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Equidistribution of Translates of Curves on Homogeneous Spaces and Dirichlet's Approximation

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Abstract

Understanding the limiting distributions of translates of measures on submanifolds of homogeneous spaces of Lie groups lead to very interesting number theoretic and geometric applications. We explore this theme in various generalities, and in specific cases. Our main tools are Ratner's theorems on unipotent flows, nondivergence theorems of Dani and Margulis, and dynamics of linear actions of semisimple groups.

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1. Introduction

Several problems in number of theory and geometry involve more than one groups of symmetries or invariance in a direct or an indirect manner. Understanding the dynamics associated to interactions between these groups equips us with deeper new insights into these problems. The proof of Oppenheim conjecture on values of quadratic forms at integral points due to Margulis[16] via study of unipotent flows provided great impetus to the approach of solving number theoretic problems via homogeneous flows techniques. The work of Ratner [17, 18] on classification of invariant measures and orbit closures for unipotent flows as conjectured by Raghunathan and Dani [3] has created the foundation for this area. Since than significant progress and success have been

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achieved in this field by several authors in terms of deep number theoretic and dynamical theorems and powerful techniques. We will discuss a class of such results which are based on describing the limit distributions of sequences of translates of smooth measures on submanifolds in homogeneous spaces of Lie groups.

2. Counting Integral Points on Varieties and Translates of Closed Orbits of Subgroups

Let V be an affine algebraic subvariety of $\mathbb{R}^n$ defined over $\mathbb{Q}$. Let B be a bounded open convex set in $\mathbb{R}^{n-1}$ with smooth boundary. For $T > 0$, define

$$N(T, V) = \text{Cardinality}(V \cap \mathbb{Z}^n \cap TB).$$

In general, it is a difficult problem to estimate $N(T, V)$ as $T \to \infty$.

In [9] Duke, Rudnick and Sarnak observed that when V is an orbit of an algebraic semisimple $\mathbb{Q}$-group G acting linearly on $\mathbb{R}^n$, due a theorem of Borel and Harish-Chandra, $V \cap \mathbb{Z}^n$ is a union of finitely many orbits of a finite index subgroup, say Γ, of $G(\mathbb{Z})$. And hence, if $p \in V \cap \mathbb{Z}^n \neq \emptyset$, we want to obtain asymptotic estimate of

$$N(T, \Gamma p) = \text{Cardinality}(\Gamma p \cap TB)$$

as a function of T for large $T > 0$. Recognizing the role of symmetry and invariance groups in this problem, they noted that if H denotes the stabilizer of p, then under some natural conditions we might expect the following limit to hold:

$$\lim_{T\to\infty} \frac{N(T, \Gamma p)}{\mathrm{Vol}_{G/H}(\{gH \in G/H : g \in G,\ gp \in TB\})} = 1, \tag{1}$$

were the G-invariant $\mathrm{Vol}_{G/H}$ on G/H is determined by the choices of Haar measures on G and H such that $\mathrm{Vol}(G/\Gamma) = \mathrm{Vol}(H/H \cap \Gamma) = 1$.

In [9], they verified this limit for affine symmetric varieties V by introducing a counting technique, and relating it to the following equidistribution result.

Theorem 2.1 (Duke-Rudnick-Sarnak). *Let G be a non-compact simple Lie group, and H be a symmetric subgroup of G; that is, H is the fixed point set of an involutive automorphism (for example, a Cartan involution) of G. Let Γ be a lattice in G, and suppose that $H \cap \Gamma$ is a lattice in H. Let μ_G denote the G-invariant probability measure on G/Γ, and μ_H denote the H-invariant probability measure on G/Γ supported on $H\Gamma/\Gamma \cong H/H \cap \Gamma$. Then for any sequence $\{g_i\}$ in G which is divergent modulo H, we have*

$$\int_{g_i H\Gamma/\Gamma} f\, d(g_i\mu_H) := \int_{y \in H\Gamma/\Gamma} f(g_i y)\, d\mu_H(y) \xrightarrow{i\to\infty} \int_{G/\Gamma} f\, d\mu_G,$$

for any bounded continuous function f on G/Γ.

In other words, the sequence of translated measures $g_i\mu_H$ converge to μ_G in the space of probability measures on G/Γ with respect to the weak-$*$ topology.

The proof of this result in [9] is based on deep results of harmonic analysis of $L^2(G/\Gamma)$. Later Eskin and McMullen [10] deduced Theorem 2.1 as a geometric or a Lie theoretic consequence the mixing property of the sequence of g_i-actions on G/Γ.

The above counting problem and the equidistribution theorem, in view of Ratner's theorem[17] on unipotent flows, motivated the following more general result of [11].

Theorem 2.2 (Eskin-Mozes-Shah). *Let G and $H \subset G$ be connected real algebraic groups defined groups over $\mathbb{Q}$ and admitting no nontrivial $\mathbb{Q}$-characters. Let $\Gamma \subset G(\mathbb{Q})$ be a lattice in G. Let μ_G and μ_H be invariant probability measures G/Γ and $H\Gamma/\Gamma$, respectively. Suppose that for a sequence $\{g_i\}$ in G, the sequence of translated measures $g_i\mu_H$ converges to a probability measure λ on G/Γ with respect to the weak-* topology. Then there exists a $\mathbb{Q}$-subgroup L of G containing H and $c \in G$ such that*

(i) $\lambda = c\mu_L$, were μ_L is the L-invariant probability measure on $L\Gamma/\Gamma$; and

(ii) there exist sequences $\{\gamma_i\} \subset \Gamma$ and $c_i \to c$ in G such that $g_iH = c_i\gamma_iH$ and $\gamma_iH \subset L\gamma_i$ for all large i.

Thus any limit measure is algebraically defined, and the obstruction for this measure to be G-invariant can be algebraically explained.

To prove this theorem one shows that except for the case when g_i is bounded modulo $Z(H) \cap \Gamma$, there exists a sequence $X_i \in \mathrm{Lie}(H)$ such that $X_i \to 0$ and $(\mathrm{Ad}\, g_i)X_i \to Y \neq 0$ in $\mathrm{Lie}(G)$, and λ is invariant under the action of the one-parameter subgroup $\{\exp(tY) : t \in \mathbb{R}\}$. Since 0 is the only eigenvalue of Y, the measure λ is invariant under a unipotent one-parameter subgroup. Now Ratner's theorem describing such measures become applicable to this question.

In [11], using the counting technique introduced by Duke, Rudnick, and Sarnak, the above result was used for proving (1) under appropriate conditions for a wide class of varieties V, and in particular, when H is a maximal Q-subgroup of G. For example, we show the following:

Let $p(x) \in \mathbb{Z}[x]$ be an irreducible monic polynomial. Then the cardinality of the set of $n \times n$ integral matrices of norm at most T and having $p(x)$ as the characteristic polynomial is asymptotically equivalent to $cT^{n(n-1)/2}$, where $c > 0$ is a constant which can be described in terms of class number, regulator, and discriminant associated to the number field generated by a root of $p(x)$ (cf. [22]).

2.1. Expanding translates of smooth measures on horospherical leaves.

The work of Eskin and McMullen [10] also motivated the following result [21]:

Theorem 2.3 (Shah). *Let G be a noncompact simple Lie group, and $g \in G$ be a semisimple element not contained in a compact subgroup of G . Let $U = \{u \in G : g^{-n}ug^n \to e \text{ as } n \to \infty\}$ denote the expanding horospherical subgroup of g. Let L be a Lie group containing G, and Γ a lattice in L such that Gx_0 is dense in L/Γ, where $x_0 = e\Gamma$. Let λ be a probability measure on U which is absolutely continuous with respect to a Haar measure on U. Let $\bar{\lambda}$ be the pushforward of λ on Gx_0 under the map $h \mapsto hx_0$ from G to L/Γ. Then as $n \to \infty$, $a^n\bar{\lambda}$ converges weakly to μ_L, the L-invariant probability measure on L/Γ. In other words, for any bounded continuous function f on L/Γ,*

$$\lim_{n\to\infty} \int_{h\in U} f(g^n hx_0)\, d\lambda(h) = \int_{L/\Gamma} f\, d\mu_L.$$

The above result can be generalized as follows: Let

$$P^- = \{b \in G : \overline{\{g^n bg^{-n} : n \in \mathbb{N}\}} \text{ is compact}\}$$

denote the stable subgroup for g. Let λ be any probability measure on G such that the pushforward of λ on $P^-\backslash G$ is absolutely continuous. Let $\bar{\lambda}$ denote the pushforward of λ on Gx_0. Then $g^n\bar{\lambda}$ converges weakly to μ_L.

As a special case one generalizes Theorem 2.1 as follows: Let H be a symmetric subgroup of G, λ be a probability measure which is absolutely continuous with respect to a Haar measure on H, and $\bar{\lambda}$ denote the pushforward of λ on Hx_0. Then for any sequence $\{g_i\} \subset G$, which diverges modulo H, the sequence $g_i\bar{\lambda}$ converges weakly to μ_L as $i \to \infty$. This result has interesting consequences to equidistribution of dense orbits of lattices on homogeneous spaces [13, 12].

3. Limits of measures on stretching translates of submanifolds

In view of the results and notation of subsection 2.1, we ask the following question: Let M be an immersed submanifold of U with $\dim(M) < \dim(U)$ and λ be a probability measure on M which is absolutely continuous with respect to a smooth measure on M. Let $\bar{\lambda}$ denote the pushforward of λ on Gx_0. Under what condition on the geometric shape of M we have that $g^n\bar{\lambda} \to \mu_L$ as $n \to \infty$?

3.0.1. An algebraic obstruction to the limit of $g^n\bar{\lambda}$ being equal to μ_L. Define

$$P_L^- = \{b \in L : \overline{\{g^n bg^{-n} : n > 0\}} \text{ is compact}\}.$$

Suppose that H is a proper subgroup of L containing g, and $q \in L$ is such that the orbit Hqx_0 is closed and carries a finite H-invariant measure. Suppose that $M \subset U \cap P_L^- Hq$. Then any weak-* limit of probability measures $g^n\bar{\lambda}$ is a direct integral of measures which are supported on closed sets of the form

$bHqx_0$, where $b \in P_L^-$ is such that $\overline{\{g^n b g^{-n} : n < 0\}}$ is compact. Such limiting measures are concentrated on strictly low dimensional submanifolds of L/Γ.

We ask if this is the only condition on the geometric shape of M. In the remaining article we will show that this is indeed the case in certain specific situations, and obtain new number theoretic and geometric consequences.

3.1. Translates of a finite arc under geodesic flow.

Let $G = \mathrm{SO}(n,1)$ and $\{a_t\}$ be a connected maximal $\mathbb{R}$-diagonalizable subgroup of G. Let $P^- = \{b \in G : \overline{\{a_t b a_{-t} : t > 0\}}$ is compact$\}$ and U be the corresponding expanding horospherical subgroup of G. Here $P^-\backslash G \cong \mathbb{S}^{n-1}$ and $U \cong \mathbb{R}^{n-1}$, and the map $u \mapsto P^- u$ from U to $P^-\backslash G$ correspond to the inverse-stereographic projection, and the right action of G on $P^-\backslash G \cong \mathbb{S}^{n-1}$ is via conformal transformations. If H is a proper closed subgroup of G containing $\{a_t\}$ and some nontrivial unipotent subgroup, then $P^- H$ correspond to a proper subsphere of $\mathbb{S}^{n-1}$. Therefore $U \cap P^- H g$ is an affine subspace or a subsphere in $U \cong \mathbb{R}^{n-1}$. In [23] we show the following:

Theorem 3.1 (Shah). *Let $\phi : (0,1) \to U$ be an analytic map such that $\phi(0,1)$ is not contained in a proper subsphere or a proper affine subspace. Then for any lattice Γ in G, $x \in G/\Gamma$ and any bounded continuous function f on G/Γ,*

$$\lim_{t \to \infty} \int_0^1 f(a_t \phi(s) x)\, ds = \int_{G/\Gamma} f \, d\mu_G$$

where μ_G is the G-invariant probability measure on G/Γ.

The above result was generalized for smooth maps in [24]. We can obtain its following geometric application:

Let $\mathbb{H}^n$ denote the hyperbolic n-ball. Let $\Gamma \subset \mathrm{SO}(n,1)$ be a torsion free discrete group of isometries of $\mathbb{H}^n$ such that the hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ has finite Riemannian volume. Let $\pi : T^1(\mathbb{H}^n) \to T^1(M)$ denote the natural quotient map of the unit tangent bundles, and let g_t denote the geodesic flow on $T^1(M)$. For $v \in T^1(\mathbb{H}^n)$, let $v^+ \in \partial\mathbb{H}^n$ denote the end of the directed geodesic starting from v.

Theorem 3.2 (Shah). *Let $\phi : [0,1] \to T^1(\mathbb{H}^n)$ be a continuous map such that the map $s \mapsto \phi(s)^+ : (0,1) \to \partial\mathbb{H}^n$ is C^1 and its derivative $d\phi(s)^+/ds$ is Lipschitz and nonzero for almost all s. Suppose that the set $\{s \in (0,1) : \phi(s)^+ \in S\}$ has zero Lebesgue measure for any proper subsphere $S \subset \partial\mathbb{H}^n$ such that S is the boundary of an isometric copy of $\mathbb{H}^k$ $(2 \leq k < n)$ in $\mathbb{H}^n$ whose image on M is a closed subset. Then for any bounded continuous function f on $T^1(M)$,*

$$\lim_{t \to \infty} \int_0^1 f(g_t \pi(\phi(s))\, ds = \int_{T^1(M)} f \, d\tilde{\mu}_M,$$

where $\tilde{\mu}_M$ is the probability measure on $T^1(M)$ corresponding to the natural Riemannian volume form on $T^1(M)$.

When ϕ is analytic, the condition of the theorem holds if the image of ϕ^+ is not contained in a proper subsphere of $\partial\mathbb{H}^n$.

4. Applications to Diophantine approximation

The above study was also prompted by the following result due to Kleinbock and Margulis [14]: Let $n \geq 2$ and $\Omega := \{g\mathbb{Z}^n : g \in \mathrm{SL}(n,\mathbb{R})\} \cong \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ denote the space unimodular lattices in $\mathbb{R}^n$. Given $\epsilon > 0$, define $\Omega(\epsilon) = \{\Lambda \in \Omega : \|\boldsymbol{v}\| \geq \epsilon, \forall \boldsymbol{v} \in \Lambda \smallsetminus \{0\}\}$. Then $\Omega(\epsilon)$ is compact, and $\cup_{\epsilon>0}\Omega(\epsilon) = \Omega$.

For $\boldsymbol{t} = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $\boldsymbol{v} = (v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$, define

$$a(\boldsymbol{t}) = \begin{bmatrix} e^{t_1+\cdots+t_{n-1}} & & & \\ & e^{-t_1} & & \\ & & \ddots & \\ & & & e^{-t_{n-1}} \end{bmatrix}, \quad u(\boldsymbol{v}) = \begin{bmatrix} 1 & v_1 & \dots & v_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Theorem 4.1 (Kleinbock-Margulis). *Let $\phi : (0,1) \to \mathbb{R}^{n-1}$ be a non-degenerate C^n-map; that is, for almost all $t \in (0,1)$, the derivatives $\phi^{(i)}(t)$, $1 \leq i \leq n-1$, span $\mathbb{R}^{n-1}$. Then there exist constants $C > 0$ and $\alpha > 0$ such that*

$$\ell(\{s \in (0,1) : a(\boldsymbol{t})u(\phi(s))\mathbb{Z}^n \notin \Omega(\epsilon)\}) \leq C\epsilon^\alpha, \quad \forall \epsilon > 0, \forall \boldsymbol{t} \in \mathbb{R}_+^{n-1}.$$

Kleinbock and Margulis [14] used this result to settle conjectures on metric properties of diophantine approximation on submanifolds of $\mathbb{R}^n$ due to Mahler, Sprindzuk and Baker.

The result raises the following dynamical question: Let ν denote the push-forward of the Lebesgue measure on $(0,1)$ under the map $s \mapsto u(\phi(s))x_0$ on Ω. Let $\boldsymbol{t}_i \in \mathbb{R}_+^{n-1}$ be a sequence such that all coordinates of $\boldsymbol{t}_i$ tend to infinity. Then as $i \to \infty$, does the measure $a(\boldsymbol{t}_i)\nu$ tend to μ, the unique $\mathrm{SL}(n,\mathbb{R})$-invariant probability measure on Ω?

It was observed by Kleinbock and Weiss [15] that an affirmative answer to this question would resolve a problem proposed by Davenport and Schmidt [7] in the late 60's on non-improvability of Dirichlet's simultaneous approximation theorem. To describe the problem, consider the following definition:

Given $\lambda > 0$ we say that $\boldsymbol{\xi} \in \mathbb{R}^k$ is $\mathrm{DT}(\lambda)$ if for all but finitely many $N \in \mathbb{N}$, there exist $0 \neq \boldsymbol{q} = (q_1, \dots, q_k) \in \mathbb{Z}^k$ and $p \in \mathbb{Z}$ such that

$$|\boldsymbol{q} \cdot \boldsymbol{\xi} + p| \leq \lambda/N^k \quad \text{and} \quad |q_i| \leq N, \forall i. \tag{2}$$

Similarly, we say that $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$ is $\mathrm{DT}'(\lambda)$ if for all but finitely many $N \in \mathbb{N}$, there exist $0 \neq q \in \mathbb{Z}$ and $\boldsymbol{p} \in \mathbb{Z}^k$ such that

$$|q\xi_i + \boldsymbol{p}| \leq \lambda/N, \ \forall i, \quad \text{and} \quad |q| \leq N^k.$$

Dirichlet's simultaneous approximation theorem states that every $\boldsymbol{\xi} \in \mathbb{R}^k$ is $\mathrm{DT}(1)$ and $\mathrm{DT}'(1)$. Davenport and Schmidt [6] showed that for any $\lambda < 1$, almost every $\boldsymbol{\xi} \in \mathbb{R}^k$ is not DT(λ) and not DT$'$(λ). In [7] they showed that for almost any $\xi \in \mathbb{R}$ the vector (ξ, ξ^2) is not DT(1/4). The result was generalized by Baker [1] for points on more general curves on $\mathbb{R}^2$, by Dodson, Rynne and Vickers [8] for points on 'low co-dimensional curved submanifolds' of $\mathbb{R}^n$, by Bugeaud [2] for the curve $(\xi, \xi^2, \ldots, \xi^n)$, and by Kleinbock and Weiss [15] for all nondegenerate curves on $\mathbb{R}^k$. In each case, it was proved that almost all points of the parametrized submanifold with respect to the parameter measure are not DT(λ) for some very small value of $\lambda > 0$ depending on the submanifold.

In [25] we provide the following answer to the above problem:

Theorem 4.2 (Shah). *Let B be a ball in $\mathbb{R}^d$ for some $d \geq 1$, and $\phi : B \to \mathbb{R}^k$ be an analytic map whose image is not contained in a proper affine subspace of R^k. Then for almost every $b \in B$, the point $\phi(b)$ is neither* DT*(λ) nor* DT$'$*(λ) for any $\lambda < 1$.*

The above statement is a consequence of the following equidistribution result [25]:

Theorem 4.3 (Shah). *Let L be any Lie group and $\rho : G = \mathrm{SL}(n, \mathbb{R}) \to L$ be a continuous homomorphism. Let Γ be a lattice in L. Let B be a bounded open subset in $\mathbb{R}^d$ ($d \geq 1$). Let $\phi : B \to \mathrm{SL}(n, \mathbb{R})$ be an analytic map such that the image of the first row of this map is not contained in a proper subspace of $\mathbb{R}^n$. Put $a_t = a((t, t, \ldots, t)) \in \mathrm{SL}(n, \mathbb{R})$ ($t \in \mathbb{R}$). Let $x \in L/\Gamma$ and suppose that $\rho(G)x$ is dense in L/Γ. Then for a bounded continuous function f on L/Γ,*

$$\lim_{t \to \infty} \frac{1}{\mathrm{Vol}(B)} \int_{b \in B} f(\rho(a_t u(\phi(b))x)\, db = \int_{L/\Gamma} f \, d\mu_L, \tag{3}$$

where db denotes the Lebesgue integral on $\mathbb{R}^d$, and μ_L is the L-invariant probability measure on L/Γ.

4.0.1. Expanding translates of shrinking submanifolds. Fix any $b \in B$ and let B_t denote a ball of radius e^{-t} about b. If B is replaced by the shrinking balls B_t in (3) then we still expect the limiting measure to be μ_L. This has been verified in the case of $n = 3$. This type of result would allows us to deduce the above theorem when ϕ to is a non-degenerate C^n curve as in Theorem 4.1.

4.1. Multiplicative Dirichlet-Minkowski approximation.

The following generalization of Dirichlet's theorem is known as Minkowski's theorem on simultaneous approximation of Linear forms: For $n \geq 2$, let $(\phi_{ij}) \in \mathrm{SL}(n, \mathbb{R})$. Let $\alpha_1, \ldots, \alpha_n > 0$ be such that $\alpha_1 \cdots \alpha_n = 1$. Then there exist $x_1, \ldots, x_n \in \mathbb{Z}$, not all 0s, such that

$$|\phi_{11}x_1 + \cdots + \phi_{1n}x_n| \leq \alpha_1; \; |\phi_{i1}x_1 + \cdots + \phi_{in}x_n| < \alpha_i \quad (i \geq 2). \tag{4}$$

By putting $\phi_{11} = \cdots = \phi_{nn} = 1$ and $\phi_{ij} = 0$ for $i \geq 2$ and $j \neq i$, we get a multiplicative version Dirichlet's theorem. Now we define the corresponding

λ-version: For $k = n-1$, let $\mathcal{N} \subset \mathbb{N}^k$ be an infinite sequence and $0 < \lambda \leq 1$. We say that $(\xi_1, \ldots, \xi_k) \in \mathbb{R}^k$ is MDT(λ) along $\mathcal{N}$ if for all but finitely many $(N_1, \ldots, N_k) \in \mathcal{N}$, there exist $q_1, \ldots, q_k \in \mathbb{Z}$, not all zero, and $p \in \mathbb{Z}$ such that

$$|p + q_1\xi_1 + \cdots + q_k\xi_k| \leq \lambda/(N_1 N_2 \ldots N_k) \text{ and } |q_i| < N_i,\ \forall i. \tag{5}$$

We also define MDT$'(\lambda)$ in a similar way. Minkowski's result implies that all points are MDT(1) and MDT$'$(1) along any $\mathcal{N}$.

Kleinbock and Weiss [15] proved that if each coordinate projection of $\mathcal{N}$ is a divergent sequence then almost all $\boldsymbol{\xi} \in \mathbb{R}^k$ are neither MDT(λ) nor MDT$'(\lambda)$ along $\mathcal{N}$ for any $\lambda < 1$. They also showed that given a non-degenerate smooth curve in $\mathbb{R}^k$, there exists a very small $\lambda > 0$ so that for almost every $\boldsymbol{\xi}$ on this curve is not MDT(λ) along $\mathcal{N}$.

For analytic curves not contained in proper affine subspaces of $\mathbb{R}^k$ we extend their result for any $\lambda < 1$ in [26] as follows:

Theorem 4.4 (Shah). *Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}^k$. Let B be an open ball in $\mathbb{R}^d$ and $\phi : B \to \mathbb{R}^k$ be an analytic map whose image is not contained in a proper affine subspace. Then for almost all $b \in B$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and $\lambda < 1$ there exist infinitely many $(N_1, \ldots, N_k) \subset \mathcal{N}$ such that both the following sets of inequalities are simultaneously insoluble:*

$$|q_1\phi_1(b) + \cdots + q_k\phi_k(b) + p| \leq \lambda/(N_1 \cdots N_k), \quad |q_i| \leq N_i\ (\forall i), \tag{6}$$

for $p, q_1, \ldots, q_k \in \mathbb{Z}$, not all zeros; and

$$|q\phi_i(b) + p_i| \leq \lambda N_i^{-1}\ (\forall i), \quad |q| \leq N_1 N_2 \ldots N_k, \tag{7}$$

for $p_1, \ldots, p_k, q \in \mathbb{Z}$, not all zeros.

In particular, $\phi(b)$ is neither MDT(λ) nor MDT$'(\lambda)$ along $\mathcal{N}$ for any $\lambda < 1$ and almost all $b \in B$.

It may be noted that, due to a theorem of Minkowski and Hajosh on critical lattices, the analogue of the above theorem on multiplicative non-improvability along $\mathcal{N}$ fails to hold if we take an unbounded sequence $\mathcal{N}$ contained $(\mathbb{R}_+)^k$ such that one of the coordinates of $\mathcal{N}$ converges to an element of $\mathbb{R} \setminus \mathbb{N}$ (see [26]).

The deductions of the above results are based on the following relation between the approximation inequality and matrix action on the space of unimodular lattices in $\mathbb{R}^{k+1}$ (see[4, 14, 15]); that is, the inequalities (5) are equivalent to

$$\begin{bmatrix} N_1\cdots N_k & & & \\ & N_1^{-1} & & \\ & & \ddots & \\ & & & N_k^{-1} \end{bmatrix} \begin{bmatrix} 1 & \xi_1 & \ldots & \xi_{n-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} p \\ q_1 \\ \vdots \\ q_k \end{bmatrix} \in [-\lambda, \lambda] \times [-1, 1]^k,$$

or in other words $a(\boldsymbol{t})u(\boldsymbol{\xi})x_0 \in L_\lambda$, where $\boldsymbol{t} = (\log N_1, \ldots, \log N_k)$, $x_0 = \mathbb{Z}^n \in \Omega$, and

$$L_\lambda = \{g\mathbb{Z}^n \in \Omega : g \in \mathrm{SL}(n, \mathbb{R}),\ g\mathbb{Z}^n \cap [-\lambda, \lambda] \times [-1, 1]^k \neq \{0\}\}$$

is the complement of a nonempty open subset of Ω if $0 < \lambda < 1$. In view of this relation, the dynamical result needed to prove theorem 4.4 is as follows [26]: Given an unbounded sequence $\{\boldsymbol{t}_i\}$ in $\mathbb{R}_+^{n-1}$, after permuting coordinates and passing to a subsequence, we will assume that its first m coordinate projections are divergent sequences ($1 \leq m \leq n-1$), and its remaining $(n-1-m)$ coordinate projections are convergent sequences. Let

$$Q = \left\{ (g_{i,j}) \in \mathrm{SL}(n, \mathbb{R}) : \text{for } i > m+1, \begin{array}{ll} g_{i,j} = 0 & \text{if } j \neq i \\ g_{i,i} = 1 & \end{array} \right\}. \tag{8}$$

Then as $i \to \infty$, $a(\boldsymbol{t}_i)Q \to a(\boldsymbol{t}_0)Q$ in $\mathrm{SL}(n,\mathbb{R})/Q$ for some $\boldsymbol{t}_0 \in \mathbb{R}^{n-1}$. In particular, if all coordinates of $\boldsymbol{t}_i$ are divergent then $Q = G$ and $\boldsymbol{t}_0 = 0$.

Theorem 4.5 (Shah). *Let B be a bounded open subset of $\mathbb{R}^d$ $(d < n)$. Let $\phi : B \to \mathbb{R}^{n-1}$ be an analytic map whose image is not contained in a proper affine subspace. Let L be a Lie group, $\rho : \mathrm{SL}(n, \mathbb{R}) \to L$ be a continuous homomorphism, and Γ be a lattice in L. Let $\{\boldsymbol{t}_i\}$ be a sequence as above. Let $x \in L/\Gamma$. Then for any bounded continuous function f on L/Γ,*

$$\lim_{i\to\infty} \frac{1}{\mathrm{Vol}(B)} \int_B f(\rho(a(\boldsymbol{t}_i)u(\phi(b)))x)\, db = \int_{y \in Hx} f(\rho(a(\boldsymbol{t}_0))y)\, d\mu_H(y),$$

where H is the smallest closed subgroup of L containing $\rho(Q)$ such that Hx is closed and admits an H-invariant probability measure, say μ_H.

5. Unipotent flows, Linearization and Linear dynamics

To prove the above dynamical results one shows that if λ is the normalized parameter measure on the submanifold $\rho(u(\phi(B)))x$ of L/Γ, which is being translated by a sequence $g_i = \rho(a(\boldsymbol{t}_i))$, and if we prove that $g_i\lambda$ converges to a measure μ on L/Γ, then μ turns out to be a direct integral of finite measures which are invariant under actions unipotent subgroups of G. Due to Ratner's measure classification theorem, if μ is not L-invariant, then μ is strictly positive on the image of a proper algebraic subvariety, say $\mathcal{V}$ of L projected to L/Γ. This variety is right invariant under certain subgroup, say N, containing unipotents and such that $N\Gamma$ is closed. At this stage one applies linearization technique [19, 5, 20] in conjunction with functions of (C, α)-growth (as introduced in [14]) to show that for each $a(\boldsymbol{t}_i)$ there exists $\gamma_i \in L$ stabilizing x such that $\rho(a(\boldsymbol{t}_i)u(\phi(B)))\gamma_i$, a lift of the entire translated trajectory, lives in a thin neighbourhood of the subvariety $\mathcal{V}$ in L modulo N . At this stage we invoke the following new observation of linear dynamical nature, to deduce that there exist some fixed $\gamma \in L$ stabilizing x such that $\rho(a(\boldsymbol{t}_i)u(\phi(B))\gamma$ gets arbitrarily close to $\mathcal{V}$ in L modulo N. The linear dynamical observation, which turns out to be one of the most crucial part of the argument, is as follows [23, 24, 25, 26]:

Theorem 5.1 (Shah). *Let $\phi : (0,1) \to \mathbb{R}^{n-1}$ be a C^1-map such that for some interval $B \subset (0,1)$, $\phi(B)$ is not contained in a proper affine subspace of $\mathbb{R}^{n-1}$. Suppose that* $\mathrm{SL}(n,\mathbb{R})$ *acts linearly on a finite dimensional vector space V. Let a sequence $\{\boldsymbol{t}_i\}$ and the associated subgroup Q be as in* (8). *Then for any $v \in V$ which is not fixed by Q, and any compact set $C \subset V$,*

$$a_{\boldsymbol{t}_i} u(\phi(B))v \not\subset C \qquad \text{for all large } i. \tag{9}$$

Note that if v is fixed by Q then $a_{\boldsymbol{t}_i} u(\phi(B))v = a_{\boldsymbol{t}_i} v \to a_{\boldsymbol{t}_0} v$ as $i \to \infty$.

Our proof of this result uses the description of finite dimensional representations of $\mathrm{SL}(2,\mathbb{R})$ to understand the intertwined linear dynamics of various copies of $\mathrm{SL}(2,\mathbb{R})$s and $\mathrm{SL}(m,\mathbb{R})$s sitting in $\mathrm{SL}(n,\mathbb{R})$.

In the case when ϕ is a nondegenerate C^n-map, we expect that (9) will hold even if we put B_i in place of B where B_i's are intervals around some $s \in (0,1)$ shrinking at some specific rate depending on $a(\boldsymbol{t}_i)$. For example, in the case when $\boldsymbol{t}_i = (t_i, \dots, t_i)$ (all same coordinates) then we can shrink B_i (around any s except for finitely many $s \in B$) at the rate of e^{-t_i} as $i \to \infty$, and (9) can be expected to hold.

The basic strategy behind the dynamical theorems of the previous section is that in very general situations, using Ratner's theorem and Linearization techniques we can reduce the equidistribution problem to a problem about 'Dynamics of subgroup actions on finite dimensional linear representations'. At that stage n we need to prove the results that are very similar to Theorem 5.1, possibly with B also shrinking at a very specific rate as $i \to \infty$. Proving a suitable linear dynamical result remains to be the the main difficulty in describing the limiting distributions of stretching translates of submanifolds on homogeneous spaces of very general Lie groups.

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References

[1] R. C. Baker. Dirichlet's theorem on Diophantine approximation. *Math. Proc. Cambridge Philos. Soc.*, 83(1):37–59, 1978.

[2] Yann Bugeaud. Approximation by algebraic integers and Hausdorff dimension. *J. London Math. Soc. (2)*, 65(3):547–559, 2002.

[3] S. G. Dani. Invariant measures of horospherical flows on noncompact homogeneous spaces. *Invent. Math.*, 47(2):101–138, 1978.

[4] S. G. Dani. Divergent trajectories of flows on homogeneous spaces and Diophantine approximation. *J. Reine Angew. Math.*, 359:55–89, 1985.

[5] S. G. Dani and G. A. Margulis. Limit distributions of orbits of unipotent flows and values of quadratic forms. In *I. M. Gelfand Seminar*, pages 91–137. Amer. Math. Soc., Providence, RI, 1993.

[6] H. Davenport and W. M. Schmidt. Dirichlet's theorem on diophantine approximation. II. *Acta Arith.*, 16:413–424, 1969/1970.

[7] H. Davenport and Wolfgang M. Schmidt. Dirichlet's theorem on diophantine approximation. In *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, pages 113–132. Academic Press, London, 1970.

[8] M. M. Dodson, B. P. Rynne, and J. A. G. Vickers. Dirichlet's theorem and Diophantine approximation on manifolds. *J. Number Theory*, 36(1):85–88, 1990.

[9] W. Duke, Z. Rudnick, and P. Sarnak. Density of integer points on affine homogeneous varieties. *Duke Math. J.*, 71(1):143–179, 1993.

[10] Alex Eskin and Curt McMullen. Mixing, counting, and equidistribution in Lie groups. *Duke Math. J.*, 71(1):181–209, 1993.

[11] Alex Eskin, Shahar Mozes, and Nimish Shah. Unipotent flows and counting lattice points on homogeneous varieties. *Ann. of Math. (2)*, 143(2):253–299, 1996.

[12] Alex Gorodnik and Barak Weiss. Distribution of lattice orbits on homogeneous varieties. *Geom. Funct. Anal.*, 17(1):58–115, 2007.

[13] Alexander Gorodnik and Hee Oh. Orbits of discrete subgroups on a symmetric space and the Furstenberg boundary. *Duke Math. J.*, 139(3):483–525, 2007.

[14] D. Y. Kleinbock and G. A. Margulis. Flows on homogeneous spaces and Diophantine approximation on manifolds. *Ann. of Math. (2)*, 148(1):339–360, 1998.

[15] Dimitry Kleinbock and Barak Weiss. Dirichlet's theorem on diophantine approximation and homogeneous flows. *Journal of Modern Dynamics (JMD)*, 2(1):43–62, 2008.

[16] G. A. Margulis. Discrete subgroups and ergodic theory. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pages 377–398. Academic Press, Boston, MA, 1989.

[17] Marina Ratner. On Raghunathan's measure conjecture. *Ann. of Math. (2)*, 134(3):545–607, 1991.

[18] Marina Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991.

[19] Nimish A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289(2):315–334, 1991.

[20] Nimish A. Shah. Limit distributions of polynomial trajectories on homogeneous spaces. *Duke Math. J.*, 75(3):711–732, 1994.

[21] Nimish A. Shah. Limit distributions of expanding translates of certain orbits on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 106(2):105–125, 1996.

[22] Nimish A. Shah. Counting integral matrices with a given characteristic polynomial. *Sankhyā Ser. A*, 62(3):386–412, 2000. Ergodic theory and harmonic analysis (Mumbai, 1999).

[23] Nimish A. Shah. Limiting distributions of curves under geodesic flow on hyperbolic manifolds. *Duke Math. J.*, 148(2):251-279, 2009.

[24] Nimish A. Shah. Asymptotic evolution of smooth curves under geodesic flow on hyperbolic manifolds. *Duke Math. J.*, 148(2):281-304, 2009.

[25] Nimish A. Shah. Equidistribution of expanding translates of curves and Dirichlets theorem on Diophantine approximation. *Invent. Math.*, 177(3):509-532, 2009.

[26] Nimish A. Shah. Expanding translates of curves and Dirichlet-Minkowski theorem on linear forms. *J. Amer. Math. Soc.*, 23:563–589, 2010.