a sequence $k_n \in \mathbb{N} \backslash \mathcal{T}_p$, $k_n \to \infty$, such that

$$\sum_{n=1}^{\infty} \mathrm{vol}_{\mathbb{R}^{d-1}}(\mathcal{U}_{k_n}(\psi_1)) = \infty.$$

Let $r_0 > 0$ be as in Theorem 3.1, and $p > 0$ is such that $A_p := \{a_h : |h| < p\} \subset G_{r_0}$. Define

$$\mathcal{V}_t(\psi_1) = \bigcap_{|h|<p} e^{-h} \mathcal{U}_{t+h}(\psi_1).$$

Since $k_n \notin \mathcal{T}_p$, it follows from (19) that for sufficiently large n, $\mathcal{U}_{k_n+h}(\psi_1) = \tilde{\mathcal{U}}_{k_n+h}(\psi_1)$ when $|h| < p$, where $\tilde{\mathcal{U}}_t(\psi_1)$ is defined in (17). Hence, using that ψ_1 is quasi-conformal, we deduce that

$$\mathcal{V}_{k_n}(\psi_1) = \bigcap_{|h|<p} e^{-h} \tilde{\mathcal{U}}_{k_n+h}(\psi_1) = \bigcap_{|h|<p} \tilde{\mathcal{U}}_{k_n+h}(e^{-h}\psi_1) \supset \tilde{\mathcal{U}}_{k_n}(c_1\psi_1)$$

for some $c_1 > 0$. In particular,

$$\sum_{n=1}^{\infty} \mathrm{vol}_{\mathbb{R}^{d-1}} (\mathcal{V}_{k_n}(\psi_1)) = \infty. \tag{42}$$

Let $\Psi_n = u(\mathcal{V}_{k_n}(\psi_1))$, $B = A_p H_{r_0}$, and $B_n = \Psi_n B$ for all $n \in \mathbb{N}$. One can check that B is open in ZH, and in particular, it has positive measure. By Theorem 1.4, the set

$$E = \{z \in G/\Gamma : \#\{n \in \mathbb{N} : z \in a_{k_n} B_n g_0 \Gamma\} = \infty\}.$$

has positive measure. It follows that the set $\tilde{E} = \{g \in G : g\Gamma \in E\}$ has positive measure as well.

Let $g \in \tilde{E}$. There exist infinitely many $n \in \mathbb{N}$ such that $g\Gamma \cap a_{k_n} B_n g_0 \Gamma \neq \emptyset$. Hence, there are infinitely many elements of Γ in the set $\cup_{n \geq 1} g^{-1} a_{k_n} B_n g_0$. By (41),

$$a_{k_n} B_n g_0 x_0 = a_{k_n} B_n w_0 = a_{k_n} \Psi_n A_p w_0 = \bigcup_{|h|<p} a_{k_n+h} u(e^h \mathcal{V}_{k_n}(\psi_1)) w_0$$
$$\subset \bigcup_{|h|<p} a_{k_n+h} u(\mathcal{U}_{k_n+h}(\psi_1)) w_0 \subset C(\bar{f}_1, c\psi_1).$$

By Lemma 2.2, there exists $\kappa = \kappa(g) \geq 1$ such that

$$g^{-1} C(\bar{f}_1, c\psi_1) \subset C(\pi(g^{-1} f_1), \kappa c \psi_1).$$

Hence,

$$\#(\Gamma x_0 \cap C(\pi(g^{-1} f_1), \kappa c \psi_1)) = \infty.$$

This shows that $\pi(g^{-1} f_1) \in F$ for every $g \in \tilde{E}$, where

$$F = \{v \in \partial X : \exists \kappa \geq 1 \text{ such that } \#(\Gamma x_0 \cap C(v, \kappa c \psi_1)) = \infty\}.$$

Since $\tilde{E}$ has positive measure, we conclude that F has positive measure. It follows from Lemma 2.2 that F is Γ-invariant. Therefore, by Proposition 4.2, F has full measure.

For $v \in F$, there exists a sequence $x_n \in \Gamma x_0$ such that $\|x_n\| \to \infty$ and $x_n \in C(v, \kappa c \psi_1)$ for some $\kappa \geq 1$, i.e.,

$$\|\pi(x_n) - v\| \leq \kappa c \psi_1(\|x_n\|), \quad \forall n \in \mathbb{N}.$$

Since $\frac{\psi(t)}{\psi_1(t)} \to \infty$ as $t \to \infty$, it follows that for all sufficiently large n,

$$\|\pi(x_n) - v\| \leq \psi(\|x_n\|).$$

This shows that every element of F is (X, ψ)-approximable and completes the proof of the first part of Theorem 1.2. □

Proof of Theorem 1.2(ii) Suppose that

$$\int_0^\infty (e^t \psi(e^t))^{d-1} \, dt < \infty.$$

Then by Lemma 2.3(i), $e^t \psi(e^t) \to 0$ as $t \to \infty$. Let

$$W = \{v \in \partial X : \Gamma x_0 \cap C_T(v, \psi) \neq \emptyset \quad \forall T > 1\}.$$

Note that by the theorem of Borel and Harish-Chandra, the set $X(\mathbb{Z})$ is a union of finitely many Γ-orbits. Hence, the set of (X,ψ)-approximable points is a finite union of sets of the form W. It remains to show that W has measure zero.

Let $\tilde{W}=\{g\in G:\pi(gf_1)\in W\}$ and $\tilde{W}_0$ a bounded subset of $\tilde{W}$. By Lemma 2.2, there exists $\kappa>1$ such that

$$g^{-1}\Gamma x_0\cap C_T(\bar{f}_1,\kappa\psi)\neq\emptyset,\quad \forall g\in\tilde{W}_0,\quad T>1,$$

Then by Lemma 2.4, there exists $c_1>1$ such that

$$g^{-1}\Gamma\cap D_T(\bar{f}_1,c_1\psi)\neq\emptyset,\quad \forall g\in\tilde{W}_0,\quad T>1,$$

and by (15), there exists $c_2>1$ such that

$$g^{-1}\Gamma\cap\left(\bigcup_{t\geq T}a_t u(\mathcal{U}_t(c_2\psi))w_0\right)\neq\emptyset,\quad \forall g\in\tilde{W}_0,\quad T>1. \tag{43}$$

Let

$$B_T=\bigcup_{t\geq T}a_t u(\mathcal{U}_t(c_2\psi))Hg_0\Gamma/\Gamma\subset G/\Gamma.$$

Since $G\cong\mathrm{O}(p,q)$, we have that $H\cong\mathrm{O}(p-1,q)$ or $\mathrm{O}(p,q-1)$. Moreover, because $d=p+q\geq 4$, H is a semisimple group. Hence, $H_0=\mathrm{Stab}_G(x_0)=g_0^{-1}Hg_0$ is a semisimple group defined over $\mathbb{Q}$, and the space $Hg_0\Gamma/\Gamma=g_0H_0\Gamma/\Gamma$ admits a finite H-invariant measure. Then there exist constants $\kappa_1,\kappa_2>1$ such that

$$\mathrm{vol}_{G/\Gamma}(B_T)\leq\kappa_1\int_T^\infty\mathrm{vol}_{\mathbb{R}^{d-1}}(\tilde{\mathcal{U}}_t(c_2\psi))\,dt\leq\kappa_2\int_T^\infty(e^t\psi(e^t))^{d-1}<\infty.$$

Hence, $\mathrm{vol}_{G/\Gamma}(B_T)\to 0$ as $T\to\infty$. By (43), $\tilde{W}_0^{-1}\Gamma\subset B_T$ for all $T>1$. Therefore, $\mathrm{vol}_{G/\Gamma}(\tilde{W}_0^{-1}\Gamma)=0$. This implies that $\mathrm{vol}_G(\tilde{W}_0)=\mathrm{vol}_G(\tilde{W}_0^{-1})=0$. And hence $\mu_\infty(\pi(\tilde{W}f_1))=\mu_\infty(W)=0$. This completes the proof of Theorem 1.2. □

5 Examples

We give examples to illustrate that the main theorem holds only on a set of points of full measure, but not everywhere.

Let Q be a positive definite quadratic form with rational coefficients in d variables. In this section, we say that a vector $v\in\{Q=1\}$ is (ϵ,s)-approximable if there exist integer solutions of

$$\left\|v-\frac{x}{y}\right\|<\frac{\epsilon}{|y|^s},\quad Q(x)-y^2=-1 \tag{44}$$

with $|y| \to \infty$. When $d \geq 3$, our main theorem implies that for $\epsilon > 0$ and $s \in [0, 1]$, almost every $v \in \{Q = 1\}$ is (ϵ, s)-approximable, and for $\epsilon > 0$ and $s > 1$, almost every $v \in \{Q = 1\}$ is not (ϵ, s)-approximable.

Example 5.1 Let $d \geq 4$. Then there exist $\epsilon > 0$ and a vector $v \in \{Q = 1\}$ which is not $(\epsilon, 1)$-approximable.

Since $d \geq 4$, by the Meyer theorem, there exists a rational vector v such that $Q(v) = 1$. Let $k \in \mathbb{N}$ be such that $kv \in \mathbb{Z}^d$ and

$$\epsilon < \min\left\{\|z\| : z \in \frac{1}{k}\mathbb{Z}^d - \{0\}\right\}.$$

If for some $(x, y) \in \mathbb{Z}^{d+1}$,

$$\|yv - x\| < \epsilon,$$

then $x = yv$ and $Q(x) = y^2$. Hence, (44) fails.

Example 5.2 Let $d \geq 2$ and $Q(x) = \sum_{i=1}^{d} x_i^2$. Then there exists a vector $v \in \{Q = 1\}$ which is $(\epsilon, 2)$-approximable for $\epsilon > 2/\sqrt{7}$.

Let $v = (\sqrt{7}/4, 3/4, 0, \dots, 0)$. We claim that there are infinitely many integer solutions of

$$y^2 \sum_{i=1}^{d} (x_i - yv_i)^2 < \epsilon^2, \quad \sum_{i=1}^{d} (x_i^2 - y^2 v_i^2) = 1. \tag{45}$$

We use that there are infinitely many integer solutions (k_j, l_j), $k_j, l_j \to \infty$, of the Pell equation $k^2 - 7l^2 = 1$ and take

$$x_1^{(j)} = k_j, \quad y^{(j)} = 4l_j, \quad x_i^{(j)} = y^{(j)} v_i, \quad i > 1.$$

Then $(x^{(j)}, y^{(j)})$ satisfies the second condition in (45) and $x_1^{(j)} - y^{(j)} v_1 \to 0$ as $j \to \infty$. For $\epsilon > (2v_1)^{-1} = 2/\sqrt{7}$ and sufficient large j,

$$y^{(j)} = (2v_1)^{-1}((x_1^{(j)} + v_1 y^{(j)}) - (x_1^{(j)} - v_1 y^{(j)})) < \epsilon(x_1^{(j)} + v_1 y^{(j)}).$$

Then $(x^{(j)}, y^{(j)})$ satisfies the first condition in (45). This shows that v is $(\epsilon, 2)$-approximable.

Acknowledgments We would like to thank Anish Ghosh for useful discussions. N.S. would like to thank Caltech for hospitality where this work was started. A.G. would like to thank Princeton University for hospitality during the academic year 2007–2008 where part of this project was completed.

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