Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

EQUIDISTRIBUTION OF CURVES IN HOMOGENEOUS SPACES AND DIRICHLET'S APPROXIMATION THEOREM FOR MATRICES

Nimish Shah

Department of Mathematics, The Ohio State University
Columbus, OH 43210, USA

Lei Yang*

College of Mathematics, Sichuan University
Chengdu, Sichuan, 610065, China

(Communicated by Shouchuan Hu)

Abstract. In this paper, we study an analytic curve $\varphi : I = [a,b] \to \mathrm{M}(m \times n, \mathbb{R})$ in the space of m by n real matrices, and show that if φ satisfies certain geometric condition, then for almost every point on the curve, the Diophantine approximation given by Dirichlet's Theorem can not be improved. To do this, we embed the curve into a homogeneous space G/Γ, and prove that under the action of some expanding diagonal subgroup $A = \{a(t) : t \in \mathbb{R}\}$, the translates of the curve tend to be equidistributed in G/Γ, as $t \to +\infty$. The proof relies on the linearization technique and representation theory.

1. INTRODUCTION.

1.1. Diophantine approximation for matrices. In 1842, Dirichlet proved the following result on simultaneous approximation of a matrix of real numbers by integral vectors: *Given two positive integers m and n, a matrix $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$, and any $N > 0$, there exist integral vectors $\mathbf{p} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $\mathbf{q} \in \mathbb{Z}^m$ such that*

$$\|\mathbf{p}\| \leq N^m \quad \text{and} \quad \|\Phi\mathbf{p} - \mathbf{q}\| \leq N^{-n}, \tag{1.1}$$

where $\|\cdot\|$ denotes the supremum norm; that is, $\|\mathbf{x}\| := \max_{1 \leq i \leq k} |x_i|$ for $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$.

Now we consider the following finer question: for a particular m by n matrix Φ, could we improve Dirichlet's Theorem? By improving Dirichlet's Theorem, we mean that there exists a constant $0 < \mu < 1$, such that for all large $N > 0$, there exists nonzero integer vector $\mathbf{p} \in \mathbb{Z}^n$ with $\|\mathbf{p}\| \leq \mu N^m$, and integer vector $\mathbf{q} \in \mathbb{Z}^m$ such

E-mail address: `shah@math.osu.edu, lyang861028@gmail.com`.

2010 *Mathematics Subject Classification.* Primary: 11J13, 11J83; Secondary: 22E40, 37A17.

Key words and phrases. Dirichlet's theorem, Diophantine approximation, homogeneous spaces, equidistribution, Ratner's theorem.

This material is based upon work supported by the National Science Foundation under Grant Nos. 1301715 and 1700394.

The second author is supported in part by ISF grant 2095/15, ERC grant AdG 267259, NSFC grant 11743006, 11801384 and the Fundamental Research Funds for the Central Universities YJ201769.

* Corresponding author: Lei Yang.

that $\|\Phi\mathbf{p}-\mathbf{q}\| \leq \mu N^{-n}$. If such constant μ exists, then we say Φ is DT_μ-improvable. If Φ is DT_μ-improvable for some $0 < \mu < 1$, then we say Φ is DT-improvable (here DT stands for Dirichlet's Theorem).

This problem was first studied by Davenport and Schmidt in [7], in which they proved that almost every matrix $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is not DT-improvable. In [7], they also proved the following result. For $m = 1$ and $n = 2$, $\mathrm{M}(1 \times 2, \mathbb{R}) = \mathbb{R}^2$, one considers the curve $\phi(s) = (s, s^2)$ in $\mathbb{R}^2$. Then for almost every $s \in \mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$, $\phi(s)$ is not $DT_{1/4}$ improvable. This result was generalized by Baker in [3]: for any smooth curve in $\mathbb{R}^2$ satisfying some curvature condition, almost every point on the curve is not DT_μ improvable for some $0 < \mu < 1$ depending on the curve. Bugeaud [4] generalized the result of Davenport and Schmidt in the following sense: for $m = 1$, and general n, almost every point on the curve $\varphi(s) = (s, s^2, \ldots, s^n)$ is not DT_μ-improvable for some small constant $0 < \mu < 1$. Their proofs are based on the technique of regular systems introduced in [7].

Recently, based on an observation of Dani [5], as well as Kleinbock and Margulis [11], Kleinbock and Weiss [10] studied this Diophantine approximation problem in the language of homogeneous dynamics, and proved the following result: for $m = 1$ and arbitrary n, if an analytic curve in $\mathrm{M}(1 \times n, \mathbb{R}) \cong \mathbb{R}^n$ is not contained in any proper affine subspace, then almost every point on the curve is not DT_μ-improvable for some small constant $0 < \mu < 1$ depending on the curve. Based on the same correspondence, Nimish Shah [18] proved the following stronger result: for $m = 1$ and general n, if an analytic curve $\varphi : I = [a, b] \to \mathbb{R}^n$ is not contained in a proper affine subspace, then almost every point on the curve is not DT-improvable. For $m = n$, Lei Yang [20] provided a geometric condition and proved that if an analytic curve $\varphi : I = [a, b] \to \mathrm{M}(n \times n, \mathbb{R})$ satisfies the condition, then almost every point on φ is not DT-improvable. The geometric condition given there provides some hint on solving the problem for general (m, n), and will be discussed in detail later.

The purpose of this paper is to give a geometric condition for each (m, n), and show that if an analytic curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

satisfies the condition, then almost every point on φ is not DT-improvable.

The geometric conditions called *generic* condition and *supergeneric* condition are defined as follows:

Definition 1.1. For any m and n, let

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

denote an analytic curve.

For $m = n$, we say φ is *generic* at $s_0 \in I$ if there exists a subinterval $J_{s_0} \subset I$ such that for any $s \in J_{s_0} \setminus \{s_0\}$, $\varphi(s) - \varphi(s_0)$ is invertible.

In order to define *supergeneric* condition, we need additional notation.

We consider the following two embeddings from $\mathrm{M}(m \times m, \mathbb{R})$ to the Lie algebra $\mathfrak{sl}(2m, \mathbb{R})$ of $\mathrm{SL}(2m, \mathbb{R})$:

$$\mathfrak{n}^+ : X \in \mathrm{M}(m \times m, \mathbb{R}) \mapsto \begin{bmatrix} \mathbf{0} & X \\ & \mathbf{0} \end{bmatrix} \in \mathfrak{sl}(2m, \mathbb{R}),$$

and

$$\mathfrak{n}^- : X \in \mathrm{M}(m \times m, \mathbb{R}) \mapsto \begin{bmatrix} \mathbf{0} & \\ X & \mathbf{0} \end{bmatrix} \in \mathfrak{sl}(2m, \mathbb{R}).$$

Let

$$\mathcal{E}_m := \begin{bmatrix} \mathrm{I}_m & \\ & -\mathrm{I}_m \end{bmatrix} \in \mathfrak{sl}(2m, \mathbb{R}). \tag{1.2}$$

If $X \in M(m \times m, \mathbb{R})$ is invertible, then the triple $\{\mathfrak{n}^+(X), \mathcal{E}_m, \mathfrak{n}^-(X^{-1})\}$ forms a standard basis of a copy of $\mathfrak{sl}(2, \mathbb{R})$.

A Lie subgroup L of $H = \mathrm{SL}(2m, \mathbb{R})$ is called *observable* if there exists a finite dimensional linear representation V of H and a nonzero vector $v \in V$ such that the subgroup of H stabilizing v is equal to L. A Lie subalgebra $\mathfrak{l}$ is called an *observable* Lie subalgebra of $\mathfrak{sl}(2m, \mathbb{R})$, if it is the Lie algebra of some observable Lie subgroup $L \subset H$.

Further we are interested in an observable subalgebra $\mathfrak{l}$ containing $\mathcal{E}_m$. Let

$$\mathcal{L}^{\pm} = \{X \in M(m \times m, \mathbb{R}) : \mathfrak{n}^{\pm}(X) \in \mathfrak{l}\}.$$

If $X \in \mathcal{L}^{\pm}$ is invertible, then $\mathfrak{n}^{\pm}(X)v = 0$ and $\mathcal{E}_m v = 0$. Therefore by the basic property of $\mathfrak{sl}_2$-triples, we get $\mathfrak{n}^{\mp}(X^{-1})v = 0$. Therefore $X^{-1} \in \mathcal{L}^{\mp}$. In particular, if $\mathcal{L}^- = M(m \times m, \mathbb{R})$, then $\mathfrak{l} = \mathfrak{sl}(2m, \mathbb{R})$.

For the case of $m = n$, the curve φ is called *supergeneric* at $s_0 \in I$ if it is *generic* at s_0 (with subinterval $J_{s_0} \subset I$), and for any proper observable subalgebra $\mathfrak{l}$ of $\mathfrak{sl}(2m, \mathbb{R})$ containing $\mathcal{E}_m$, we have

$$\{(\varphi(s_1) - \varphi(s_0))^{-1} - (\varphi(s_2) - \varphi(s_0))^{-1}) : s_1, s_2 \in J_{s_0} \setminus \{s_0\}\} \not\subset \mathcal{L}^-. \tag{1.3}$$

When $m \neq n$, we will define what it means to say that φ is *generic* (*supergeneric*) at $s_0 \in I$ by induction on $m + n$ as follows:

For $m < n$, we express $\varphi(s) = [\varphi_1(s); \varphi_2(s)]$, where $\varphi_1(s)$ is the first m by m block, and $\varphi_2(s)$ is the rest m by $n - m$ block. We say φ is *generic* (*supergeneric*) at $s_0 \in I$, if there exists a subinterval $J_{s_0} \subset I$ such that for any $s \in J_{s_0} \setminus \{s_0\}$, $\varphi_1(s) - \varphi_1(s_0)$ is invertible; and if we define $\psi : J_{s_0} \setminus \{s_0\} \to \mathrm{M}(m \times (n - m), \mathbb{R})$, by

$$\psi(s) := (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$$

then ψ is *generic* (*supergeneric*) at some $s_1 \in J_{s_0} \setminus \{s_0\}$.

For $m > n$, φ is called *generic* (*supergeneric*) at $s_0 \in I$ if its transpose

$$\varphi^{\mathrm{T}} : I = [a, b] \to \mathrm{M}(n \times m, \mathbb{R})$$

is *generic* (*supergeneric*) at s_0.

We say that φ is *generic* (*supergeneric*) or satisfies *generic* (*supergeneric*) condition, if φ is *generic* (*supergeneric*) at some $s_0 \in I$. Since φ is analytic, if it is *generic* (*supergeneric*) at one point of I then it will be *generic* (*supergeneric*) at all but finitely many points of I.

Remark 1.2. We will discuss the *generic* condition and the *supergeneric condition* in detail in Appendix A. Here we list several important statements.

1. If m and n are coprime, then the *generic* condition is the same as the *supergeneric* condition (see Proposition A.1).
2. If $m = 1$ or $n = 1$, the *generic* condition (which is the same as the *supergeneric* condition) is equivalent to the condition that the curve is not contained in any proper affine subspace (see Proposition A.2).

3. In [20], it is proved that for $m = n$, if there exists $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that the derivative $\varphi'(s_0)$ is invertible, $\varphi(s) - \varphi(s_0)$ is invertible for any $s \in J_{s_0} \setminus \{s_0\}$, and
$$\{(\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0} \setminus \{s_0\}\}$$
is not contained in any proper affine subspace of $\mathrm{M}(m \times m, \mathbb{R})$, then almost every point on the curve is not DT-improvable. Clearly this condition implies the *supergeneric* condition.
4. For any m and n, the set of *supergeneric* curves in $\mathrm{M}(m \times n, \mathbb{R})$ is open and dense in the set of analytic curves in $\mathrm{M}(m \times n, \mathbb{R})$ (see Proposition A.6).

In this paper we will prove the following result:

Theorem 1.3. *For any m and n, if an analytic curve*
$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$
is supergeneric*, then almost every point on φ is not DT-improvable. If $(m, n) = 1$, then the same result holds for* generic *analytic curves.*

1.2. **Dirichlet's approximation and homogeneous dynamics.** Let $G = \mathrm{SL}(m+n, \mathbb{R})$, and let $\Gamma = \mathrm{SL}(m + n, \mathbb{Z})$. The homogeneous space G/Γ can be identified with the space of unimodular lattices of $\mathbb{R}^{m+n}$. Every point $[g] = g\Gamma$ corresponds to the unimodular lattice $g\mathbb{Z}^{m+n}$. Let $0 < \mu < 1$, let B_μ denote the open sup-norm ball of radius μ, and
$$K_\mu = \{\Lambda \in G/\Gamma : \Lambda \cap B_\mu = \{\mathbf{0}\}\}.$$
Then K_μ contains an open neighborhood of $\mathbb{Z}^{m+n}$ in G/Γ and by Mahler's criterion K_μ is compact. So $\mu_G(K_\mu) > 0$, where μ_G is the G-invariant probability measure on G/Γ.

Define the diagonal subgroup $A = \{a(t) : t \in \mathbb{R}\}$ and the embedding $u : \mathrm{M}(m \times n, \mathbb{R}) \to G$ by
$$a(t) := \begin{bmatrix} e^{nt}\mathrm{I}_m & \\ & e^{-mt}\mathrm{I}_n \end{bmatrix} \text{ and } \Phi \mapsto u(\Phi) = \begin{bmatrix} \mathrm{I}_m & \Phi \\ & \mathrm{I}_n \end{bmatrix}. \tag{1.4}$$

Suppose $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is DT_μ-improvable. Then by (1.1), for each large $N > 0$, there exist $\mathbf{p} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{q} \in \mathbb{Z}^m$ such that $\|\mathbf{p}\| \le \mu N^m$ and $\|\Phi\mathbf{p} - \mathbf{q}\| \le \mu N^{-n}$. Then
$$a(\log N)u(\Phi)(-\mathbf{q}, \mathbf{p}) = \begin{bmatrix} e^{n \log N}(-\mathbf{q} + \Phi\mathbf{p}) \\ e^{-m \log N}\mathbf{p} \end{bmatrix} \in B_\mu,$$
and hence $a(\log N)u(\Phi)\mathbb{Z}^{m+n} \notin K_\mu$. Thus
$$E := \{s \in I : \varphi(s) \text{ is } DT_\mu\text{-improvable}\} = \{s \in I : a(\log N)u(\varphi(s))[e] \notin K_\mu \text{ for all large } N\}.$$

Fix $N_0 \in \mathbb{N}$, and let
$$E_{N_0} = \{s \in I : a(\log N)u(\varphi(s))[e] \notin K_\mu \text{ for all } N \ge N_0\}.$$

Suppose that $|E_{N_0}| > 0$, where $|\cdot|$ denotes the Lebesgue measure. By Lebesgue density theorem, we can pick an interval J of I such that
$$|E_{N_0} \cap J|/|J| \ge 1 - \mu_G(K_\mu)/2. \tag{1.5}$$

Suppose we can proved the following:

1.2.1. *Claim. The expanding curve* $a(t)u(\varphi(J))[e]$ *gets equidistributed in* G/Γ *as* $t \to +\infty$. *In particular, for all large* N,

$$\frac{1}{|J|}|\{s \in J : a(\log N)u(\varphi(s))[e] \in K_\mu\}| > \mu_G(K_\mu)/2.$$

Then by the definition of E_{N_0} we conclude that

$$|E_{N_0} \cap J|/|J| < 1 - \mu_G(K_\mu)/2,$$

which contradicts the choice of J as in (1.5). This proves that $|E_{N_0}| = 0$ for all $N_0 \in \mathbb{N}$. Hence $|E| = 0$; that is, $\varphi(s)$ is not DT_μ-improvable for almost every $s \in I$.

1.3. **Equidistribution of expanding curves in homogeneous spaces.** It turns out that the equidistribution result described in the previous section holds in a much more general setting. In fact, we will prove the following result:

Theorem 1.4. *Let* G *be a Lie group containing* $H = \mathrm{SL}(m+n, \mathbb{R})$, *and* $\Gamma < G$ *be a lattice of* G. *Let* μ_G *denote the unique* G*-invariant probability measure on the homogeneous space* G/Γ. *Take* $x = g\Gamma \in G/\Gamma$ *such that its* H*-orbit* Hx *is dense in* G/Γ. *Let us fix the diagonal group*

$$A = \left\{ a(t) = \begin{bmatrix} e^{nt}\mathrm{I}_m & \\ & e^{-mt}\mathrm{I}_n \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Let $\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$ *be an analytic curve. We embed the curve into* H *by*

$$u : X \in \mathrm{M}(m \times n, \mathbb{R}) \mapsto u(X) = \begin{bmatrix} \mathrm{I}_m & X \\ & \mathrm{I}_n \end{bmatrix}.$$

For $t > 0$, *let* μ_t *denote the normalized parameteric measure on the curve* $a(t)u(\varphi(I))x \subset G/\Gamma$; *that is, for a compactly supported continuous function* $f \in C_c(G/\Gamma)$,

$$\int f \mathrm{d}\mu_t := \frac{1}{|I|} \int_{s \in I} f(a(t)u(\varphi(s))x)\mathrm{d}s.$$

If φ *is* generic, *then every weak-*$*$ *limit measure* μ_∞ *of* $\{\mu_t : t > 0\}$ *is still a probability measure. If the curve* φ *is* supergeneric, *then* $\mu_t \to \mu_G$ *as* $t \to +\infty$ *in weak-*$*$ *topology; that is, for any function* $f \in C_c(G/\Gamma)$,

$$\lim_{t \to +\infty} \frac{1}{|I|} \int_{s \in I} f(a(t)u(\varphi(s))x)\mathrm{d}s = \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

Moreover, if $(m, n) = 1$, *then* generic *property will imply that* $\mu_t \to \mu_G$ *as* $t \to +\infty$.

Remark 1.5.

1. As we explained in §1.2, Theorem 1.3 follows from Claim 1.2.1, which in turn follows from Theorem 1.4 with $G = H = \mathrm{SL}(m+n, \mathbb{R})$, $\Gamma = \mathrm{SL}(m+n, \mathbb{Z})$, $x = [e] = \mathbb{Z}^{m+n} \in G/\Gamma$, $I = J$, and a choice of $f \in C_c(G/\Gamma)$ supported on K_μ such that $0 \le f \le 1$ and $\int f \, d\mu_G > \mu_G(K)/2$.
2. Even in the case $G = H = \mathrm{SL}(m+n, \mathbb{R})$, Theorem 1.4 is much stronger than Theorem 1.3, since it applies to an arbitrary lattice $\Gamma < G$.
3. It is also interesting to consider the following question: Given any nontrivial analytic curve
$$\varphi : I \to \mathrm{M}(m \times n, \mathbb{R}),$$
is it true that for almost every $X \in \mathrm{M}(m \times n, \mathbb{R})$, the equidistribution result as in Theorem 1.4 holds for $\varphi + X$? We conjecture that the statement is true.

The study of limit distributions of evolution of curves translated by diagonalizable subgroups in homogeneous spaces has its own interest and has a lot of interesting connections to geometry and Diophantine approximation. One can summarize this type of problems as follows:

Problem 1.6. Let H be a semisimple Lie group, generated by its unipotent subgroups. Fix a diagonalizable one parameter subgroup $A = \{a(t) : t \in \mathbb{R}\} \subset H$, and let $U^+(A)$ denote the expanding horospherical subgroup of A in H. Let G be a Lie group containing H, and let Γ be a lattice of G.

Let

$$\phi : I = [a, b] \to H$$

be a piece of analytic curve in H with nonzero projection on $U^+(A)$ (this will make sure that the translates of $\phi(I)$ by $\{a(t) : t > 0\}$ expand). Given a point $x = g\Gamma \in G/\Gamma$, Ratner's topological theorem (cf. [14]) tells that the closure of Hx is a finite volume homogeneous subspace Fx, where F is a Lie subgroup of G containing H. Let μ_F denote the unique probability F-invariant measure supported on Fx. One can ask whether the expanding curves $\{a(t)\phi(I)x : t > 0\}$ tend to be equidistributed in Fx, i.e., as $t \to +\infty$, the normalized parametric measure supported on $a(t)\phi(I)x$ approaches μ_F in weak-$*$ topology.

Remark 1.7. Without loss of generality, in this paper, we always assume that Hx is dense in G/Γ. If Hx is not dense, suppose its closure is Fx, then we may replace G by F, Γ by $F \cap x\Gamma x^{-1}$ (which is a lattice of F by the closeness of Fx).

Nimish Shah [17] and [19] studied the case $H = \mathrm{SO}(n, 1)$ and $G = \mathrm{SO}(m, 1)$ where $m \geq n$. In this case the diagonalizable subgroup $\{a(t) : t \in \mathbb{R}\}$ is a fixed maximal $\mathbb{R}$-split Cartan subgroup of H. In [19] it is proved that given an analytic curve

$$\phi : I = [a, b] \to H,$$

and a point $x = g\Gamma \in G/\Gamma$, unless the natural visual map

$$\mathrm{Vis} : \mathrm{SO}(n, 1)/\mathrm{SO}(n-1) \cong \mathrm{T}^1(\mathbb{H}^n) \to \partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$$

sends the curve $\phi(I)$ to a proper subsphere of $\mathbb{S}^{n-1}$, the translates $\{a(t)\phi(I)x : t > 0\}$ of $\phi(I)x$ tend to be equidistributed as $t \to +\infty$. In [17], the same result is proved when ϕ is only C^n differentiable. In [17] and [19], the obstruction of equidistribution is discussed and the limit measure is given when the equidistribution fails. This result is generalized by Yang [21] in the following sense: for $H = \mathrm{SO}(n, 1)$ and arbitrary Lie group G containing H, if the same condition on the curve holds, then the expanding curve $a(t)\phi(I)x$ tends to be equidistributed as $t \to +\infty$. Shah [18] studied the case $m = 1$ of the problem we consider in this paper, and proved that if the analytic curve $\varphi : I \to \mathrm{M}(1 \times n, \mathbb{R}) = \mathbb{R}^n$ is not contained in a proper affine subspace of $\mathbb{R}^n$, then the equidistribution holds. It turns out that this condition is the same as *generic* condition for $m = 1$. Later Yang [20] studied the case $m = n$.

When the *generic* condition holds but *supergeneric* condition does not, we want to understand the obstruction of equidistribution and describe the limit measures of $\{\mu_t : t > 0\}$ to some extent. This requires more subtle argument. In [17] and [19], obstruction of equiditribution and description of limit measures are clearly given unconditionally for the case $H = \mathrm{SO}(n, 1)$ and $G = \mathrm{SO}(m, 1)$ in the set up of Problem 1.6. In our case, the problem becomes much more complicated. In this paper, we only discuss the case $n = km$, and we conjecture that similar result remains true for general (m, n) such that $(m, n) > 1$ (for the case $(m, n) = 1$, *generic* is the same as *supergeneric*, so there is nothing in between).

1.4. **Relation to extremality of submanifolds in homogeneous spaces.** Another direction to study Diophantine properties of a real matrix $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is to determine whether Φ is very well approximable. We say $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is very well approximable if there exists some constant $\delta > 0$ such that there exist infinitely many nonzero integer vectors $\mathbf{p} \in \mathbb{Z}^n$ and integer vectors $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\Phi \mathbf{p} - \mathbf{q}\| \leq \|\mathbf{p}\|^{-n/m-\delta}.$$

A submanifold $\mathcal{U} \subset \mathrm{M}(m \times n, \mathbb{R})$ is called extremal if with respect to the Lebesgue measure on $\mathcal{U}$, almost every point is not very well approximable. Based on the same correspondence due to Dani [5] and Kleinbock and Margulis [11], this problem can also be studied through homogenous dynamics. Kleinbock and Margulis [11] proved that if a submanifold $\mathcal{U} \subset \mathrm{M}(1 \times n, \mathbb{R})$ is nondegenerate, then $\mathcal{U}$ is extremal. Kleinbock, Margulis and Wang [9] later gave a necessary and sufficient condition of a submanifold of $\mathrm{M}(m \times n, \mathbb{R})$ being extremal. The condition is stated in terms of a particular representation of $H = \mathrm{SL}(m + n, \mathbb{R})$ and can not be translated to a geometric condition. Recently, Aka, Breuillard, Rosenzweig and de Saxcé [2] gave a family of subvarieties of $\mathrm{M}(m \times n, \mathbb{R})$ called constraining pencils, and proved that if a submanifold $\mathcal{U} \subset \mathrm{M}(m \times n, \mathbb{R})$ is not contained in a constraining pencil, then $\mathcal{U}$ is extremal. The result was previously annouced in [1]. It turns out that the *generic* condition implies the condition given in [2]. We will discuss it in detail in Appendix A (see Proposition A.9).

1.5. **Organization of the paper.** The paper is organized as follows: In §2, assuming the *generic* condition on φ, we will relate a unipotent invariance to limit measures of $\{\mu_t : t > 0\}$, and show that every limit measure is still a probability measure. This allows us to apply Ratner's theorem. In §3, we will apply Ratner's theorem and the linearization technique to study the limit measure via a particular linear representation of H. Finally we will get a linear algebraic condition on φ. Assuming some technical lemmas proved in §4, we prove Theorem 1.4. In §4, we will recall and prove some basic lemmas on linear representations, which are essential in our proof. In §5, assuming the *generic* condition, we will study the obstruction of equidistribution and limit measures of $\{\mu_t : t > 0\}$. We will only discuss the case $n = km$, and give a conjecture for general case. In the appendix, we will discuss the *generic* condition and the *supergeneric* condition in detail.

Notation 1.8. In this paper, we will use the following notation.

For $\epsilon > 0$ small, and quantities Q_1 and Q_2, $Q_1 \overset{\epsilon}{\approx} Q_2$ means that $|Q_1 - Q_2| \leq \epsilon$. Fix a right G-invariant metric $d(\cdot, \cdot)$ on G, then for $x_1, x_2 \in G/\Gamma$, and $\epsilon > 0$, $x_1 \overset{\epsilon}{\approx} x_2$ means $x_2 = gx_1$ such that $d(g, e) < \epsilon$.

For two related variable quantanties Q_1 and Q_2, $Q_1 \ll Q_2$ means there exists a constant $C > 0$ such that $Q_1 \leq CQ_2$, and $Q_1 \gg Q_2$ means $Q_2 \ll Q_1$. $O(Q_1)$ denotes some quantity $\ll Q_1$ or some vector whose norm is $\ll Q_1$.

2. NON-DIVERGENCE OF THE LIMIT MEASURES AND UNIPOTENT INVARIANCE.

2.1. **Preliminaries on Lie group structures.** We first recall some basic facts on the group

$$H = \mathrm{SL}(m+n, \mathbb{R}).$$

Without loss of generality, throughout this paper we always assume $m \leq n$.

The centralizer of the diagonal subgroup A, $Z_H(A)$, has the following form:

$$Z_H(A) = \left\{ \begin{bmatrix} B & \\ & C \end{bmatrix} : B \in \mathrm{GL}(m, \mathbb{R}), C \in \mathrm{GL}(n, \mathbb{R}), \text{ and } \det B \det C = 1 \right\}.$$

The expanding horospherical subgroup of A, $U^+(A)$ has the following form:

$$U^+(A) := \left\{ u(X) := \begin{bmatrix} \mathrm{I}_m & X \\ & \mathrm{I}_n \end{bmatrix} : X \in \mathrm{M}(m \times n, \mathbb{R}) \right\}.$$

Similarly, the contracting horospherical subgroup $U^-(A)$ has the following form:

$$U^-(A) := \left\{ u^-(X) := \begin{bmatrix} \mathrm{I}_m & \\ X & \mathrm{I}_n \end{bmatrix} : X \in \mathrm{M}(n \times m, \mathbb{R}) \right\}. \tag{2.1}$$

For any $z \in Z_H(A)$ and $u(X) \in U^+(A)$, $zu(X)z^{-1} = u(z \cdot X)$ where $z \cdot X$ is defined as follows:

$$\text{if } z = \begin{bmatrix} B & \\ & C \end{bmatrix} \in Z_H(A) \text{ and } X \in \mathrm{M}(m \times n, \mathbb{R}) \text{ then } z \cdot X := BXC^{-1}. \tag{2.2}$$

This defines an action of $Z_H(A)$ on $\mathrm{M}(m \times n, \mathbb{R})$.

Similarly we can define the action of $Z_H(A)$ on $\mathrm{M}(n \times m, \mathbb{R})$ induced by the conjugate action of $Z_H(A)$ on $U^-(A)$.

Let $P^-(A) := Z_H(A)U^-(A)$ denote the maximal parabolic subgroup of H associated with A.

Definition 2.1. For any $X \in \mathrm{GL}(m, \mathbb{R})$, we consider the following three elements in the Lie algebra $\mathfrak{h}$ of H:

$$\mathfrak{n}^+(X) := \begin{bmatrix} \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathfrak{n}^-(X^{-1}) := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ X^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathfrak{a} := \begin{bmatrix} \mathrm{I}_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then $\{\mathfrak{n}^+(X), \mathfrak{n}^-(X^{-1}), \mathfrak{a}\}$ makes a $\mathfrak{sl}(2, \mathbb{R})$- *triple*; that is, they satisfy the following relations

$$[\mathfrak{a}, \mathfrak{n}^+(X)] = 2\mathfrak{n}^+(X) \quad [\mathfrak{a}, \mathfrak{n}^-(X^{-1})] = -2\mathfrak{n}^-(X^{-1}), \quad [\mathfrak{n}^+(X), \mathfrak{n}^-(X^{-1})] = \mathfrak{a}.$$

Therefore, there is an embedding of $\mathrm{SL}(2, \mathbb{R})$ into H that sends $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to $\exp(\mathfrak{n}^+(X))$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to $\exp(\mathfrak{n}^-(X^{-1}))$, and $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ to $\exp(t\mathfrak{a})$. We denote the image of this $\mathrm{SL}(2, \mathbb{R})$ embedding by $\mathrm{SL}(2, X) \subset H$. Let us denote

$$\sigma(X) := \begin{bmatrix} \mathbf{0} & X & \mathbf{0} \\ -X^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathrm{SL}(2, X).$$

It is easy to see that $\sigma(X)$ corresponds to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

2.2. **Unipotent invariance.** Throughout this paper, we always assume that

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is analytic.

Recall that for $t > 0$, μ_t denotes the normalized parametric measure on the curve $a(t)u(\varphi(I))x$, and μ_G denotes the unique G invariant probability measure on G/Γ. Our aim is to prove that $\mu_t \to \mu_G$ as $t \to +\infty$. We first modify the measures μ_t to another measure λ_t and show that if $\lambda_t \to \mu_G$, then $\mu_t \to \mu_G$ as well. Then we can study $\{\lambda_t : t > 0\}$ instead. The motivation for this modification is that any accumulation point of $\{\lambda_t : t > 0\}$ is invariant under a unipotent subgroup.

The measure λ_t is defined as follows:

Definition 2.2 (cf. [15, (5.2)])**.** Without loss of generality, we may assume that $\varphi'(s) \neq \mathbf{0}$ for all $s \in I$. Since φ is analytic, there exists some integer $1 \leq b \leq m$, such that the derivative $\varphi'(s)$ has rank b for all $s \in I$ but finitely many points. Let $\mathrm{E}_b(m)$ be the m by m matrix defined as follows:

$$\mathrm{E}_b(m) := \begin{cases} \begin{bmatrix} \mathrm{I}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \text{if } b < m, \\ \\ \quad \mathrm{I}_m & \text{if } b = m. \end{cases}$$

Given a closed subinterval $J \subset I$ such that $\varphi'(s)$ has rank b for all $s \in J$, we define an analytic curve $z : J \to Z_H(A)$ such that

$$z(s) \cdot \varphi'(s) = [\mathrm{E}_b(m); \mathbf{0}], \ \forall s \in J. \tag{2.3}$$

For $t > 0$, we define λ_t^J to be the normalized parametric measure on $\{z(s)a(t)u(\varphi(s))x : s \in J\}$; that is, for $f \in C_c(G/\Gamma)$,

$$\int f \mathrm{d}\lambda_t^J := \frac{1}{|J|} \int_{s \in J} f(z(s)a(t)u(\varphi(s))x)\mathrm{d}s. \tag{2.4}$$

Remark 2.3. For any subinterval $J \subset I$, we can similarly define μ_t^J to be the normalized parameter measure on $a(t)u(\varphi(J))x$.

Proposition 2.4. *Suppose that for any closed subinterval $J \subset I$ such that λ_t^J is defined, we have $\lambda_t^J \to \mu_G$ as $t \to +\infty$. Then $\mu_t \to \mu_G$ as $t \to +\infty$.*

Proof. Let $s_1, s_2, \ldots, s_l \in I$ be all the points where $\varphi'(s)$ does not have rank b. For any fixed $f \in C_c(G/\Gamma)$ and $\epsilon > 0$, we want to show that for $t > 0$ large enough,

$$\int f \mathrm{d}\mu_t \overset{4\epsilon}{\approx} \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

For each $i \in \{1, 2, \ldots, l\}$, one can choose a small open subinterval $B_i \subset I$ containing s_i such that

$$\left| (\sum_{i=1}^{l} |B_i|) \int_{G/\Gamma} f \mathrm{d}\mu_G \right| \leq \epsilon |I|, \tag{2.5}$$

and for any $t > 0$,

$$\left| \int_{\cup_{i=1}^{l} B_i} f(a(t)u(\varphi(s))x)\mathrm{d}s \right| \leq \epsilon |I|. \tag{2.6}$$

Since f is uniformly continuous, there exists a constant $\delta > 0$, such that if $x_1 \overset{\delta}{\approx} x_2$ then $f(x_1) \overset{\epsilon}{\approx} f(x_2)$. We cut $I \setminus \cup_{i=1}^{l} B_i$ into several small closed subintervals $J_1, J_2, \ldots, J_p$ such that for every J_r,

$$z^{-1}(s_1)z(s_2) \overset{\delta}{\approx} e \text{ for any } s_1, s_2 \in J_r.$$

Now for a fixed J_r, we choose $s_0 \in J_r$ and define $f_0(x) = f(z^{-1}(s_0)x)$. Then for any $s \in J_r$, because $z^{-1}(s_0)z(s)a(t)u(\varphi(s))x \overset{\delta}{\approx} a(t)u(\varphi(s))x$, we have

$$f_0(z(s)a(t)u(\varphi(s))x) = f(z^{-1}(s_0)z(s)a(t)u(\varphi(s))x) \overset{\epsilon}{\approx} f(a(t)u(\varphi(s))x).$$

Therefore

$$\int f_0 \mathrm{d}\lambda_t^{J_r} \overset{\epsilon}{\approx} \int f \mathrm{d}\mu_t^{J_r}.$$

Because $\int f_0 \mathrm{d}\lambda_t^{J_r} \to \int_{G/\Gamma} f_0(x)\mathrm{d}\mu_G(x)$ as $t \to +\infty$, and

$$\begin{aligned} & \int_{G/\Gamma} f_0(x)\mathrm{d}\mu_G(x) \\ = & \int_{G/\Gamma} f(z^{-1}(s_0)x)\mathrm{d}\mu_G(x) \\ = & \int_{G/\Gamma} f(x)\mathrm{d}\mu_G \text{ (because } \mu_G \text{ is } G\text{-invariant)}, \end{aligned}$$

we have that there exists a constant $T_r > 0$, such that for $t > T_r$,

$$\int f_0 \mathrm{d}\lambda_t^{J_r} \overset{\epsilon}{\approx} \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

Therefore, for $t > T_r$,

$$\int f \mathrm{d}\mu_t^{J_r} \overset{2\epsilon}{\approx} \int_{G/\Gamma} f \mathrm{d}\mu_G,$$

i.e.,

$$\int_{J_r} f(a(t)u(\varphi(s))x)\mathrm{d}s \overset{2\epsilon|J_r|}{\approx} |J_r| \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

Then for $t > \max_{1\le r\le p} T_r$, we can sum up the above approximations for $r = 1, 2, \ldots, p$ and get

$$\int_{I\setminus \cup_{i=1}^{l} B_i} f(a(t)u(\varphi(s))x)\mathrm{d}s \overset{2\epsilon|I|}{\approx} (|I| - \sum_{i=1}^{l} |B_i|) \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

Combined with (2.5) and (2.6), the above approximation implies that

$$\int_I f(a(t)u(\varphi(s))x)\mathrm{d}s \overset{4\epsilon|I|}{\approx} |I| \int_{G/\Gamma} f \mathrm{d}\mu_G,$$

which is equivalent to

$$\int f \mathrm{d}\mu_t \overset{4\epsilon}{\approx} \int_{G/\Gamma} f \mathrm{d}\mu_G.$$

Because $\epsilon > 0$ can be arbitrarily small, we complete the proof. □

By this proposition, if we could prove the equidistribution of $\{\lambda_t := \lambda_t^I : t > 0\}$ as $t \to +\infty$ assuming that $\varphi'(s)$ has rank b for all $s \in I$, then the equidistribution of $\{\mu_t : t > 0\}$ as $t \to +\infty$ will follow. Therefore, later in this paper, we will assume that $\varphi'(s)$ has rank b for all $s \in I$ and define λ_t to be the normalised parametric measure on the curve $\{z(s)a(t)u(\varphi(s))x : s \in I\}$.

We will show that any limit measure of $\{\lambda_t : t > 0\}$ is invariant under the unipotent subgroup

$$W := \{u(r[\mathrm{E}_b(m); \mathbf{0}]) : r \in \mathbb{R}\}. \tag{2.7}$$

Proposition 2.5 (See [19]). *Let $t_i \to +\infty$ be a sequence such that $\lambda_{t_i} \to \mu_\infty$ in weak-$*$ topology, then μ_∞ is invariant under W-action.*

Proof. Given any $f \in C_c(G/\Gamma)$, and $r \in \mathbb{R}$, we want to show that

$$\int f(u(r[\mathrm{E}_b(m); \mathbf{0}])x) \mathrm{d}\mu_\infty = \int f(x) \mathrm{d}\mu_\infty.$$

Since $z(s)$ and $\varphi(s)$ are analytic and defined on the closed interval $I = [a, b]$, there exists a constant $T_1 > 0$ such that for $t \geq T_1$, $z(s)$ and $\varphi(s)$ can be extended to analytic curves defined on $[a - |r|e^{-(m+n)t}, b + |r|e^{-(m+n)t}]$. Throughout the proof, we always assume that $t_i \geq T_1$. Then $z(s + re^{-(m+n)t_i})$ and $\varphi(s + re^{-(m+n)t_i})$ are both well defined for all $s \in I$.

We define $f_1 \in C_c(G/\Gamma)$ as follows:

$$f_1(y) := f(u(r[\mathrm{E}_b(m); \mathbf{0}])y) \text{ for any } y \in G/\Gamma.$$

From the definition of μ_∞, we have

$$\begin{aligned}
\int f(u(r[\mathrm{E}_b(m); \mathbf{0}])x) \mathrm{d}\mu_\infty &= \int f_1(x) \mathrm{d}\mu_\infty \\
&= \lim_{t_i \to +\infty} \frac{1}{|I|} \int_{s \in I} f_1(z(s)a(t_i)u(\varphi(s))x) \mathrm{d}s \\
&= \lim_{t_i \to +\infty} \frac{1}{|I|} \int_{s \in I} f(u(r[\mathrm{E}_b(m); \mathbf{0}])z(s)a(t_i)u(\varphi(s))x) \mathrm{d}s.
\end{aligned}$$

We want to show that

$$u(r[\mathrm{E}_b(m); \mathbf{0}])z(s)a(t_i)u(\varphi(s)) \approx z(s + re^{-(m+n)t_i})a(t_i)u(\varphi(s + re^{-(m+n)t_i})). \tag{2.8}$$

Since $z(s + re^{-(m+n)t_i}) \approx z(s)$ for t_i large enough, it suffices to show that

$$u(r[\mathrm{E}_b(m); \mathbf{0}])z(s)a(t_i)u(\varphi(s)) \approx z(s)a(t_i)u(\varphi(s + re^{-(m+n)t_i})).$$

Note that $t \in \mathbb{R}$ and $L, M \in \mathrm{M}(m \times n, \mathbb{R})$,

$$u(L + M) = u(L)u(M) \text{ and } a(t)u(M)a(-t) = u(e^{(m+n)t}M),$$

and by Taylor's theorem

$$\varphi(s + re^{-(m+n)t_i}) = \varphi(s) + re^{-(m+n)t_i}\varphi'(s) + O(e^{-2(m+n)t_i}).$$

Then we have

$$\begin{aligned}
&z(s)a(t_i)u(\varphi(s + re^{-(m+n)t_i})) \\
&= z(s)a(t_i)u(\varphi(s) + re^{-(m+n)t_i}\varphi'(s) + O(e^{-2(m+n)t_i})) \\
&= z(s)a(t_i)u(O(e^{-2(m+n)t_i}) + re^{-(m+n)t_i}\varphi'(s))u(\varphi(s)) \\
&= z(s)a(t_i)u(O(e^{-2(m+n)t_i}) + re^{-(m+n)t_i}\varphi'(s))a(-t_i)a(t_i)u(\varphi(s)) \\
&= z(s)u(O(e^{-(m+n)t_i}) + r\varphi'(s))a(t_i)u(\varphi(s)) \\
&= z(s)u(O(e^{-(m+n)t_i}) + r\varphi'(s))z(s)^{-1}z(s)a(t_i)u(\varphi(s)) \\
&= u(O(e^{-(m+n)t_i}) + rz(s) \cdot \varphi'(s))z(s)a(t_i)u(\varphi(s)), \quad \text{by (2.2)}, \\
&= u(O(e^{-(m+n)t_i}))u(r[\mathrm{E}_b(m); \mathbf{0}])z(s)a(t_i)u(\varphi(s)), \quad \text{by (2.3)}.
\end{aligned}$$

When t_i is large enough, $u(O(e^{-(m+n)t_i}))$ can be ignored. Therefore, for any $\delta > 0$, there exists $T > 0$, such that for $t_i > T$,

$$u(r[\mathrm{E}_b(m); \mathbf{0}])z(s)a(t_i)u(\varphi(s)) \overset{\delta}{\approx} z(s + re^{-(m+n)t_i})a(t_i)u(\varphi(s + re^{-(m+n)t_i})).$$

Now fix $\epsilon > 0$. We choose $\delta > 0$ such that whenever $x_1 \overset{\delta}{\approx} x_2$, we have $f(x_1) \overset{\epsilon}{\approx} f(x_2)$. Then from the above argument, we have for $t_i > T$,

$$f(u(r[\mathrm{E}_b(m);\mathbf{0}])z(s)a(t_i)u(\varphi(s))x) \overset{\epsilon}{\approx} f(z(s+re^{-(m+n)t_i})a(t_i)u(\varphi(s+re^{-(m+n)t_i}))x).$$

Therefore,

$$\begin{array}{rl} & \frac{1}{|I|}\int_{s\in I} f(u(r[\mathrm{E}_b(m);\mathbf{0}])z(s)a(t_i)u(\varphi(s))x)\mathrm{d}s \\ \overset{\epsilon}{\approx} & \frac{1}{|I|}\int_{s\in I} f(z(s+re^{-(m+n)t_i})a(t_i)u(\varphi(s+re^{-(m+n)t_i}))x)\mathrm{d}s \\ = & \frac{1}{|I|}\int_{a+re^{-(m+n)t_i}}^{b+re^{-(m+n)t_i}} f(z(s)a(t_i)u(\varphi(s))x)\mathrm{d}s. \end{array}$$

Since $|f|$ is bounded, when $t_i > 0$ is large enough,

$$\frac{1}{|I|}\int_{a+re^{-(m+n)t_i}}^{b+re^{-(m+n)t_i}} f(z(s)a(t_i)u(\varphi(s))x)\mathrm{d}s \overset{\epsilon}{\approx} \frac{1}{|I|}\int_a^b f(z(s)a(t_i)u(\varphi(s))x)\mathrm{d}s.$$

Therefore, for t_i large enough,

$$\int f(u(r[\mathrm{E}_b(m);\mathbf{0}])x)\mathrm{d}\lambda_{t_i} \overset{2\epsilon}{\approx} \int f(x)\mathrm{d}\lambda_{t_i}.$$

Letting $t_i \to +\infty$, we have

$$\int f(u(r[\mathrm{E}_b(m);\mathbf{0}])x)\mathrm{d}\mu_\infty \overset{2\epsilon}{\approx} \int f(x)\mathrm{d}\mu_\infty.$$

Since the above approximation is true for arbitrary $\epsilon > 0$, we have that μ_∞ is W-invariant. □

2.3. **Non-divergence of limit measures.** We also need to show that any limit measure μ_∞ of $\{\lambda_t : t > 0\}$ is still a probability measure of G/Γ, i.e., no mass escapes to infinity as $t \to +\infty$. To do this, it suffices to show the following proposition:

Proposition 2.6. *Suppose $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$ is* generic. *For any $\epsilon > 0$, there exists a compact subset $\mathcal{K}_\epsilon \subset G/\Gamma$ such that*

$$\lambda_t(\mathcal{K}_\epsilon) \geq 1 - \epsilon \textit{ for all } t > 0.$$

This proposition will be proved via linearization technique combined with a lemma in linear dynamics as in [18].

Definition 2.7. Let $\mathfrak{g}$ denote the Lie algebra of G, and denote $d = \dim G$. We define

$$V = \bigoplus_{i=1}^{d} \bigwedge^{i} \mathfrak{g},$$

and let G act on V via $\bigoplus_{i=1}^{d} \bigwedge^{i} \mathrm{Ad}(G)$. This defines a linear representation of G:

$$G \to \mathrm{GL}(V).$$

Remark 2.8. In this paper, we will treat V as a representation of H.

The following theorem is the basic tool to prove that there is no mass-escape when we pass to a limit measure:

Theorem 2.9 (see [18, Proposition 3.4])**.** *Fix a norm $\|\cdot\|$ on V. There exist finitely many vectors*

$$v_1, v_2, \ldots, v_r \in V$$

such that for each $i = 1, 2, \ldots, r$, the orbit Γv_i is discrete, and moreover, the following holds: for any $\epsilon > 0$ and $R > 0$, there exists a compact set $K \subset G/\Gamma$ such that for any $t > 0$ and any subinterval $J \subset I$, one of the following holds:

S.1 *There exist $\gamma \in \Gamma$ and $j \in \{1, \ldots, r\}$ such that*
$$\sup_{s \in J} \|a(t)u(\varphi(s))g\gamma v_j\| < R,$$

S.2
$$|\{s \in J : a(t)u(\varphi(s))x \in K\}| \geq (1 - \epsilon)|J|.$$

Remark 2.10. The above theorem follows from the argument as in [16, Theorem 2.2] (see [18, Proposition 3.4] for the proof). It relies on the work of Dani and Margulis [6] and its extension due to Kleinbock and Margulis [11]. To get such a result, it is crucial to find constants $C > 0$ and $\alpha > 0$ such that in this particular representation, all the coordinate functions of $a(t)u(\varphi(\cdot))$ are (C, α)-good. Here a function $f : I \to \mathbb{R}$ is called (C, α)-good if for any subinterval $J \subset I$ and any $\epsilon > 0$, the following holds:
$$|\{s \in J : |f(s)| < \epsilon\}| \leq C \left(\frac{\epsilon}{\sup_{s \in J} |f(s)|}\right)^{\alpha} |J|.$$

Notation 2.11. Let $\mathcal{V}$ be a finite dimensional linear representation of a Lie group F. Then for a one-parameter diagonal subgroup $D = \{d(t) : t \in \mathbb{R}\}$ of F, we can decompose $\mathcal{V}$ as the direct sum of eigenspaces of D; that is,
$$\mathcal{V} = \bigoplus_{\lambda \in \mathbb{R}} \mathcal{V}^{\lambda}(D),$$
where $\mathcal{V}^{\lambda}(D) = \{v \in \mathcal{V} : d(t)v = e^{\lambda t}v\}$.

We define
$$\mathcal{V}^{+}(D) = \bigoplus_{\lambda > 0} \mathcal{V}^{\lambda}(D), \quad \mathcal{V}^{-}(D) = \bigoplus_{\lambda < 0} \mathcal{V}^{\lambda}(D), \quad \mathcal{V}^{\pm 0}(D) = \mathcal{V}^{\pm}(D) + \mathcal{V}^{0}(D).$$
For a vector $v \in \mathcal{V}$, we denote by $v^{+}(D)$ ($v^{\lambda}(D)$, $v^{-}(D)$, $v^{0}(D)$, $v^{+0}(D)$ and $v^{-0}(D)$ respectively) the projection of v to $\mathcal{V}^{+}(D)$ ($\mathcal{V}^{\lambda}(D)$, $\mathcal{V}^{-}(D)$, $\mathcal{V}^{0}(D)$, $\mathcal{V}^{+0}(D)$ and $\mathcal{V}^{-0}(D)$ respectively) with respect to the above direct sums.

The proof of Proposition 2.6 depends on the following property of finite dimensional representations of $\mathrm{SL}(m+n, \mathbb{R})$:

Lemma 2.12 (Basic Lemma)**.** *Let V be a finite dimensional representation of $\mathrm{SL}(m+n, \mathbb{R})$, and let $A = \{a(t) : t \in \mathbb{R}\} \subset \mathrm{SL}(m+n, \mathbb{R})$ denote the diagonal subgroup as in* (1.4)*. If an analytic curve*
$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$
is generic*, then for any nonzero vector $v \in V$, there exists some $s \in I$ such that*
$$u(\varphi(s))v \notin V^{-}(A).$$

A proof of this linear dynamical lemma is one of the most important technical contributions of this paper, and we will postpone its proof to §4.

Proof of Proposition 2.6 assuming Lemma 2.12. Let V be as in Definition 2.7. Since $A \subset H$ is a diagonal subgroup, we have the following decomposition:
$$V = \bigoplus_{\lambda \in \mathbb{R}} V^{\lambda}(A)$$

where $V^\lambda(A)$ is defined as in Notation 2.11. Choose the norm $\|\cdot\|$ on V to be the maximum norm associated to some choices of norms on $V^\lambda(A)$'s.

For contradiction we assume that there exists a constant $\epsilon > 0$ such that for any compact subset $\mathcal{K} \subset G/\Gamma$, there exist some $t > 0$ such that $\lambda_t(\mathcal{K}) < 1 - \epsilon$. Now we fix a sequence $\{R_i > 0 : i \in \mathbb{N}\}$ tending to zero. By Theorem 2.9, for any R_i, there exists a compact subset $\mathcal{K}_i \subset G/\Gamma$, such that for any $t > 0$, one of the following holds:

S1. There exist $\gamma \in \Gamma$ and $j \in \{1, \ldots, r\}$ such that

$$\sup_{s\in I} \|a(t)u(\varphi(s))g\gamma v_j\| < R_i,$$

S2.

$$|\{s \in I : a(t)u(\varphi(s))x \in \mathcal{K}_i\}| \geq (1 - \epsilon)|I|.$$

From our hypothesis, for each $\mathcal{K}_i$, there exists some $t_i > 0$ such that *S2.* does not hold. So there exist $\gamma_i \in \Gamma$ and $v_{j(i)}$ such that

$$\sup_{s\in I} \|a(t_i)u(\varphi(s))g\gamma_i v_{j(i)}\| < R_i. \tag{2.9}$$

By passing to a subsequence of $\{i \in \mathbb{N}\}$, we may assume that $v_{j(i)} = v_j$ remains the same for all i.

Since Γv_j is discrete in V, we have $t_i \to \infty$ as $i \to \infty$ and there are the following two cases:

Case 1. By passing to a subsequence of $\{i \in \mathbb{N}\}$, $\gamma_i v_j = \gamma v_j$ remains the same for all i.

Case 2. $\|\gamma_i v_j\| \to \infty$ along some subsequence.

For *Case 1.*: We have $a(t_i)u(\varphi(s))g\gamma v_j \to \mathbf{0}$ as $i \to \infty$ for all $s \in I$. This implies that

$$\{u(\varphi(s))g\gamma v_j\}_{s\in I} \subset V^-(A),$$

which contradicts Lemma 2.12.

For *Case 2.*: After passing to a subsequence, we have

$$v := \lim_{i\to\infty} g\gamma_i v_j/\|g\gamma_i v_j\|, \quad \|v\| = 1, \text{ and } \lim_{i\to\infty} \|g\gamma_i v_j\| = \infty. \tag{2.10}$$

By Lemma 2.12, let $s \in I$ be such that $u(\varphi(s))v \notin V^-(A)$. Then by (2.10) there exists $\delta_0 > 0$ and $i_0 \in \mathbb{N}$ such that

$$\|(u(\varphi(s))g\gamma_i v_j)^{0+}\| \geq \delta_0 \|g\gamma_i v_j\|, \quad \forall i \geq i_0.$$

Then

$$\|a(t_i)u(\varphi(s))g\gamma_i v_j\| \geq \delta_0\|g\gamma_i v_j\| \to \infty, \text{ as } i \to \infty,$$

which contradicts (2.9). Thus Cases 1 and 2 both lead to contradictions. □

Remark 2.13. The same proof also shows that any limit measure of $\{\mu_t : t > 0\}$ is still a probability measure, which is the non-divergence part of Theorem 1.4.

3. RATNER'S THEOREM AND THE LINEARIZATION TECHNIQUE.

In this section, let us assume that φ is *supergeneric*. Take any convergent subsequence $\lambda_{t_i} \to \mu_\infty$. By Proposition 2.5 and Proposition 2.6, μ_∞ is a W-invariant probability measure on G/Γ, where W is a unipotent one-parameter subgroup given by (2.7). We will apply Ratner's theorem and the linearization technique to understand the measure μ_∞.

Definition 3.1. Let $\mathcal{L}$ be the collection of proper analytic subgroups $L < G$ such that $L \cap \Gamma$ is a lattice of L. Then $\mathcal{L}$ is a countable set ([13]).

For $L \in \mathcal{L}$, define

$$\begin{aligned} N(L, W) &= \{g \in G : g^{-1}Wg \subset L\}, \text{ and} \\ S(L, W) &= \bigcup_{L' \in \mathcal{L}, L' \subsetneq L} N(L', W). \end{aligned} \tag{3.1}$$

We formulate Ratner's measure classification theorem as follows (cf. [12]):

Theorem 3.2 ([13])**.** *Let $\pi : G \to X = G/\Gamma$ denote the natural projection sending $g \in G$ to $g\Gamma \in X$. Given the W-invariant probability measure μ on G/Γ, if μ is not G-invariant then there exists $L \in \mathcal{L}$ such that*

$$\mu(\pi(N(L, W))) > 0 \quad \text{and} \quad \mu(\pi(S(L, W))) = 0. \tag{3.2}$$

Moreover, almost every W-ergodic component of μ on $\pi(N(L, W))$ is a measure of the form $g\mu_L$ where $g \in N(L, W) \backslash S(L, W)$, μ_L is a finite L-invariant measure on $\pi(L)$, and $g\mu_L(E) = \mu_L(g^{-1}E)$ for all Borel sets $E \subset G/\Gamma$. In particular, if $L \lhd G$, then the restriction of μ on $\pi(N(L, W))$ is L-invariant.

We want to show that $\mu_\infty = \mu_G$. For contradiction, let us assume that $\mu_\infty \neq \mu_G$. Then by Ratner's Theorem, there exists $L \in \mathcal{L}$ such that

$$\mu_\infty(\pi(N(L, W))) > 0 \text{ and } \mu_\infty(\pi(S(L, W))) = 0. \tag{3.3}$$

Now we want to apply the linearization technique to obtain algebraic consequences of this statement.

Definition 3.3. Let V be the finite dimensional representation of G defined as in Definition 2.7, for $L \in \mathcal{L}$, we choose a basis $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_l$ of the Lie algebra $\mathfrak{l}$ of L, and define

$$p_L = \wedge_{i=1}^{l} \mathfrak{e}_i \in V.$$

Note that the stabilizer of p_L is $N_G^1(L)$ where

$$N_G^1(L) := \{g \in G : gLg^{-1} = L \text{ and } \det(\mathrm{Ad}(g)|_{\mathfrak{l}}) = 1\}. \tag{3.4}$$

Define

$$\Gamma_L := \{\gamma \in \Gamma : \gamma p_L = \pm p_L\}.$$

From the action of G on p_L, we get a map:

$$\begin{aligned} &\eta : G \to V, \\ &g \mapsto g p_L. \end{aligned}$$

Let $\mathcal{A}$ denote the Zariski closure of $\eta(N(L, W))$ in V. Then $N(L, W) = G \cap \eta^{-1}(\mathcal{A})$.

Using the fact that φ is analytic, we obtain the following consequence of the linearization technique (cf. [19, 18, 15]).

Proposition 3.4 ([18, Proposition 5.5])**.** *Let $x = g\Gamma$ be as in Theorem 1.4 and C be a compact subset of $N(H, W) \setminus S(H, W)$. Given $\epsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ such that, given a relatively compact neighborhood Φ of $\mathcal{D}$ in V, there exists a neighborhood $\mathcal{O}$ of $C\Gamma$ in G/Γ such that for any $t \in \mathbb{R}$ and subinterval $J \subset I$, one of the following statements holds:*

SS1. $|\{s \in J : a(t)u(\varphi(s))g\Gamma \in \mathcal{O}\}| \leq \epsilon |J|$.

SS2. There exists $\gamma \in \Gamma$ such that $a(t)z(s)u(\varphi(s))g\gamma p_L \in \Phi$ for all $s \in J$.

The following proposition provides the obstruction to the limiting measure not being G-invariant in terms of linear actions of groups, and it is a key result for further investigations.

Proposition 3.5. *Let $x = g\Gamma$ be as in Theorem 1.4. There exists a $\gamma \in \Gamma$ such that*

$$\{u(\varphi(s))g\gamma p_L : s \in I\} \subset V^{-0}(A). \tag{3.5}$$

Proof (assuming Lemma 2.12). By (3.3), there exists a compact subset $C \subset N(L, W)) \setminus S(L, W)$ and $\epsilon > 0$ such that $\mu_\infty(C\Gamma) > \epsilon > 0$. Apply Proposition 3.4 to obtain $\mathcal{D}$, and choose any Φ, and obtain a $\mathcal{O}$ so that either *SS1.* or *SS2.* holds. Since $\lambda_{t_i} \to \mu_\infty$, we conclude that *SS1.* does not hold for $t = t_i$ for all $i \geq i_0$. Therefore for every $i \geq i_0$, *SS2.* holds and there exists $\gamma_i \in \Gamma$ such that

$$\{a(t_i)z(s)u(\varphi(s))g\gamma_i p_L : s \in I\} \subset \Phi. \tag{3.6}$$

Since Γp_L is discrete in V, by passing to a subsequence, there are two cases:

Case 1. $\gamma_i p_L = \gamma p_L$ for some $\gamma \in \Gamma$ for all i large enough; or
Case 2. $\|\gamma_i p_L\| \to \infty$ as $i \to \infty$.

In *Case 1*, since Φ is bounded in (3.6), we deduce that $z(s)u(\varphi(s))g\gamma p_L \subset V^{-0}(A)$ for all $s \in I$. Since $V^{-0}(A)$ is $Z_H(A)$-invartiant, (3.5) holds.

In *Case 2*, by arguing as in the *Case 2.* of the Proof of Proposition 2.6, using genericity of φ and Lemma 2.12, we obtain that $\|a(t_i)u(\varphi(s))g\gamma_i p_L\| \to \infty$. This contradicts (3.6), because $z(s) \subset Z_H(A)$ and Φ is bounded. Thus *Case 2* does not occur. $\square$

We will need the following analogue of the Basic lemma 2.12.

Lemma 3.6. *Let V be an irreducible representation of $H = \mathrm{SL}(m+n, \mathbb{R})$. Let*

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

be a supergeneric *analytic curve. Then if there is a nonzero vector $v \in V$ such that*

$$\{u(\varphi(s))v : s \in I\} \subset V^{-0}(A),$$

then V is a trivial representation.

We will postpone its proof to §4.

Proof of Theorem 1.4 assuming Lemma 3.6. Suppose $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$ is *supergeneric*, and the normalized parametric measures $\{\lambda_t : t > 0\}$ do not tend to the Haar measure μ_G along some subsequence $t_i \to +\infty$. By Proposition 3.5, there exists some $L \in \mathcal{L}$ and $\gamma \in \Gamma$ such that

$$u(\varphi(s))g\gamma p_L \in V^{-0}(A)$$

for all $s \in I$. Then by Lemma 3.6, we have that $v := g\gamma p_L$ is fixed by the whole group H. Hence p_L is fixed by the action of $\gamma^{-1}g^{-1}Hg\gamma$. Thus

$$\begin{aligned} \Gamma p_L &= \overline{\Gamma p_L}, \text{ since } \Gamma p_L \text{ is discrete} \\ &= \overline{\Gamma\gamma^{-1}g^{-1}Hg\gamma p_L} \\ &= \overline{\Gamma g^{-1}Hg\gamma p_L} \\ &= Gg\gamma p_L, \text{ since } \overline{Hg\Gamma} = G \\ &= Gp_L. \end{aligned}$$

This implies $G_0 p_L = p_L$ where G_0 is the connected component of e. In particular, $\gamma^{-1}g^{-1}Hg\gamma \subset G_0$ and $G_0 \subset N^1_G(L)$. By [17, Theorem 2.3], there exists a closed subgroup $F_1 \subset N^1_G(L)$ containing all Ad-unipotent one-parameter subgroups of G contained in $N^1_G(L)$ such that $F_1 \cap \Gamma$ is a lattice in F_1 and $F_1\Gamma$ is closed. If we put $F = g\gamma F_1 \gamma^{-1} g^{-1}$, then $H \subset F$ since H is generated by its unipotent one-parameter subgroups. Moreover, $Fx = g\gamma F_1\Gamma$ is closed and admits a finite F-invariant measure. Then since $\overline{Hx} = G/\Gamma$, we have $F = G$. This implies $F_1 = G$ and thus $L \lhd G$. Therefore $hLh^{-1} = L$ for all $h \in G$. Therefore, since $N(L, W) \neq \emptyset$, by (3.1) we have $W \subset L$ and $N(L, W) = G$. Therefore, $L \cap H$ is a normal subgroup of H containing W. Since H is a simple group, we have $H \subset L$. Since L is a normal subgroup of G and $L\Gamma$ is a closed orbit with finite L-invariant measure, every orbit of L on G/Γ is also closed and admits a finite L-invariant measure, in particular, Lx is closed. But since Hx is dense in G/Γ, Lx is also dense. This shows that $L = G$, which contradicts our hypothesis that the limit measure is not μ_G. This completes the proof. $\square$

4. **SOME LINEAR DYNAMICAL RESULTS.** We shall start with a dynamical lemma about finite dimensional representations of $\mathrm{SL}(2,\mathbb{R})$ which sharpens the earlier results due to Shah [19, Lemma 2.3] and Yang [21, Lemma 5.1].

Lemma 4.1. *Let V be a finite dimensional linear representation of* $\mathrm{SL}(2,\mathbb{R})$. *Let*

$$A = \left\{ a(t) := \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\},$$

$$U = \left\{ u(s) := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} : s \in \mathbb{R} \right\}, \text{ and } U^- = \left\{ u^-(s) := \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} : s \in \mathbb{R} \right\}.$$

Express V as the direct sum of eigenspaces with respect to the action of A:

$$V = \bigoplus_{\lambda \in \mathbb{R}} V^\lambda(A), \text{ where } V^\lambda(A) := \{v \in V : a(t)v = e^{\lambda t} v : \forall t \in \mathbb{R}\}.$$

For any $v \in V \setminus \{0\}$ and $\lambda \in \mathbb{R}$, let $v^\lambda = v^\lambda(A)$ denote the $V^\lambda(A)$-component of v,

$$\lambda^{\max}(v) = \max\{\lambda : v^\lambda \neq 0\},$$

and $v^{\max} = v^{\lambda^{\max}}(v)$. Then for any $r \neq 0$,

$$\lambda^{\max}(u(r)v) \geq -\lambda^{\max}(v). \tag{4.1}$$

In particular,

$$\lambda^{\max}(v) < 0 \text{ then } \lambda^{\max}(u(r)v) > 0, \forall r \neq 0. \tag{4.2}$$

Moreover, if the equality holds in (4.1) then

$$v = u^-(-r^{-1})v^{\max} \text{ and } (u(r)v)^{\max} = \sigma(r)v^{\max}, \text{ where } \sigma(r) = \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix}. \tag{4.3}$$

Proof. Observe that $u(1)u^-(-1)u(1) = \sigma(1)$, $u(-1)u^-(1)u(-1) = \sigma(-1)$ and for $r \neq 0$, conjugating all terms of these equalities by $a(\log(|r|/2))$ we get $u(r)u^-(-r^{-1})u(r) = \sigma(r)$, and hence

$$u(r) = \sigma(r)u(-r)u^-(r^{-1}), \forall r \neq 0. \tag{4.4}$$

Since $\sigma(r)a(t)\sigma(r)^{-1} = a(-t)$ for all $r \neq 0$, we have that

$$\sigma(r)V^\lambda(A) = V^{-\lambda}(A), \text{ for all } \lambda.$$

Hence for any $v \in V \setminus \{\mathbf{0}\}$,

$$\lambda^{\max}(\sigma(r)v) = -\lambda^{\min}(v), \text{ and } (\sigma(r)v)^{\max} = \sigma(r)v^{\min}. \tag{4.5}$$

For any $r \in \mathbb{R}$, since $u(r)$ is unipotent and $a(t)u(r)a(-t) = u(e^{2t}r)$, we have that

$$\lambda^{\min}(u(r)v) = \lambda^{\min}(v). \tag{4.6}$$

Similarly, for any $s \in \mathbb{R}$, we have $a(t)u^-(s)a(-t) = u^-(e^{-2t}s)$, and hence

$$\lambda^{\max}(u^-(s)v) = \lambda^{\max}(v). \tag{4.7}$$

Using the above relations (4.4), (4.5), (4.6) and (4.7), we get

$$\begin{aligned}\lambda^{\max}(u(r)v) &= \lambda^{\max}(\sigma(r)u(-r)u^-(r^{-1})v) \\ &= -\lambda^{\min}(u(-r)u^-(r^{-1})v) \\ &= -\lambda^{\min}(u^-(r^{-1})v) \\ &\geq -\lambda^{\max}(u^-(r^{-1})v) \\ &= -\lambda^{\max}(v).\end{aligned}$$

Further if there are all equalities in the above relation, then

$$\lambda^{\min}(u^-(r^{-1})v) = \lambda^{\max}(u^-(r^{-1})v) = \lambda^{\max}(v).$$

Therefore,

$$u^-(r^{-1})v = (u^-(r^{-1})v)^{\max} = v^{\max}; \text{ that is, } v = u^-(-r^{-1})v^{\max},$$

and

$$\begin{aligned}(u(r)v)^{\max} = \sigma(r)(u(-r)u^-(r^{-1})v)^{\min} &= \sigma(r)(u^-(r^{-1})v)^{\min} \\ &= \sigma(r)(u^-(r^{-1})v)^{\max} = \sigma(r)v^{\max}.\end{aligned}$$

□

Lemma 4.1 immediately implies the following statement:

Corollary 4.2. *Let the notation be as in Lemma 4.1. If $v, u(r)v \in V^{-0}(A)$ for some $r \neq 0$, then*

$$\lambda^{\max}(v) = 0 \text{ and } v = u^-(-r^{-1})v^0(A).$$

4.1. **Linear dynamical lemmas for $\mathrm{SL}(m+n,\mathbb{R})$ representations.** First we give the proof of the basic lemma (Lemma 2.12) that we have used more than once in previous sections. The new techniques developed in this section form the core of this paper, and we expect these techniques to be valuable for other problems.

In order to clearly explain the main idea in the proof, we first prove Lemma 2.12 for the following baby case: $(m,n) = (1,2)$.

Proof of Lemma 2.12 for $(m,n) = (1,2)$. Here $\varphi(s) = (\varphi_1(s), \varphi_2(s))$, where $\varphi_1(s), \varphi_2(s) \in \mathbb{R}$.

For a contradiction, let us assume that

$$u(\varphi(s))v \in V^-(A) \qquad \text{for all } s \in I. \tag{4.8}$$

In view of Notation 2.11, for $s \in I$, let

$$\mu_0(s) = \max\{\lambda : (u(\varphi(s))v)^{\lambda}(A) \neq 0\} \text{ and } \mu_0 = \max\{\mu_0(s) : s \in I\}.$$

Since φ is analytic, we have $\mu_0(s) = \mu_0$ for all but finitely many $s \in I$. By our assumption, $\mu_0 < 0$.

Let us fix $s_0 \in I$ with $\mu_0(s_0) = \mu_0$, and denote $\Delta(s) = (\Delta_1(s), \Delta_2(s)) := \varphi(s) - \varphi(s_0)$. Since φ is generic, there exists a subinterval $J_{s_0} \subset I$ such that for any $s \in J_{s_0} \setminus \{s_0\}$,

$$\Delta_1(s) \neq 0, \qquad \text{and } \frac{\Delta_2(s)}{\Delta_1(s)} \text{ is not constant.}$$

Let us denote $\psi(s) := \frac{\Delta_2(s)}{\Delta_1(s)} \in \mathbb{R}$. By choosing smaller J_{s_0}, we get $\mu_0(s) = \mu_0$ for all $s \in J_{s_0}$.

Let us fix $s \in J_{s_0} \setminus \{s_0\}$. Let us denote $v_0 := u(\varphi(s_0))v$ and $v_s := u(\varphi(s))v$. Then $v_s = u(\Delta(s))v_0$.

Let us write $a(t) = a_1(2t)a_2(t)$, where

$$a_1(t) := \begin{bmatrix} e^t & & \\ & e^{-t} & \\ & & 1 \end{bmatrix}, \quad \text{and} \quad a_2(t) := \begin{bmatrix} 1 & & \\ & e^t & \\ & & e^{-t} \end{bmatrix}. \tag{4.9}$$

Let us denote $A_1 := \{a_1(t) : t \in \mathbb{R}\}$ and $A_2 := \{a_2(t) : t \in \mathbb{R}\}$. Then we can decompose V as the direct sum of common eigenspaces of A_1 and A_2:

$$V = \bigoplus_{\delta_1, \delta_2} V^{\delta_1, \delta_2}, \text{ where } V^{\delta_1,\delta_2} := \{v \in V : a_1(t)v = e^{\delta_1 t}v, \text{ and } a_2(t)v = e^{\delta_2 t}v\}.$$

Then

$$V^\lambda(A) = \sum_{2\delta_1 + \delta_2 = \lambda} V^{\delta_1, \delta_2}.$$

Since $\Delta_1(s) \in \mathbb{R} \setminus \{0\} = \mathrm{GL}_m(\mathbb{R})$ for $m = 1$, by Definition 2.1 we have

$$\mathrm{SL}(2, \Delta_1(s)) = \left\{ \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} : g \in \mathrm{SL}(2, \mathbb{R}) \right\}.$$

Let us decompose V as the direct sum of irreducible sub-representations of $A \ltimes \mathrm{SL}(2, \Delta_1(s))$. For any such sub-representation $W \subset V$, let $p_W : V \to W$ denote the A-equivariant projection. By basic facts on $\mathrm{SL}(2,\mathbb{R})$-representations (see [8, Claim 11.4], for example), we have that every irreducible sub-representation $W \subset V$ admits a *standard basis*: $\{w_0, w_1, \ldots, w_r\}$, such that

$$a_1(t)w_i = e^{(r-2i)t}w_i, \quad \text{for } 0 \leq i \leq r.$$

We claim that each w_i is also an eigenvector for A. In fact,

$$a(t) = a_1(3t/2)b(t), \qquad \text{where } b(t) = \begin{bmatrix} e^{t/2} & & \\ & e^{t/2} & \\ & & e^{-t} \end{bmatrix}.$$

Note that $b(t)$ commutes $\mathrm{SL}(2, \Delta_1(s))$, so $b(t)$ acts on W as a scalar $e^{\delta t}$ for some $\delta \in \mathbb{R}$. Therefore,

$$a(t)w_i = e^{(3(r-2i)/2+\delta)t}w_i, \quad \text{for } 1 \leq i \leq r. \tag{4.10}$$

For $k < i$, the A-weight of w_k is strictly greater than the A-weight of w_i.

Let us denote

$$u'(r) := \begin{bmatrix} 1 & & \\ & 1 & r \\ & & 1 \end{bmatrix}.$$

It is straightforward to verify that

$$u(\Delta(s)) = u'(-\psi(s))u(\Delta_1(s), 0)u'(\psi(s)).$$

Therefore,

$$v_s = u(\Delta(s))v_0 = u'(-\psi(s))u(\Delta_1(s),0)u'(\psi(s))v_0.$$

Note that $u'(r)$ commutes with $a(t)$, and thus preserves every eigenspace of A. Therefore, since the highest A weight of $v_s = u(\varphi(s))v$ is μ_0, we have that the highest A-weight of $u'(\psi(s))v_s$ is μ_0 and

$$(u'(\psi(s))v_s)^{\mu_0}(A) = u'(\psi(s)) \cdot (v_s)^{\mu_0}(A).$$

Since

$$u'(\psi(s))v_s = u(\Delta_1(s),0)u'(\psi(s))v_0,$$

we have that the highest A-weight of $u(\Delta_1(s),0)u'(\psi(s))v_0$ is μ_0. Applying the same argument to v_0 and $u'(\psi(s))v_0$, we have that the highest A-weight of $u'(\psi(s))v_0$ is μ_0 and

$$(u'(\psi(s))v_0)^{\mu_0}(A) = u'(\psi(s)) \cdot (v_0)^{\mu_0}(A).$$

For any irreducible $A \ltimes \mathrm{SL}(2,\Delta_1(s))$-sub-representation $W \subset V$ such that

$$p_W(u'(\psi(s)) \cdot (v_0)^{\mu_0}(A)) \neq \mathbf{0}.$$

Then

$$p_W(u'(\psi(s)) \cdot (v_0)^{\mu_0}(A)) = a_i w_i, \text{ for some } 0 \leq i \leq r, \ 0 \neq a_i \in \mathbb{R}.$$

By (4.10) we have $\mu_0 = 3(r-2i)/2 + \delta$, and that for $k < i$, A-weight of w_k is larger than the A-weight of w_i, which is μ_0. We know that the maximum A-weight any A-eigen-component of $p_W(u'(\psi(s))v_0)$ is μ_0. Therefore

$$p_W(u'(\psi(s))v_0) = \sum_{k \geq i} a_k w_k, \text{ for some real } a_k\text{'s}.$$

Note that W is also an irreducible $\mathrm{SL}(2,\Delta_1(s))$-sub-representation of V. We have

$$p_W(u(\Delta_1(s),0)u'(\psi(s))v_0) = u(\Delta_1(s),0)p_W(u'(\psi(s))v_0).$$

We know that the A-weight of any eigen component of $u(\Delta_1(s),0)u'(\psi(s))v_0$ is at most μ_0. Hence the same holds for $p_W(u(\Delta_1(s),0)u'(\psi(s))v_0)$. Combined with the fact that for $k < i$, the A-weight of w_k is greater than the A-weight of w_i, and that the A-weight of w_i is μ_0, we conclude that

$$u(\Delta_1(s),0)p_W(u'(\psi(s))v_0) = u(\Delta_1(s),0)(\sum_{k \geq i} a_k w_k) = \sum_{k \geq i} b_k w_k, \text{ for some real } b_k\text{'s}.$$

We claim that the A_1-weight of $p_W(u'(\psi(s))v_0)$, which is $r - 2i$, is nonnegative. In fact, if $r - 2i < 0$,

$$p_W(u'(\psi(s))v_0) \in V^-(A_1) \text{ and } u(\Delta_1(s),0)p_W(u'(\psi(s))v_0) \in V^-(A_1).$$

This contradicts Corollary 4.2 for $SL(2,\Delta_1(s))$ representation W, $p_W(u'(\psi(s))v_0)$ playing the role of v and $\{u(\Delta_1(s),0)\}$ being the corresponding unipotent one-parameter subgroup and $A_1 = \{a_1(t) : t \in \mathbb{R}\}$ the corresponding expanding diagonal subgroup. This proves the claim that $r - 2i \geq 0$.

Therefore, by (4.9), the A_2-weight of w_i, which is $\mu_0 - 2(r-2i)$, is negative, because $\mu_0 < 0$. In other words,

$$p_W((u'(\psi(s))v_0)^{\mu_0}(A)) \in V^-(A_2).$$

Since this holds for every irreducible $A \ltimes \mathrm{SL}(2,\Delta_1(s))$-sub-representation W of V, we have

$$(u'(\psi(s))v_0)^{\mu_0}(A) \in V^-(A_2).$$

Note that $u'(\psi(s))$ preserves every eigenspace of A, we have

$$(u'(\psi(s))v_0)^{\mu_0}(A) = u'(\psi(s)) \cdot (v_0)^{\mu_0}(A).$$

Therefore,

$$u'(\psi(s)) \cdot (v_0)^{\mu_0}(A) \in V^-(A_2), \qquad \text{for any } s \in J_{s_0} \setminus \{s_0\}.$$

Since ψ is not constant, we can choose $s_1, s_2 \in J_{s_0} \setminus \{s_0\}$ such that $\psi(s_1) \neq \psi(s_2)$. Note that $u'(r)$, $a_2(t)$ are both contained in $H_2 \cong \mathrm{SL}(2,\mathbb{R})$, where

$$H_2 := \left\{ \begin{bmatrix} 1 & \\ & h_2 \end{bmatrix} : h_2 \in \mathrm{SL}(2,\mathbb{R}) \right\}.$$

Therefore,

$$u'(\psi(s_1))(v_0)^{\mu_0}(A) \in V^-(A_2) \text{ and } u'(\psi(s_1))(v_0)^{\mu_0}(A) \in V^-(A_2).$$

This contradicts Corollary 4.2 for the $H_2 \cong \mathrm{SL}(2,\mathbb{R})$ action on V with $u'(\psi(s_1))(v_0)^{\mu_0}(A)$ playing the role of v, and $u'(r)$ playing the role of u for $r = \psi(s_2) - \psi(s_1) \neq 0$, and the u' expanding diagonal subgroup A_2. Therefore our assumption that $\mu_0 < 0$, or equivalently (4.8), is false. This completes the proof. □

Now let us prove the general case of Lemma 2.12.

Proof of Lemma 2.12. We use induction to complete the proof. For the case $m = n$, the lemma is due to Yang [20]. We provide a proof here.

When $m = n$, we take a point s_0 and a subinterval $J_{s_0} \subset I$ such that for all $s \in J_{s_0} \setminus \{s_0\}$,

$$\varphi(s) - \varphi(s_0) \in \mathrm{GL}(m,\mathbb{R}).$$

Then we consider the subgroup $\mathrm{SL}(2, \varphi(s)-\varphi(s_0)) \cong \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2m,\mathbb{R})$ for some fixed $s \in J_{s_0} \setminus \{s_0\}$ (see Definition 2.1), and apply Corollary 4.2 for $\mathrm{SL}(2,\mathbb{R})$ replaced by $\mathrm{SL}(2, \varphi(s)-\varphi(s_0))$, v replaced by $u(\varphi(s_0))v$ and $u(r)$ replaced by $u(\varphi(s)-\varphi(s_0))$. Then by (4.2),

$$u(\varphi(s_0))v \notin V^-(A) \text{ or } \lambda^{\max}((\varphi(s))v) \notin V^{-1}(A).$$

This completes the proof of the Lemma for the case of $m = n$.

If $m > n$, then by applying a suitable inner automorphism of $\mathrm{SL}(m+n,\mathbb{R})$ given by a coordinate permutation $\sigma_{m,n}$, we can convert this problem to the case of $m < n$. Therefore we will assume that $m < n$.

As inductive hypothesis, we assume that for all (m', n') such that

$$m' \leq m, n' \leq n \text{ and } m' + n' < m + n,$$

the conclusion of the Lemma holds. We want to prove that the conclusion holds for (m, n).

For contradiction, we assume that for some nonzero vector $v \in V$,

$$u(\varphi(s))v \in V^-(A)$$

for all $s \in I$. For $s \in I$, let $\mu_0(s) = \max\{\lambda : (u(\varphi(s))v)^\lambda(A) \neq 0\}$ and $\mu_0 = \max\{\mu_0(s) : s \in I\}$. Since φ is analytic, we have $\mu_0(s) = \mu_0$ for all but finitely many $s \in I$. By our assumption

$$\mu_0 < 0. \tag{4.11}$$

Fix $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that $\mu_0(s) = \mu_0(s_0) = \mu_0$ for all $s \in J_{s_0}$ and if we write $\varphi(s) = [\varphi_1(s); \varphi_2(s)]$, then $\varphi_1(s) - \varphi_1(s_0) \in \mathrm{GL}(m, \mathbb{R})$ for $s \in J_{s_0} \setminus \{s_0\}$. Let

$\psi : J_{s_0} \setminus \{s_0\} \to \mathrm{M}(m \times (n-m), \mathbb{R})$, be defined by $\psi(s) := (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$.

Then ψ is *generic* by the of genericity of φ (see Definition 1.1). Replacing v by $u(\varphi(s_0))v$ and $\varphi(s)$ by $\varphi(s) - \varphi(s_0)$, we may assume that $\varphi(s_0) = \mathbf{0}$.

For any fixed $s \in J_{s_0} \setminus \{s_0\}$, it is straightforward to verify that

$$u(\varphi(s)) = u'(-\psi(s))u([\varphi_1(s); \mathbf{0}])u'(\psi(s)), \text{ where} \tag{4.12}$$

$$u'(Y) := \begin{bmatrix} \mathrm{I}_m & & \\ & \mathrm{I}_m & Y \\ & & \mathrm{I}_{n-m} \end{bmatrix} \in Z_H(A) \text{ for } Y \in M(m \times (n-m), \mathbb{R}). \tag{4.13}$$

Therefore $u(\varphi(s))v \in V^-(A)$ implies that

$$u([\varphi_1(s); \mathbf{0}])u'(\psi(s))v \in V^-(A).$$

Let us denote

$$A_1 := \left\{ a_1(t) := \begin{bmatrix} e^t \mathrm{I}_m & & \\ & e^{-t}\mathrm{I}_m & \\ & & \mathrm{I}_{n-m} \end{bmatrix} : t \in \mathbb{R} \right\},$$

and

$$A_2 := \left\{ a_2(t) := \begin{bmatrix} \mathrm{I}_m & & \\ & e^{(n-m)t}\mathrm{I}_m & \\ & & e^{-mt}\mathrm{I}_{n-m} \end{bmatrix} : t \in \mathbb{R} \right\}.$$

We express V as the direct sum of common eigenspaces of A_1 and A_2:

$$V = \bigoplus_{\delta_1, \delta_2} V^{\delta_1, \delta_2}, \text{ where } V^{\delta_1, \delta_2} := \left\{ v \in V : a_1(t)v = e^{\delta_1 t}v, a_2(t)v = e^{\delta_2 t}v \text{ for all } t \in \mathbb{R} \right\}. \tag{4.14}$$

Then because $a(t) = a_1(nt)a_2(t)$, we have

$$V^\lambda(A) = \bigoplus_{n\delta_1 + \delta_2 = \lambda} V^{\delta_1, \delta_2}. \tag{4.15}$$

For any vector $v \in V$, let v^{δ_1, δ_2} denote the projection of v onto the eigenspace V^{δ_1, δ_2}.

We also decompose V as the direct sum of irreducible sub-representations of $A \ltimes \mathrm{SL}(2, \varphi_1(s))$. For any such sub-representation $W \subset V$, let $p_W : V \to W$ denote the A-equivariant projection. By the theory of finite dimensional irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ (see [8, Claim 11.4]), there exists a basis $\{w_0, w_1, \ldots, w_r\}$ of W such that

$$a_1(t)w_i = e^{(r-2i)t}w_i, \quad \text{for } 0 \le i \le r. \tag{4.16}$$

We claim that each w_i is also an eigenvector for A. In fact,

$$a(t) = a_1((m+n)t/2)b(t), \quad \text{where } b(t) = \begin{bmatrix} e^{\frac{n-m}{2}t}\mathrm{I}_{2m} & \\ & e^{-mt}\mathrm{I}_{n-m} \end{bmatrix} \in Z_H(\mathrm{SL}(2, \varphi_1(s)),$$

and hence $b(t)$ acts on W as a scalar $e^{\delta t}$ for some $\delta \in \mathbb{R}$. Therefore,

$$a(t)w_i = e^{((r-2i)(m+n)/2 + \delta)t}w_i, \quad \text{for } 1 \le i \le r. \tag{4.17}$$

Since $(m+n)/2 > 0$, if $k < i$ then the A-weight of w_k is strictly greater than the A-weight of w_i.

Since $u'(\psi(s)) \in Z_H(A)$, μ_0 is the highest A-weight for v, we have that μ_0 is also the highest A-weight for $u'(\psi(s))v$ and

$$(u'(\psi(s))v)^{\mu_0}(A) = u'(\psi(s))v^{\mu_0}(A).$$

Now suppose that W as above is such that $p_W(u'(\psi(s))v^{\mu_0}(A)) \neq \mathbf{0}$. Then

$$p_W(u'(\psi(s))v^{\mu_0}(A)) = a_i w_i, \text{ for some } 0 \leq i \leq r,\ 0 \neq a_i \in \mathbb{R};$$

by (4.17) $\mu_0 = (r-2i)(m+n)/2 + \delta$. For $k < i$, the weight of w_k for A_1 is greater than that of w_i, so the A-weight of w_k is greater than the A-weight of w_i which equals μ_0. Since the projection p_W is A-equivariant and μ_0 is the highest A-weight, we have

$$p_W(u'(\psi(s))v) = \sum_{k \geq i} a_k w_k, \quad \text{where } a_k \in \mathbb{R}.$$

We claim that $r - 2i \geq 0$. In fact, if $r - 2i < 0$, then by (4.16), $p_W(u'(\psi(s))v) \in V^-(A_1)$. By Corollary 4.2,

$$V^{-0}(A_1) \not\ni u([\varphi_1(s); \mathbf{0}])p_W(u'(\psi(s))v) = p_W(u([\varphi_1(s); \mathbf{0}])u'(\psi(s))v).$$

So $p_W(u([\varphi_1(s); \mathbf{0}])u'(\psi(s))v)$ must have nonzero projection on $\mathbb{R}w_k$ for some $k < i$. Hence

$$u([\varphi_1(s); \mathbf{0}])u'(\psi(s))v$$

has nonzero projection $V^\mu(A)$ for some $\mu > \mu_0$. Now since $u'(-\psi(s)) \in Z_H(A)$, the projection of $u(\varphi(s))v = u'(-\psi(s))u([\varphi_1(s); \mathbf{0}])u'(\psi(s))v$ on $V^\mu(A)$ is nonzero for $\mu > \mu_0$. This contradicts our choice of μ_0 and proves the claim that $r - 2i \geq 0$.

This claim implies that for any (δ_1, δ_2), if $(u'(\psi(s))v^{\mu_0}(A))^{\delta_1,\delta_2} \neq \mathbf{0}$ then $\delta_1 \geq 0$. Since $\mu_0 = n\delta_1 + \delta_2 < 0$, we have $\delta_2 < 0$. In other words,

$$\{u'(\psi(s))v^{\mu_0}(A) : s \in J_{s_0} \setminus \{s_0\}\} \subset V^-(A_2).$$

Now $u'(\psi(s))$ and A_2 are both contained in

$$\begin{bmatrix} \mathrm{I}_m & \\ & \mathrm{SL}(n, \mathbb{R}) \end{bmatrix} \cong \mathrm{SL}(m + (n-m), \mathbb{R}).$$

Our inductive hypothesis for $(m, n-m)$ tells that this is impossible because ψ is *generic*.

This finishes the proof. $\square$

Let us prove Lemma 3.6.

We first prove the following statement.

Lemma 4.3. *Let V be a finite dimensional representation of* $\mathrm{SL}(m+n, \mathbb{R})$ *and let*

$$A := \left\{ a(t) := \begin{bmatrix} e^{nt}\mathrm{I}_m & \\ & e^{-mt}\mathrm{I}_n \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Let

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

be an analytic curve. Suppose there exists a nonzero vector $v \in V$ such that

$$\{u(\varphi(s))v : s \in I\} \subset V^{-0}(A).$$

Then for all $s \in I$, $(u(\varphi(s))v)^0(A)$ is invariant under the unipotent subgroup

$$\{u(h\varphi'(s)) : h \in \mathbb{R}\}.$$

Proof of Lemma 4.3. For any $h \in \mathbb{R}$, on the one hand,

$$a(t)u(\varphi(s+e^{-(m+n)t}h))v = (u(\varphi(s+e^{-(m+n)t}h))v)^0(A) + O(e^{-\lambda(m,n)t}),$$

for some $\lambda(m,n) > 0$ depending on m and n. As $t \to +\infty$,

$$(u(\varphi(s+e^{-(m+n)t}h))v)^0(A) \to (u(\varphi(s))v)^0(A), \text{ and } O(e^{-\lambda(m,n)t}) \to \mathbf{0}.$$

Thus, as $t \to +\infty$,

$$a(t)u(\varphi(s+e^{-(m+n)t}h))v \to (u(\varphi(s))v)^0(A).$$

On the other hand,

$$\begin{aligned} & a(t)u(\varphi(s+e^{-(m+n)t}h))v \\ = \ & a(t)u(he^{-(m+n)t}\varphi'(s))u(O(e^{-2(m+n)t}))u(\varphi(s))v \\ = \ & a(t)u(he^{-(m+n)t}\varphi'(s))a(-t)a(t)u(O(e^{-2(m+n)t}))a(-t)a(t)u(\varphi(s))v \\ = \ & u(h\varphi'(s))u(O(e^{-(m+n)t}))a(t)u(\varphi(s))v. \end{aligned}$$

As $t \to +\infty$, $u(O(e^{-(m+n)t})) \to \mathrm{id}$, $a(t)u(\varphi(s))v \to (u(\varphi(s))v)^0(A)$. Therefore, as $t \to +\infty$,

$$a(t)u(\varphi(s+e^{-(m+n)t}h))v \to u(h\varphi'(s))(u(\varphi(s))v)^0(A).$$

This shows that $(u(\varphi(s))v)^0(A)$ is invariant under $\{u(h\varphi'(s)) : h \in \mathbb{R}\}$. □

Proof of Lemma 3.6. The strategy of the proof is similar to that of Lemma 2.12.

We begin with the case $m = n$. This case is studied in [20] but the statement proved there is weaker than the statement here.

Fix a point $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that $\varphi(s) - \varphi(s_0)$ is invertible for all $s \in J_{s_0} \setminus \{s_0\}$ and moreover, $\{\mathfrak{n}^-((\varphi(s_1)-\varphi(s_0))^{-1} - (\varphi(s_2)-\varphi(s_0))^{-1}) : s_1, s_2 \in J_{s_0} \setminus \{s_0\}\}$ is not contained in any proper observable subalgebra of $\mathfrak{sl}(2m,\mathbb{R})$. By replacing $\varphi(s)$ by $\varphi(s)-\varphi(s_0)$, we may assume that $\varphi(s_0) = \mathbf{0}$.

In the isomorphism $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SL}(2,\varphi(s))$ (see Definition 2.1), $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ corresponds to $u(\varphi(s))$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ corresponds to $u^-(\varphi^{-1}(s))$, and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ corresponds to $\sigma(\varphi(s))$. By Corollary 4.2, we have that $v, u(\varphi(s))v \in V^{-0}(A)$ implies that

$$v = u^-(-\varphi^{-1}(s))v^0(A).$$

In particular, $v^0(A) \neq \mathbf{0}$.

Taking any $s_1, s_2 \in J_{s_0} \setminus \{s_0\}$, we have

$$u^-(-\varphi^{-1}(s_1))v^0(A) = v = u^-(-\varphi^{-1}(s_2))v^0(A).$$

This shows that $v^0(A)$ is fixed by $u^-(\varphi^{-1}(s_1) - \varphi^{-1}(s_2))$ for all $s_1, s_2 \in J_{s_0} \setminus \{s_0\}$. By definition, $v^0(A)$ is also fixed by A. Let L denote the subgroup of H stabilizing $v^0(A)$, and $\mathfrak{l}$ denote its Lie algebra. Then from the above argument we have $\mathfrak{l}$ is observable and contains $\mathcal{E} \in \mathrm{Lie}(A)$ (see (1.2)) and

$$\{\mathfrak{n}^-((\varphi(s_1)-\varphi(s_0))^{-1} - (\varphi(s_2)-\varphi(s_0))^{-1}) : s_1, s_2 \in J_{s_0} \setminus \{s_0\}\};$$

recall that earlier we had replaced $\varphi(s)$ by $\varphi(s)-\varphi(s_0)$ and assumed that $\varphi(s_0) = \mathbf{0}$ for notational simplicity. Because φ is *supergeneric*, in view of (1.3) we have that $L = H$. Since V is an irreducible representation of H, V is trivial.

This finishes the proof for $m = n$.

For the general case we give the proof by an inductive argument. Suppose the statement holds for all (m', n') such that $m' \leq m$, $n' \leq n$ and $m' + n' < m + n$. We want to prove the statement for (m, n).

We choose a point s_0 and a subinterval $J_{s_0} \subset I$ such that the following statements hold:

1. If we write $\varphi(s) = [\varphi_1(s); \varphi_2(s)]$ where $\varphi_1(s)$ is the first m by m block, and $\varphi_2(s)$ is the rest m by $n-m$ block, then for any $s \in J_{s_0} \setminus \{s_0\}$, $\varphi_1(s) - \varphi_1(s_0)$ is invertible.
2. The curve $\psi(s) = (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$ is *supergeneric* as a curve from $J_{s_0} \setminus \{s_0\}$ to $\mathrm{M}(m \times (n-m), \mathbb{R})$.

Without loss of generality we may assume that $\varphi(s_0) = \mathbf{0}$ and $v \in V^{-0}(A)$. The notations such that $u'(\cdot)$, A_2 and $v^{\mu_0}(A)$ have the same meaning as in the proof of Lemma 2.12. Using the same argument as the proof of Lemma 2.12, we can deduce that

$$\{u'(\psi(s))v^{\mu_0}(A) : s \in J_{s_0} \setminus \{s_0\}\} \subset V^{-0}(A_2).$$

By inductive hypothesis, we conclude that $v^{\mu_0}(A)$ is fixed by the whole

$$H' = \begin{bmatrix} \mathrm{I}_m & \\ & \mathrm{SL}(n, \mathbb{R}) \end{bmatrix} \cong \mathrm{SL}(n, \mathbb{R}).$$

In particular, $v^{\mu_0}(A)$ is fixed by A_2. Let the direct sum

$$V^{\mu_0}(A) = \bigoplus_{n\delta_1 + \delta_2 = \mu_0} V^{\delta_1, \delta_2}$$

be as in the proof of Lemma 2.12. From the proof of Lemma 2.12 we know that any nonzero projection $(v^{\mu_0}(A))^{\delta_1, \delta_2}$ of $v^{\mu_0}(A)$ with respect to this direct sum satisfies δ_1 (the eigenvalue for A_1) is non-negative. Because we have $\delta_2 = 0$ and $n\delta_1 + \delta_2 \le 0$, we conclude that $\delta_1 = \delta_2 = 0$. This implies that $\mu_0 = 0$. By Lemma 4.3, we have $v^0(A)$ is invariant under $\{u(h\varphi'(s_0)) : h \in \mathbb{R}\}$. By our assumption, $\varphi'(s_0)$ has rank b. By conjugating it with elements in H', we have that $u(X)$ fixes $v^0(A)$ for any X with rank b. Note that the space spanned by all rank b matrices is the whole space $\mathrm{M}(m \times n, \mathbb{R})$. This shows that $v^0(A)$ is invariant under the whole $U^+(A)$. Since $v^0(A)$ is also invariant under A, $v^0(A)$ is invariant under the whole group H. Since we assume that V is an irreducible representation of H, we conclude that V is trivial.

This completes the proof. □

Lemma 3.6 is sufficient to prove the equidistribution result under the *supergeneric* condition.

Now we consider the case $n = km$ and the curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is *generic*, and possibly not *supergeneric*. In this case, we will prove the following result which can be thought of as a generalization of Corollary 4.2. It will be applied to describe the obstruction to equidistribution for *generic* curves as done in §5.

Let us denote

$$\begin{aligned} P^-(A) &= \{h \in H : \lim_{t \to \infty} a(t) h a(t)^{-1} \text{ exists in } H\} \\ &= \left\{ \begin{bmatrix} g_1 & \\ X & g_2 \end{bmatrix} \in H : g_1 \in \mathrm{GL}(m, \mathbb{R}),\ g_2 \in \mathrm{GL}(n, \mathbb{R}),\ X \in \mathrm{M}(n \times m, \mathbb{R}) \right\} \\ &= U^-(A) Z_H(A). \end{aligned} \tag{4.18}$$

Note that $P^-(A)$ is a maximal parabolic subgroup of H.

Lemma 4.4. *Let $n = km$ and*

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

be an analytic generic *curve. Let V be an irreducible representation of $H = \mathrm{SL}(m+n, \mathbb{R})$ and $v \in V$ be a nonzero vector of V. Let A denote the diagonal subgroup as before. Suppose*

$$\{u(\varphi(s))v : s \in I\} \subset V^{-0}(A).$$

Then for all $s_0 \in I$ satisfying the generic *condition, there exists $\xi(s_0) \in P^-(A)$ such that*

$$(u(\varphi(s_0))v)^0(A) = \xi(s_0)u(\varphi(s_0))v.$$

Definition 4.5. Assume $n = km$, then we can write $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ as $[\Phi_1; \Phi_2; \ldots; \Phi_k]$ where Φ_i denotes the i-th m by m block of Φ. An analytic curve $\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$ is called *standard* at $s_0 \in I$ if there exist k points $s_1, \ldots, s_k \in I$ such that for $i = 1, \ldots, k$, we have

$$\varphi(s_i) - \varphi(s_0) = [\mathbf{0}; \ldots; \varphi_i(s_i) - \varphi_i(s_0); \ldots; \mathbf{0}],$$

where $\varphi_i(s_i) - \varphi_i(s_0)$ is invertible, it appears in the i-th $m \times m$ block and all other blocks are $\mathbf{0}$.

In order to prove Lemma 4.4, we will need the following lemma.

Lemma 4.6. *Assume $n = km$. For any analytic curve*

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

which is generic *at $s_0 \in I$, there exists an element $z' = z'(s_0) \in Z_H(A)$ depending analytically on s_0, such that the conjugated curve*

$$\phi := z' \cdot \varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is standard *at s_0; where the action of $Z_H(A)$ on $\mathrm{M}(m \times n, \mathbb{R})$ is given by (2.2).*

Proof. Replacing $\varphi(s)$ by $\varphi(s) - \varphi(s_0)$, we may assume that $\varphi(s_0) = \mathbf{0}$.

We will prove the statement by induction on k.

When $k = 1$, the statement follows from the definition of *generic* property.

Suppose the statement holds for all $k' < k$. Then we will prove the statement for $n = km$.

We write

$$\varphi(s) = [\varphi_1(s); \varphi_2(s); \ldots; \varphi_k(s)], \tag{4.19}$$

where $\varphi_i(s)$ is the i-th m by m block of $\varphi(s)$. From the definition of *generic* property (Definition 1.1), there exist a subinterval $J_{s_0} \subset I$ such that for $s \in J_{s_0} \setminus \{s_0\}$, $\varphi_1(s)$ is invertible, and the curve $\psi : J_{s_0} \setminus \{0\} \to \mathrm{M}(m \times (n - m), \mathbb{R})$ defined by

$$\psi(s) = [\psi_1(s); \psi_2(s); \ldots; \psi_{k-1}(s)],$$

where $\psi_i(s) = \varphi_1^{-1}(s)\varphi_i(s)$ is *generic*.

As before, let us denote

$$u'(\psi(s)) = \begin{bmatrix} \mathrm{I}_m & & \\ & \mathrm{I}_m & \psi(s) \\ & & \mathrm{I}_{n-m} \end{bmatrix} \in Z_H(A)$$

for $s \in J_{s_0}$. Now we fix a point $s_1 \in J_{s_0}$ and a subinterval $J_{s_1} \subset J_{s_0}$ such that ψ satisfies the *generic* condition for s_1 and J_{s_1}. Replacing φ by $u'(\psi(s_1)) \cdot \varphi$, recall (2.2), we get

$$\varphi(s_1) = [\varphi_1(s_1); \mathbf{0}; \ldots; \mathbf{0}] \text{ and } \psi(s_1) = \mathbf{0}.$$

Let

$$A' := \left\{ a'(t) := \begin{bmatrix} \mathrm{I}_m & & \\ & e^{(n-m)t}\mathrm{I}_m & \\ & & e^{-mt}\mathrm{I}_{n-m} \end{bmatrix} : t \in \mathbb{R} \right\}$$

and

$$H' := \left\{ \begin{bmatrix} \mathrm{I}_m & \\ & X \end{bmatrix} : X \in \mathrm{SL}(n, \mathbb{R}) \right\} \subset Z_H(A).$$

By inductive hypothesis, there exists $z'' \in Z_{H'}(A') \subset Z_H(A)$, such that $z'' \cdot \psi$ is *standard* at s_1. Since $\psi(s_1) = \mathbf{0}$, there exist $s_2, s_3, \ldots, s_k \in J_{s_1}$ such that

$$z'' \cdot \psi(s_i) = [\mathbf{0}; \ldots; \psi_{i-1}(s_i); \ldots; \mathbf{0}], \text{ for } i = 2, \ldots, k,$$

where the $(i-1)$-th $m \times m$ block $\psi_{i-1}(s_i)$ is invertible. Now we replace φ by $z'' \cdot \varphi$. Note that by definition, $\varphi_i(s) = \varphi_1(s)\psi_{i-1}(s)$ for $i = 2, \ldots, k$, and $s \in J_{s_0}$. Thus, we have for $i = 2, \ldots, k$,

$$\varphi(s_i) = [\varphi_1(s_i); \mathbf{0}; \ldots; \mathbf{0}; \varphi_1(s_i)\psi_{i-1}(s_i); \mathbf{0}; \ldots; \mathbf{0}].$$

Let z_1 denote the following element:

$$z_1 := \begin{bmatrix} \mathrm{I}_m & & & & \\ & \mathrm{I}_m & & & \\ & \psi_1^{-1}(s_2) & \mathrm{I}_m & & \\ & \vdots & & \ddots & \\ & \psi_{k-1}^{-1}(s_k) & \mathbf{0} & \cdots & \mathrm{I}_m \end{bmatrix} \in Z_H(A).$$

By direct calculation, we have that $z_1 \cdot \varphi$ is *standard* at s_0 with given $s_1, s_2, \ldots, s_k$.

This completes the proof.

□

Now we are ready to prove Lemma 4.4.

Proof of Lemma 4.4. By Lemma 4.6, we may conjugate the curve by some $z'(s_0) \in Z_H(A)$, such that the conjugated curve, which we still denote by φ, satisfies the following: there exist

$$s_1, s_2, \ldots, s_k \in I,$$

such that, in view of the notation in (4.19),

$$\varphi(s_i) - \varphi(s_0) = [\mathbf{0}; \ldots; \varphi_i(s_i) - \varphi_i(s_0); \mathbf{0}; \ldots; \mathbf{0}] \text{ for } i = 1, 2, \ldots, k.$$

Replacing v by $u(\varphi(s_0))v$ and $\varphi(s)$ by $\varphi(s) - \varphi(s_0)$, we may assume that $\varphi(s_0) = \mathbf{0}$ and $v \in V^{-0}(A)$. Then it suffices to show that

$$v = \xi v^0(A), \text{ for some } \xi \in P^-(A). \tag{4.20}$$

For each $i = 1, 2, \ldots, k$, let

$$A_i := \left\{ a_i(t) := \begin{bmatrix} e^t\mathrm{I}_m & & & \\ & \ddots & & \\ & & e^{-t}\mathrm{I}_m & \\ & & & \ddots \end{bmatrix} : t \in \mathbb{R} \right\},$$

where $e^{-t}\mathrm{I}_m$ appears in the $(i+1)$-th $m \times m$ diagonal block, and the dotted entries are all equal to 1. We denote its Lie algebra by

$$\mathfrak{a}_i := \{t\mathcal{A}_i : t \in \mathbb{R}\},$$

where $\mathcal{A}_i := \log a_i(1)$. Let $\mathrm{SL}(2, \varphi(s_i))$ denote the $\mathrm{SL}(2,\mathbb{R})$ copy in H containing A_i as the diagonal subgroup and $\{u(r\varphi(s_i)) : r \in \mathbb{R}\}$ as the upper triangular unipotent subgroup, and $a_i(t)u(r\varphi(s_i))a_i(-t) = u(re^{2t}\varphi(s_i))$.

We express the representation V as the direct sum of common eigenspaces of $A_1, A_2, \ldots, A_k$:

$$V = \bigoplus_{\boldsymbol{\delta}=(\delta_1,\ldots,\delta_k)\in\mathbb{Z}^k} V(\boldsymbol{\delta}), \tag{4.21}$$

where

$$V(\boldsymbol{\delta}) := \left\{v \in V : a_i(t)v = e^{\delta_i t}v \text{ for all } i = 1, 2, \ldots, k \text{ and } t \in \mathbb{R}\right\}.$$

Let $w \in V(\boldsymbol{\delta})\backslash\{\mathbf{0}\}$. We claim that for all $i = 1, 2, \ldots, k$ and $\mathbf{e}_i = (-1, \ldots, -2, \ldots, -1)$, with 2 in the i-th coordinate,

$$\mathfrak{n}(\varphi(s_i))w \in V(\boldsymbol{\delta} - \mathbf{e}_i), \tag{4.22}$$

recall that $\mathfrak{n}(\varphi(s_i)) = \log u(\varphi(s_i))$.

It is straight forward to check that

$$[\mathcal{A}_i, \mathfrak{n}(\varphi(s_i))] = 2\mathfrak{n}(\varphi(s_i)) \text{ and } [\mathcal{A}_j, \mathfrak{n}(\varphi(s_i))] = \mathfrak{n}(\varphi(s_i)) \text{ for } j \neq i.$$

Therefore,

$$\mathcal{A}_j\mathfrak{n}(\varphi(s_i))w = \mathfrak{n}(\varphi(s_i))\mathcal{A}_j w + [\mathcal{A}_j, \mathfrak{n}(\varphi(s_i))]w = \begin{cases}(\delta_j + 1)w & \text{if } j \neq i\\ (\delta_i + 2)w & \text{if } j = i.\end{cases}$$

This proves (4.22).

Let $\mathcal{A} := \log a(1)$, it is easy to see that $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_k$. Therefore,

$$V^\sigma(A) = \bigoplus_{\delta_1+\cdots+\delta_k=\sigma} V(\delta_1, \ldots, \delta_k).$$

Fix any $i = 1, \ldots, k$. Because $A_1, \ldots, A_k$ normalize $\mathrm{SL}(2, \varphi(s_i))$, we can decompose V into the direct sum of irreducible representations V_p of $\mathrm{SL}(2, \varphi(s_i))$ which are invariant under $A_1, \ldots, A_k$:

$$V = \bigoplus_p V_p. \tag{4.23}$$

As a standard fact on $\mathrm{SL}(2,\mathbb{R})$ representations (see [8, Claim 11.4], for example), every V_p admits a standard basis $\{w_0, w_1, \ldots, w_l\}$, such that for each $1 \le r \le l$, w_r is contained in some weight space $V(\delta_1, \delta_2, \cdots, \delta_k)$, and we index the basis elements such that $a_i(t)w_r = e^{(l-2r)t}w_r$; that is,

$$\text{if } w_r \in V(\delta_1, \delta_2, \cdots, \delta_k) \text{ then } \delta_i = l - 2r. \tag{4.24}$$

Moreover since $\mathfrak{n}(\varphi(s_i))w_s$ is a nonzero multiple of w_{s-1} for all $1 \le s \le l$, by (4.22) we have

$$w_{r-j} \in V(\boldsymbol{\delta} - j\mathbf{e}_i), \quad \text{for } r - l \le j \le r. \tag{4.25}$$

Let

$$\pi_p : V \to V_p$$

denote the canonical projection from V to V_p with respect to (4.23), and let

$$q(\boldsymbol{\delta}) : V \to V(\boldsymbol{\delta})$$

denote the canonical projection from V to $V(\boldsymbol{\delta})$ with respect to (4.21). Then

$$\pi_p \circ q(\boldsymbol{\delta}) = q(\boldsymbol{\delta}) \circ \pi_p. \tag{4.26}$$

We call a vector $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k) \in \mathbb{Z}^k$ *admissible* if $\delta_i \ge 0$ for all i.

Claim 4.7. For any $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k) \in \mathbb{Z}^k$, if $q(\boldsymbol{\delta})(v) \neq 0$, then $\boldsymbol{\delta}$ is *admissible.*

Proof of Claim 4.7. For $\boldsymbol{\delta} = (\delta_1, \dots, \delta_k)$, define

$$\sigma(\boldsymbol{\delta}) := \delta_1 + \cdots + \delta_k \in \mathbb{Z}.$$

Since $v \in V^{-0}(A)$, we have $\sigma(\boldsymbol{\delta}) \leq 0$. We now begin by assuming that the statement of this claim is valid for any $\boldsymbol{\delta}'$ such that $\sigma(\boldsymbol{\delta}') > \sigma(\boldsymbol{\delta})$; note that the statement is vacuously true if $\sigma(\boldsymbol{\delta}) = 0$ (in fact, in this case we have that $\boldsymbol{\delta} = \mathbf{0}$).

Let $1 \leq i \leq k$ be such that $\delta_i = \min(\delta_1, \dots, \delta_k)$. Then

$$\delta_i \leq \sigma(\boldsymbol{\delta})/k \leq 0, \text{ and if } \delta_i = 0 \text{ then } \boldsymbol{\delta} = \mathbf{0}. \tag{4.27}$$

For this choice of i, consider the decomposition (4.23) of V as $V = \bigoplus_p V_p$ with respect to the action of $\mathrm{SL}(2, \varphi(s_i))$. There exists some V_p such that $\pi_p(q(\boldsymbol{\delta})v) \neq \mathbf{0}$. If $\{w_0, w_1, \dots, w_l\}$ denotes the standard basis of V_p, then by (4.24), $\pi_p(q(\boldsymbol{\delta})v)$ is a nonzero multiple of w_r for some $0 \leq r \leq l$ such that $\delta_i = l - 2r$.

If $\pi_p(v)$ has a non-zero coefficient on w_{r-j} for some $1 \leq j \leq r$, then by (4.25), we have $w_{r-j} \in V(\boldsymbol{\delta} - j\mathbf{e}_i)$. But then $q(\boldsymbol{\delta} - j\mathbf{e}_i)(v) \neq 0$ and $\sigma(\boldsymbol{\delta} - j\mathbf{e}_i) = \sigma(\boldsymbol{\delta}) + j(k+1) > \sigma(\boldsymbol{\delta})$. By our inductive hypothesis, $\boldsymbol{\delta} - j\mathbf{e}_i$ is admissible, and hence $\boldsymbol{\delta}$ is admissible.

Now we can suppose that $\pi_p(v)$ is contained in the span of $w_r, \dots, w_l$. Then

$$a_i(t)w_{r+j} = e^{(\delta_i - 2j)t} w_{r+j} \text{ and } \delta_i - 2j \leq \delta_i, \quad \forall\, j = 0, \dots, l - r. \tag{4.28}$$

Therefore by (4.1) in Lemma 4.1 applied to V_p and the action of $\mathrm{SL}(2, \varphi_i(s_i))$, we have that

$$\lambda^{\max}(\pi_p(u(\varphi_i(s_i))v)) \geq -\lambda^{\max}(\pi_p(v)). \tag{4.29}$$

Now $\pi_p(u(\varphi_i(s_i))v) = u(\varphi_i(s_i))\pi_p(v)$ has a nonzero coefficient on w_{r-j} for some $j \in \{0, \dots, r\}$ such that

$$a_i(t)w_{r-j} = e^{(\delta_i + 2j)t} w_{r-j}$$

and by (4.28) and (4.29),

$$\delta_i + 2j \geq -\delta_i, \text{ and hence } j \geq -\delta_i. \tag{4.30}$$

By (4.25), $w_{r-j} \in V(\boldsymbol{\delta} - j\mathbf{e}_i)$. Therefore,

$$q(\boldsymbol{\delta} - j\mathbf{e}_i)(\pi_p(u(\varphi_i(s_i))v) \neq 0. \tag{4.31}$$

By (4.30) and (4.27),

$$\sigma(\boldsymbol{\delta} - j\mathbf{e}_i) = \sigma(\boldsymbol{\delta}) + j(k+1) \geq \sigma(\boldsymbol{\delta}) - (k+1)\delta_i \geq \sigma(\boldsymbol{\delta})(1 - (k+1)/k) \geq 0. \tag{4.32}$$

By our assumption, $u(\varphi_i(s_i))v \in V^{-0}(A)$. So by (4.31), $V(\boldsymbol{\delta} - j\mathbf{e}_i) \subset V^{-0}(A)$. Hence $\sigma(\boldsymbol{\delta} - j\mathbf{e}_i) \leq 0$. Therefore all terms in (4.32) are zero. Therefore $\sigma(\boldsymbol{\delta}) = 0$ and $\delta_i = 0$. Therefore by (4.27), we have that $\boldsymbol{\delta} = \mathbf{0}$, which is admissible. This completes the proof of Claim 4.7. □

Now we get back to the proof of (4.20). For $i = 0, 1, \dots, k$, let us denote

$$E_0 = \{\mathbf{0}\} \text{ and } E_i := \{c_1\mathbf{e}_1 + \cdots + c_i\mathbf{e}_i : c_1, \dots, c_i \in \mathbb{Z}_{\geq 0}\},$$

and define for any $v' \in V$,

$$v'_i := \sum_{\boldsymbol{\delta} \in E_i} q(\boldsymbol{\delta})(v'). \tag{4.33}$$

By Claim 4.7, $v = v_k$ and $v^0 = q(\mathbf{0})(v) = v_0$. Therefore, in order to prove (4.20), it is sufficient to show the following:

$$v_i \in U^-(A)v_{i-1}, \quad \text{for all } 1 \leq i \leq k. \tag{4.34}$$

To prove this, fix any $1 \le i \le k$ and consider the decomposition

$$V = \bigoplus_p V_p$$

as in (4.23) into $\mathrm{SL}(2, \varphi(s_i))$-irreducible and $A_1, \dots, A_k$-invariant subspaces V_p. Let $\pi_p : V \to V_p$ denote the canonical projection with respect to this decomposition. By (4.26) and (4.33),

$$\pi_p(v')_j = \pi_p(v'_j), \quad \text{for all } v' \in V \text{ and } j = 0, 1, \dots, k.$$

Hence

$$\pi_p(v_{i-1}) = \pi_p((v_i)_{i-1}) = \pi_p(v_i)_{i-1}.$$

Therefore

$$\text{if } \pi_p(v_i) = 0 \text{ then } \pi_p(v_{i-1}) = 0. \tag{4.35}$$

Now suppose that $\pi_p(v_i) \neq 0$. Let $\{w_0, \dots, w_l\}$ denote a standard basis of V_p; that is, (4.24) holds. Let $0 \le r \le l$ be such that

$$\pi_p(v_i) \subset \mathrm{Span}\{w_r, \dots, w_l\} \setminus \mathrm{Span}\{w_{r+1}, \dots, w_l\}. \tag{4.36}$$

In particular, $\pi_p(v_i)$ has a nonzero projection on w_r. Hence by Claim 4.7,

$$w_r \in V(c_1\mathbf{e}_1 + \cdots + c_i\mathbf{e}_i) \quad \text{for some } c_1, \dots, c_i \in \mathbb{Z}_{\ge 0}. \tag{4.37}$$

By (4.25) we have that

$$w_{r+j} \in V(c_1\mathbf{e}_1 + \cdots + c_{i-1}\mathbf{e}_{i-1} + (c_i + j)\mathbf{e}_i), \quad \text{for all } -r \le j \le l - r. \tag{4.38}$$

Therefore $\pi_p(v) \in \sum_{\boldsymbol{\delta} \in E_i} V(\boldsymbol{\delta})$. Hence

$$\pi_p(v_i) = \pi_p(v)_i = \pi_p(v). \tag{4.39}$$

By (4.38) we have

$$a_i(t) w_{r+j} = e^{-(\lambda + 2j)t} w_{r+j} \quad \text{for } -r \le j \le l - r, \text{ where } \lambda = c_1 + \cdots + c_{i-1} + 2c_i. \tag{4.40}$$

We apply Lemma 4.1 to the $\mathrm{SL}_2(\varphi(s_i))$-action on V_p and the vector $\pi_p(v_i)$. Let $-r \le j \le l - r$ be such that

$$u(\varphi(s_i))\pi_p(v_i) \subset \mathrm{Span}\{w_{r+j}, \dots, w_l\} \setminus \mathrm{Span}\{w_{r+j+1}, \dots, w_l\}. \tag{4.41}$$

Then by (4.1), (4.36) and (4.40) we get

$$-(\lambda + 2j) \ge \lambda. \tag{4.42}$$

On the other hand by (4.39) and our basic assumption we have

$$u(\varphi(s_i))\pi_p(v_i) = u(\varphi(s_i))\pi_p(v) = \pi_p(u(\varphi(s_i))v) \in V^{-0}(A).$$

Hence by (4.38) and (4.41) we have

$$0 \ge \sigma(c_1\mathbf{e}_1 + \cdots + c_{i-1}\mathbf{e}_{i-1} + (c_i + j)\mathbf{e}_i) = (\lambda - c_i + j)(-(k+1)). \tag{4.43}$$

Now combining (4.42) and (4.43), and we get

$$c_i \le \lambda + j \le 0.$$

On the other hand, by (4.37), $c_i \ge 0$. Therefore $c_i = 0$ and $j = -\lambda$. Since $c_i = 0$, by (4.38) we have that the projection of $\pi_p(v_i)$ on the line $\mathbb{R}w_r$ equals

$$\pi_p(v_i)_{i-1} = \pi_p((v_i)_{i-1}) = \pi_p(v_{i-1}).$$

And since $j = -\lambda$, we have equality in (4.42), which corresponds to equality in (4.1) of Lemma 4.1. Therefore (4.3) holds and in view of Definition 2.1, we get

$$\pi_p(v_i) = u^-(\mathbf{0}, \ldots, -\varphi_i(s_i)^{-1}, \ldots, \mathbf{0})\pi_p(v_{i-1}) = \pi_p(u^-(0, \ldots, -\varphi_i(s_i)^{-1}, \ldots, 0)v_{i-1}), \tag{4.44}$$

where

$$u^-(\mathbf{0}, \ldots, -\varphi_i(s_i)^{-1}, \ldots, \mathbf{0}) := \begin{bmatrix} I_m & & & & & \\ \mathbf{0} & I_m & & & & \\ \vdots & & \ddots & & & \\ -\varphi_i(s_i)^{-1} & & & I_m & & \\ \vdots & & & & \ddots & \\ \mathbf{0} & & & & & I_m \end{bmatrix} \in \mathrm{SL}(2, \varphi(s_i)) \cap U^-(A).$$

Therefore, due to (4.35), (4.44) holds for all p, and hence (4.34) holds.

This completes the proof. □

Remark 4.8.

1. Though our proof works for the special case $n = km$, we conjecture that the conclusion of Lemma 4.4 should hold for general (m, n).
2. From the proof we can see, if we assume $\varphi(s_0) = \mathbf{0}$ and $v \in V^{-0}(A)$, then $z'(s_0) \cdot v^0(A)$ is fixed by

$$B := \left\{ b(t_1, t_2, \ldots, t_k) := \begin{bmatrix} e^{t_1}\mathrm{I}_m & & & \\ & e^{t_2}\mathrm{I}_m & & \\ & & \ddots & \\ & & & e^{t_k}\mathrm{I}_m \end{bmatrix} : t_1 + t_2 + \cdots + t_k = 0 \right\}.$$

5. **OBSTRUCTION TO EQUIDISTRIBUTION.** We will study the obstruction of equidistribution of the expanding curves $\{a(t)u(\varphi(I))x : t > 0\}$ as $t \to +\infty$ and describe limit measures if equidistribution fails.

Our present technique is insufficient to handle *non-generic* curves. In this paper, we focus on *generic* curves.

If m and n are co-prime, the *generic* condition is the same as the *supergeneric* condition, so there is nothing to discuss in this case.

Therefore we consider the case $(m, n) > 1$ and the analytic curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is *generic* but not *supergeneric*. However, for now, we could only handle the case $n = km$ where $k > 1$ is some positive integer (see Example A.8 for an example of *generic* curve which is not *supergeneric*). To handle general (m, n), we need some general version of Lemma 4.4. With these assumptions, we want to describe the obstruction of equidistribution of $\{a(t)u(\varphi(I))x : t > 0\}$ as $t \to \infty$.

In this section, we always assume that $n = km$ and the analytic curve

$$\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$$

is *generic*.

Theorem 5.1 (See [19, Proposition 4.9]). *Let $x = g\Gamma$ be as in Theorem 1.4. Suppose the expanding curves $\{a(t)u(\varphi(s))x : t > 0\}$ do not tend to be equidistributed* *along some subsequence $t_i \to +\infty$. By Proposition 3.5, there exist $L \in \mathcal{L}$ and $\gamma \in \Gamma$ such that*

$$u(\varphi(s))g\gamma p_L \in V^{-0}(A),$$

for all $s \in I$. Then there exist $h \in H$ and some Lie subgroup F of G containing A such that $Fhg\Gamma$ is closed in G/Γ, $h^{-1}Fh$ fixes $v = g\gamma p_L$, and

$$\{u(\varphi(s)) : s \in I\} \subset P^-(A)Fh.$$

Recall that $P^-(A) = Z_H(A)U^-(A)$ denotes the maximal parabolic subgroup of H associated with A.

Proof. Let $s_0 \in I$ such that every point in a neighborhood J of s_0 satisfies the *generic* condition. By Lemma 4.4, for every $s \in I$, we have that

$$\lim_{t\to\infty} a(t)u(\varphi(s))v = (u(\varphi(s))v)^0(A) = \xi(s)u(\varphi(s))v,$$

for some $\xi(s) \in P^-(A)$. Let $p_0 = (u(\varphi(s_0))v)^0(A)$. Then $p_0 = \xi(s_0)u(\varphi(s_0))v$. This implies that

$$\lim_{t\to+\infty} a(t)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}p_0 = \xi(s)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}p_0.$$

Let $F_1 := N^1_G(L)$, then F_1 is the stabilizer of

$$p_L = (g\gamma)^{-1}v = (g\gamma)^{-1}u(-\varphi(s_0))\xi(s_0)^{-1}p_0.$$

Since the orbit Γp_L is discrete, we have that ΓF_1 is closed in G. Let $h := \xi(s_0)u(\varphi(s_0)) \in H$ and

$$F := (hg\gamma)F_1(hg\gamma)^{-1}.$$

It is easy to see that F is the stabilizer of p_0. Since p_0 is invariant under A, we have that $A \subset F$. Secondly, $Fhg\Gamma = hg\gamma F_1\Gamma$ is closed. Finally, it is easy to check that $h^{-1}Fh$ is the stabilizer of $v = g\gamma p_L$.

Since Gp_0 is open in its closure, the map $gF \mapsto gp_0 : G/F \to Gp_0$ is a homeomorphism. Thus we have that in G/F,

$$\lim_{t\to+\infty} a(t)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}F = \xi(s)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}F. \tag{5.1}$$

Since the Lie algebra of F is $\{\mathrm{Ad}(a(t)) : t \in \mathbb{R}\}$-invariant, there exists an $\{\mathrm{Ad}(a(t)) : t \in \mathbb{R}\}$-invariant subspace W of the Lie algebra of H complementary to the Lie algebra of F. We decompose $W = S^0 \oplus W^- \oplus W^+$ into the fixed point space, the contracting subspace and the expanding subspace for the action of $\mathrm{Ad}(a(t))$ as $t \to +\infty$. Then for all $s \in J$ near s_0 we have,

$$\xi(s)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}F = \exp(w^0(s))\exp(w^-(s))\exp(w^+(s))F, \tag{5.2}$$

for all $s \in J$ near s_0, where $w^0 \in W^0$, and $w^\pm \in W^\pm$. Combining (5.1) and (5.2), we get that $w^+(s) = 0$ for all s near s_0. Thus we get $\xi(s)u(\varphi(s) - \varphi(s_0))\xi(s_0)^{-1}F = \exp(w^0(s))\exp(w^-(s))F$. Let $\eta(s) := \exp(w^0(s))\exp(w^-(s))$. It is easy to see that $\eta(s) \in P^-(A)$. Hence for all $s \in J$ near s_0, we have $u(\varphi(s) - \varphi(s_0)) \in \xi(s)^{-1}\eta(s)F\xi(s_0)$. Thus, we have

$$u(\varphi(s)) \in \xi(s)^{-1}\eta(s)Fh \text{ for all } s \in J \text{ near } s_0.$$

Therefore by the analyticity of φ, we get that

$$\{u(\varphi(s)) : s \in I\} \subset P^-(A)Fh \cap U^+(A) = P^-(A)(F \cap H)h \cap U^+(A).$$

This completes the proof. □

Remark 5.2. By the above theorem, there exist analytic curves

$$\xi^- : I \to U^-(A)$$

and

$$\xi^0 : I \to Z_H(A)$$

such that $u(\varphi(s))g\Gamma \subset \xi^-(s)\xi^0(s)Fhg\Gamma$ for all $s \in I$. Then as $t \to \infty$, the distance between $a(t)u(\varphi(s))g\Gamma$ and $\xi^0(s)Fhg\Gamma$ tends to zero, since $Fhg\Gamma$ is a proper closed $\{a(t)\}$-invariant subset of G/Γ. Thus every limit measure of the sequence $\{\mu_t : t > 0\}$ as $t \to +\infty$ is a probability measure whose support is contained in $\xi^0(I)Fhg\Gamma$. Replacing F by a smaller subgroup containing A, we can actually ensure that $Fhg\Gamma$ admits a finite F-invariant measure.

We conjecture that in the general case of $(m, n) > 1$, if φ is *generic*, then Lemma 4.4, Theorem 5.1, and Remark 5.2 should hold.

Appendix A. **More discussion and examples on the *generic* and *supergeneric* condtions.** Let us discuss the *generic* condition and the *supergeneric* condition in detail so that we can understand them better.

Proposition A.1. *If m and n are coprime, then the* generic *condition is the same as the* supergeneric *condition in* $\mathrm{M}(m \times n, \mathbb{R})$*.*

Proof. Let us prove it by induction on $m + n$.

When $m = n = 1$, it is easy to see that the *generic* condition and the *supergeneric* condition are both equivalent to the condition that $\varphi : I \to \mathrm{M}(1 \times 1, \mathbb{R}) = \mathbb{R}$ is not constant.

Suppose that the statement holds for any coprime (m', n') with $m' + n' < m + n$. We will prove the statement for (m, n). Without loss of generality, let us assume that $m < n$.

Given an analytic curve $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$, let us check if φ is *generic* or *supergeneric*. We first reduce the curve to another curve $\psi : J_{s_0} \to \mathrm{M}(m \times (n - m), \mathbb{R})$. In this process, there is no difference between *genericity* and *supergenericity*; that is, if we can not construct such ψ, then we claim that φ is neither *generic* nor *supergeneric*, otherwise, we may continue. If we get ψ, then by our inductive hypothesis, ψ is *generic* if and only if it is *supergeneric*. By the inductive definition of *generic* condition and *supergeneric* condition, we conclude that φ is *generic* if and only if it is *supergeneric*.

This completes the proof. □

Let us consider the case where $m = 1$ or $n = 1$.

Proposition A.2. *If $m = 1$ or $n = 1$, the* generic *condition (which is the same as the* supergeneric *condition) is equivalent to the condition that the curve is not contained in any proper affine subspace.*

Proof. We will only prove the statement for $m = 1$. The proof for $n = 1$ is the same.

We will prove the statement by induction on n.

For $n = 1$, the statement is obvious.

Suppose that the statement holds for $n - 1$. Let us prove it for n. Given an analytic curve

$$\varphi : I \to \mathrm{M}(1 \times n, \mathbb{R}) = \mathbb{R}^n,$$

let us check if it is *generic*. Let us write

$$\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s)).$$

Without loss of generality, let us assume that $\varphi(s_0) = \mathbf{0}$ for some $s_0 \in I$. Then if $\varphi_1(s) = 0$ for every $s \in I$, we claim that φ is not *generic*. In this case we have that φ is contained in the subspace $x_1 = 0$. If not, then there exists a subinterval $J_{s_0} \subset I$ such that $\varphi_1(s) - \varphi_1(s_0) = \varphi_1(s) \neq 0$ for any $s \in J_{s_0}$. Let us define $\psi : J_{s_0} \to \mathbb{R}^{n-1}$ as follows:

$$\psi(s) := (\varphi_1^{-1}(s)\varphi_2(s), \ldots, \varphi_1^{-1}(s)\varphi_n(s)).$$

By our inductive hypothesis, $\psi(s)$ is not *generic* if and only if it is contained in some proper affine subspace

$$a_1 + a_2 x_1 + \cdots + a_n x_{n-1} = 0.$$

This is equivalent to

$$a_1 + a_2\varphi_1^{-1}(s)\varphi_2(s) + \cdots + a_n\varphi_1^{-1}(s)\varphi_n(s) = 0 \text{ for any } s \in J_{s_0},$$

which is equivalent to that $\{\varphi(s) : s \in J_{s_0}\}$ is contained in the proper affine subspace

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0.$$

Since φ is analytic, this is equivalent to that $\{\varphi(s) : s \in I\}$ is contained in a proper affine subspace.

This completes the proof. $\square$

Remark A.3. In [18], the case $m = 1$ is studied. It is proved that the obstruction to equidistribution is that the curve is contained in a proper affine subspace of $\mathbb{R}^n$. The above proposition shows that the main result in [18] is a special case of Theorem 1.4.

First we construct *supergeneric* curves for $m = n$.

It is easy to see that for any analytic curve

$$\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$$

and any $X \in \mathrm{M}(m \times n, \mathbb{R})$, $\varphi(s)$ is *supergeneric* if and only if $\varphi(s) + X$ is *supergeneric*. Therefore, we will only consider analytic curves passing through $\mathbf{0} \in \mathrm{M}(m \times n, \mathbb{R})$.

Example A.4.

1. For $m = n = 2$, the analytic curve $\varphi : I = [-1, 1] \to \mathrm{M}(2 \times 2, \mathbb{R})$ defined by

$$\varphi(s) := \begin{bmatrix} s & s^4 \\ s^2 & s^3 \end{bmatrix}$$

is *supergeneric*.

2. For $m = n = 3$, the analytic curve $\varphi : I = [-1, 1] \to \mathrm{M}(3 \times 3, \mathbb{R})$ defined by

$$\varphi(s) := \begin{bmatrix} s^{16} - s^{17} & 0 & -s^8 + s^9 \\ 0 & -s^{11} + s^{12} & s^7 - s^8 \\ -s^{13} + s^{14} & s^{10} - s^{11} & 0 \end{bmatrix}$$

is *supergeneric*.

Proof.

1. Let $s_0 = 0$, then $\varphi(s_0) = \mathbf{0}$. It is easy to check that

$$\det(\varphi(s) - \varphi(s_0)) = \det \varphi(s)$$

is not always zero and

$$\psi(s) := (\varphi(s) - \varphi(0))^{-1} = \varphi^{-1}(s)$$

is the following:

$$\psi(s) = (\det \varphi(s))^{-1} \begin{bmatrix} s^3 & -s^4 \\ -s^2 & s \end{bmatrix}.$$

It is not contained in any proper affine subspace of $\mathrm{M}(2 \times 2, \mathbb{R})$. This proves that $\varphi(s)$ is *supergeneric*.

2. It is straightforward to check that $\det \varphi(s)$ is not always zero and $\psi(s) = (\varphi(s) - \varphi(0))^{-1} = \varphi^{-1}(s)$ is the following:

$$\psi(s) = (\det \varphi(s))^{-1} \begin{bmatrix} s & s^2 & s^3 \\ s^4 & s^5 & s^7 \\ s^8 & s^{10} & s^{11} \end{bmatrix}.$$

It is not contained in any proper affine subspace of $\mathrm{M}(3 \times 3, \mathbb{R})$. This shows that $\varphi(s)$ is *supergeneric*.

□

For $m \neq n$, it is also easy to construct *supergeneric* curves.

Example A.5. For $m = 2$ and $n = 3$, the curve $\varphi : [-1, 1] \to \mathrm{M}(2 \times 3, \mathbb{R})$ defined by

$$\varphi(s) := \begin{bmatrix} s^5 & -s^3 & 1 \\ -s^2 & s & 0 \end{bmatrix}$$

is *supergeneric.*

Proof. Let us write $\varphi(s) = [\varphi_1(s); \varphi_2(s)]$ where

$$\varphi_1(s) = \begin{bmatrix} s^5 & -s^3 \\ -s^2 & s \end{bmatrix}$$

and

$$\varphi_2(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let $s_0 = 0$, then $\varphi(s_0) = \mathbf{0}$. The curve

$$\psi(s) = (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$$

is the following:

$$\psi(s) = \frac{1}{s^6 - s^5} \begin{bmatrix} s \\ s^2 \end{bmatrix}.$$

It is easy to see that $\psi(s)$ is not contained in any proper affine subspace of $\mathbb{R}^2$. Therefore, $\psi(s)$ is *supergeneric* and hence $\varphi(s)$ is *supergeneric.* □

For *supergeneric* curves, we have the following statement.

Proposition A.6. *For any $m \geq 1$ and $n \geq 1$, let us equip the set of analytic curves in $\mathrm{M}(m \times n, \mathbb{R})$ with the uniform norm $\|\cdot\|_\infty$; namely, for $\varphi_1, \varphi_2 : I \to \mathrm{M}(m \times n, \mathbb{R})$,*

$$\|\varphi_1 - \varphi_2\|_\infty := \max_{s \in I}\{|\varphi_1(s) - \varphi_2(s)|\}.$$

Then the set of supergeneric *curves in $\mathrm{M}(m \times n, \mathbb{R})$ is dense and open in the set of analytic curves in $\mathrm{M}(m \times n, \mathbb{R})$.*

Proof. Let us prove the statement by induction on $m + n$.

We first prove the statement for $m = n$. We will show the following stronger statement: the set of analytic curves satisfying the condition given in [20], denoted by $\mathcal{G}$, is open and dense.

We first claim that $\mathcal{G}$ is open. In fact, given an analytic curve $\varphi : I \to \mathrm{M}(m \times m, \mathbb{R})$ in $\mathcal{G}$, we have a point $s_0 \in I$ and a subinterval $J_{s_0} \subset I$, such that $\varphi(s) - \varphi(s_0)$ is invertible for any $s \in J_{s_0}$, and the curve

$$\psi(s) = (\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0}$$

is not contained in any proper affine subspace of $\mathrm{M}(m \times m, \mathbb{R})$. The invertibility is apparently an open condition. To see that the condition $\psi(s)$ not contained in any proper affine subspace is also open, we note that this condition is equivalent to the condition that the derivatives of $\psi(s)$ at some $s_1 \in J_{s_0}$ span the whole space. This condition is stable under perturbation. This shows that $\mathcal{G}$ is open.

Now let us prove that $\mathcal{G}$ is dense. Suppose not, then there exists an open subset $\mathcal{N}$ of the collection of analytic curves in $\mathrm{M}(m \times m, \mathbb{R})$ such that every $\varphi \in \mathcal{N}$ is not in $\mathcal{G}$. We first claim that there exists $\varphi \in \mathcal{N}$ which is *generic*. In fact, φ is not *generic* if and only if for any $s_0 \in I$, $\varphi(s) - \varphi(s_0)$ is contained in the subvariety of $\mathrm{M}(m \times m, \mathbb{R})$ defined by $\det(X) = 0$. Therefore, if $\varphi \in \mathcal{N}$ is not *generic*, we can easily perturb it to make it *generic*. This proves the claim. By replacing $\mathcal{N}$ with a smaller open set, we may assume that every $\varphi \in \mathcal{N}$ is *generic*. For $\varphi \in \mathcal{N}$, let us fix a point $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that $\varphi(s) - \varphi(s_0)$ is invertible for any $s \in J_{s_0}$. By our assumption, we have that $\psi(s) := (\varphi(s) - \varphi(s_0))^{-1}$ is contained in a proper affine subspace of $\mathrm{M}(m \times m, \mathbb{R})$. Then we have that for any $s \in J_{s_0}$, the derivatives of ψ at $s \in J_{s_0}$ do not span the whole space. Let $\mathcal{V}(\varphi, s)$ denote the linear span of the derivatives of ψ at $s \in J_{s_0}$. Let us fix $s_1 \in J_{s_0}$. We may choose $\varphi \in \mathcal{N}$ with maximal $\mathcal{V}(\varphi, s_1)$. Our plan is to find $\tilde{\varphi} : I \to \mathrm{M}(m \times m, \mathbb{R})$ close to φ such that $\mathcal{V}(\tilde{\varphi}, s_1)$ is larger than $\mathcal{V}(\tilde{\varphi}, s_1)$, which leads to a contradiction. By our assumption, $\mathcal{V}(\varphi, s_1)$ is not the whole space, then there exists $i \leq m^2$ such that $\mathcal{V}(\varphi, s_1)$ is spanned by

$$\{\psi'(s_1), \psi^{(2)}(s_1), \ldots, \psi^{(i-1)}(s_1)\}.$$

Let us write $\tilde{\varphi}(s) = \varphi(s) + \epsilon\eta(s)$ where $\eta : I \to \mathrm{M}(m \times m, \mathbb{R})$ is an analytic map and $\epsilon > 0$ is a small parameter. We may choose $\epsilon > 0$ small enough to make sure that $\tilde{\varphi} \in \mathcal{N}$. Without loss of generality, we may assume that $\varphi(s_0) = \eta(s_0) = \mathbf{0}$. Let us consider

$$\tilde{\psi}(s) := (\tilde{\varphi}(s) - \tilde{\varphi}(s_0))^{-1} = \tilde{\varphi}^{-1}(s).$$

Then we have that

$$\begin{aligned} \tilde{\psi}(s) &= \tilde{\varphi}^{-1}(s) \\ &= (\varphi(s) + \epsilon\eta(s))^{-1} \\ &= ((\mathrm{I}_m + \epsilon\eta(s)\varphi^{-1}(s))\varphi(s))^{-1} \\ &= \varphi^{-1}(s)(\mathrm{I}_m + \epsilon\eta(s)\varphi^{-1}(s))^{-1}. \end{aligned}$$

Because
$$(\mathrm{I}_m + \epsilon\eta(s)\varphi^{-1}(s))^{-1} = \mathrm{I}_m - \epsilon\eta(s)\varphi^{-1}(s) + O(\epsilon^2),$$
we have that
$$\begin{aligned}\tilde{\psi}(s) &= \varphi^{-1}(s) - \epsilon\varphi^{-1}(s)\eta(s)\varphi^{-1}(s) + O(\epsilon^2)\\ &= \psi(s) - \epsilon\varphi^{-1}(s)\eta(s)\varphi^{-1}(s) + O(\epsilon^2).\end{aligned}$$
By ignoring the error term $O(\epsilon^2)$, we can see that $\mathcal{V}(\tilde{\varphi}, s_1)$ is larger than $\mathcal{V}(\varphi, s_1)$ if the subspace spanned by derivatives of
$$f(s) := \psi(s) - \epsilon\varphi^{-1}(s)\eta(s)\varphi^{-1}(s)$$
at s_1 is larger than $\mathcal{V}(\varphi, s_1)$. Let $\eta(s) = \varphi(s)\xi(s)\varphi(s)$ where
$$\xi : I \to \mathrm{M}(m \times m, \mathbb{R})$$
is an analytic map satisfying that $\xi^{(i)}(s_1) \notin \mathcal{V}(\varphi, s_1)$ and $\xi^{(j)}(s_1) = \mathbf{0}$ for any $j \neq i$. It is easy to find such $\xi(s)$. In fact, a polynomial map with appropriate coefficients will work. Then we have that
$$f(s) = \psi(s) - \epsilon\xi(s).$$
For $1 \leq j \leq i-1$,
$$f^{(j)}(s_1) = \psi^{(j)}(s_1) - \epsilon\xi^{(j)}(s_1) = \psi^{(j)}(s_1).$$
For $j = i$, we have that
$$f^{(i)}(s_1) = \psi^{(i)}(s_1) - \epsilon\xi^{(i)}(s_1) \notin \mathcal{V}(\varphi, s_1).$$
This implies that the space spanned by $\{f^{(j)}(s_1) : 1 \leq j \leq i\}$ is larger than $\mathcal{V}(\varphi, s_1)$. By our previous discussion, this proves that $\mathcal{G}$ is a dense set, and hence finishes the proof for $m = n$.

Suppose the statement holds for any (m', n') with $m'+n' < m+n$. We will prove the statement for (m, n). Without loss of generality, let us assume that $m < n$.

Given an analytic curve $\varphi : I \to \mathrm{M}(m \times n, \mathbb{R})$, let us write
$$\varphi(s) = [\varphi_1(s); \varphi_2(s)]$$
where $\varphi_1(s)$ denotes the first m by m block and $\varphi_2(s)$ denotes the rest m by $n-m$ block.

Let us first prove that the set of *supergeneric* curves is open.

Given a supergeneric curve $\varphi = [\varphi_1; \varphi_2] : I \to \mathrm{M}(m \times n, \mathbb{R})$, we have a point $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that $\varphi_1(s) - \varphi_1(s_0)$ is invertible for any $s \in J_{s_0}$ and
$$\psi(s) := (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0)), s \in J_{s_0}$$
is *supergeneric*. By our argument in the proof of the case $m = n$, we have that for any
$$\tilde{\varphi} = [\tilde{\varphi}_1; \tilde{\varphi}_2] : I \to \mathrm{M}(m \times n, \mathbb{R})$$
close enough to φ, $\tilde{\varphi}_1(s) - \tilde{\varphi}_1(s_0)$ is invertible for any $s \in J_{s_0}$. Let us denote
$$\tilde{\psi}(s) := (\tilde{\varphi}_1(s) - \tilde{\varphi}_1(s_0))^{-1}(\tilde{\varphi}_2(s) - \tilde{\varphi}_2(s_0)) : s \in J_{s_0}.$$
It is easy to see that $\tilde{\psi}$ is close to ψ if $\tilde{\varphi}$ is close to φ. By our inductive hypothesis, we have that $\tilde{\psi}$ is *supergeneric* if $\tilde{\psi}$ is close to ψ. This shows that $\tilde{\varphi}$ is *supergeneric* for any $\tilde{\varphi}$ close enough to φ.

Let us prove that the set of *supergeneric* curves is dense. Suppose not, then there exists an open subset $\mathcal{N}$ of the set of analytic curves such that any $\varphi = [\varphi_1; \varphi_2] \in \mathcal{N}$

is not *supergeneric*. By the same reason as in the proof of the case $m = n$, we may assume that there exists a point $s_0 \in I$ and a subinterval $J_{s_0} \subset I$ such that for any $\varphi = [\varphi_1; \varphi_2] \in \mathcal{N}$ and any $s \in J_{s_0}$, $\varphi_1(s) - \varphi_1(s_0)$ is invertible. By our inductive hypothesis, for any open neighborhood $\mathcal{N}'$ of $\psi(s) := (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0))$, there exists a *supergeneric* curve of the form $\psi(s) + \epsilon\eta(s) \in \mathcal{N}'$. Let us choose $\mathcal{N}'$ small enough such that

$$\tilde{\varphi}(s) = [\varphi_1(s); \varphi_2(s) + \epsilon(\varphi_1(s) - \varphi_1(s_0))\eta(s)] \in \mathcal{N}.$$

It is easy to check that

$$\tilde{\psi}(s) := (\tilde{\varphi}_1(s) - \tilde{\varphi}_1(s_0))^{-1}(\tilde{\varphi}_2(s) - \tilde{\varphi}_2(s_0))$$

is equal to $\psi(s) + \epsilon\eta(s)$. This shows that $\tilde{\varphi}$ is *supergeneric*.

This completes the proof. □

Remark A.7. In the argument above, by replacing "analytic" with "polynomial", we can also show that for any polynomial curve

$$\varphi : I \to \mathrm{M}(m \times m, \mathbb{R}),$$

there are polynomial *supergeneric* curves arbitrarily close to φ.

In the next example, we will see that the *generic* condition is not the same as the *supergeneric* condition.

Example A.8. Let $m = n = 2$. The analytic curve $\varphi : [-1, 1] \to \mathrm{M}(2 \times 2, \mathbb{R})$ defined by

$$\varphi(s) := \begin{bmatrix} s & s^2 \\ s^2 & s \end{bmatrix}$$

is *generic* but not *supergeneric*.

Proof. It is easy to see that φ is *generic* because $\varphi(s) = \varphi(s) - \varphi(0)$ is invertible for any $s \in [1/4, 1/2]$.

Let us prove that φ is not *supergeneric*. In fact, for any $s_0 \in [-1, 1]$, we have that

$$\varphi(s) - \varphi(s_0) = \begin{bmatrix} s - s_0 & s^2 - s_0^2 \\ s^2 - s_0^2 & s - s_0 \end{bmatrix}.$$

Then $\psi(s) := (\varphi(s) - \varphi(s_0))^{-1}$ is given as follows:

$$\psi(s) = \frac{1}{(s - s_0)^2 - (s^2 - s_0^2)^2} \begin{bmatrix} s - s_0 & s_0^2 - s^2 \\ s_0^2 - s^2 & s - s_0 \end{bmatrix}.$$

It it easy to see that ψ is well defined in some subinterval J_{s_0} of I. Moreover, for any $s_1, s_2 \in J_{s_0}$, we have that $\psi(s_1) - \psi(s_2)$ is contained in the subspace $\mathcal{S}$ of $\mathrm{M}(2 \times 2, \mathbb{R})$ defined as follows:

$$\mathcal{S} := \left\{ \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \in \mathrm{M}(2 \times 2, \mathbb{R}) : x_{1,1} = x_{2,2}, x_{1,2} = x_{2,1} \right\}.$$

Recall that

$$\mathcal{E}_2 = \begin{bmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_2 \end{bmatrix} \in \mathfrak{sl}(4, \mathbb{R})$$

is defined in (1.2). Let us define the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(4, \mathbb{R})$ as follows. Let

$$E_1 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$E_2 := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For $i = 1, 2$, let us define

$$X_i := \begin{bmatrix} \mathbf{0} & E_i \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathfrak{sl}(4, \mathbb{R}),$$

$$Y_i := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ E_i & \mathbf{0} \end{bmatrix} \in \mathfrak{sl}(4, \mathbb{R}),$$

and

$$H_i := [X_i, Y_i] = \begin{bmatrix} E_i & \\ & -E_i \end{bmatrix} \in \mathfrak{sl}(4, \mathbb{R}).$$

It is easy to check that for $i = 1, 2$, $\{X_i, Y_i, H_i\}$ generate a Lie subalgebra, denoted by $\mathfrak{h}_i$, isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Moreover, for any $L_1 \in \{X_1, Y_1, H_1\}$ and $L_2 \in \{X_2, Y_2, H_2\}$, we have that $[L_1, L_2] = \mathbf{0}$. This implies that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ generate a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$. Let us define $\mathfrak{h}$ to be this Lie subalgebra. $\mathfrak{h}$ is an observable Lie subalgebra since it is semisimple.

It is easy to see that $\mathcal{E}_2$ and $\mathfrak{n}^-(\mathcal{S})$ are contained in $\mathfrak{h}$. This shows that $\varphi(s)$ is not *supergeneric*. □

It is worth explaining the condition given in [2] and its relation to our *generic* condition.

Let us denote $M(s) := [\mathrm{I}_m; \varphi(s)] \in \mathrm{M}(m \times (m+n), \mathbb{R})$. Given a subspace $W \subset \mathbb{R}^{m+n}$ and $0 < r \leq m$, we define the pencil $\mathcal{P}_{W,r}$ to be

$$\mathcal{P}_{W,r} := \{M \in \mathrm{M}(m \times (m+n), \mathbb{R}) : \dim MW = r\}.$$

If $0 < r < \frac{m \dim W}{m+n}$, then we call $\mathcal{P}_{W,r}$ a constraining pencil. In [2], the following theorem is proved: if a submanifold is not contained in any constraining pencil, then the submanifold is extremal. In our case, it says that if the curve $\{M(s) : s \in I\}$ is not contained in any constraining pencil $\mathcal{P}_{W,r}$, then the curve is extremal. It is easy to see that if W is a rational subspace, then the constraining pencil $\mathcal{P}_{W,r}$ is not extremal. So this condition is considered almost optimal.

Proposition A.9. *Suppose that the analytic curve $\varphi : I = [a, b] \to \mathrm{M}(m \times n, \mathbb{R})$ is* generic, *then the curve $\{M(s) = [\mathrm{I}_m; \varphi(s)] : s \in I\}$ is not contained in any constraining pencil $\mathcal{P}_{W,r}$.*

Proof. Without loss of generality, we may assume that every point in I satisfies the *generic* condition.

We will prove the statement by induction on (m, n). Without loss of generality, we may assume that $m \leq n$.

We first prove the statement holds for (n, n). For contradiction, suppose that there exists some subspace W and $0 < r < \frac{\dim W}{2}$ such that

$$M(s) = [\mathrm{I}_n; \varphi(s)] \in \mathcal{P}_{W,r} \text{ for all } s \in I.$$

This implies that $\mathrm{Ker}M(s) \cap W > \frac{\dim W}{2}$ for all $s \in I$. Then for any $s_1, s_2 \in I$, the dimension of $\mathrm{Ker}M(s_1) \cap \mathrm{Ker}M(s_2) \cap W$ is greater than 0, since the sum of $\dim(\mathrm{Ker}M(s_1) \cap W)$ and $\dim(\mathrm{Ker}M(s_2) \cap W)$ is greater than $\dim W$. It is easy to see that

$$\mathrm{Ker}M(s) = \{(-\varphi(s)w, w) : w \in \mathbb{R}^n\}.$$

Therefore, there exist $w_1, w_2 \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ such that $(-\varphi(s_1)w_1, w_1) = (-\varphi(s_2)w_2, w_2)$. This implies $w_1 = w_2$ and $\varphi(s_1)w_1 = \varphi(s_2)w_1$. Therefore $(\varphi(s_1) - \varphi(s_2))w_1 = \mathbf{0}$. But this is impossible since $w_1 \neq \mathbf{0}$ and $\varphi(s_1) - \varphi(s_2)$ is invertible. This contradiction shows the statement for (n, n).

Suppose the statement holds for all (m', n') such that

$$m' \leq m, n' \leq n \text{ and } m' + n' < m + n,$$

we want to prove the statement for (m, n). Suppose not, then the curve $\{M(s) = [\mathrm{I}_m; \varphi(s)] : s \in I\}$ is contained in some pencil $\mathcal{P}_{W,r}$ where $r < \frac{m \dim W}{m+n}$. Let us fix some $s_0 \in I$ and denote

$$W_0 := \mathrm{Ker} M(s_0) \cap W.$$

Then from our assumption we have that $\dim W_0 = \dim W - r$. For any $s \in I$, since

$$\dim(\mathrm{Ker} M(s) \cap W) = \dim W - r,$$

we have that

$$\begin{array}{rcl} \dim(\mathrm{Ker} M(s) \cap W_0) & = & \dim(\mathrm{Ker} M(s) \cap \mathrm{Ker} M(s_0) \cap W) \\ & \geq & 2(\dim W - r) - \dim W = \dim W - 2r = \dim W_0 - r. \end{array}$$

Therefore, $\dim M(s)W_0 \leq r$ for all $s \in I$.

We write any $w \in W \subset \mathbb{R}^{m+n}$ as (w_1, w_2) where $w_1 \in \mathbb{R}^m$ and $w_2 \in \mathbb{R}^n$. Since

$$W_0 = \mathrm{Ker} M(s_0) \cap W,$$

every $(w_1, w_2) \in W_0$ satisfies that $w_1 = -\varphi(s_0)w_2$. By identifying $(-\varphi(s_0)w_2, w_2)$ with $w_2 \in \mathbb{R}^n$, we may consider W_0 as a subspace of $\mathbb{R}^n$. By direct calculation, we have that under this identification, $M(s) : W_0 \to \mathbb{R}^m$ is defined as follows:

$$M(s) : w \in W_0 \mapsto (\varphi(s) - \varphi(s_0))w \in \mathbb{R}^m.$$

Following our previous notation, we may write $\varphi(s) = [\varphi_1(s); \varphi_2(s)]$ where $\varphi_1(s)$ denotes the first m by m block of $\varphi(s)$ and $\varphi_2(s)$ denotes the rest m by $n - m$ block. By our assumption, $\varphi_1(s) - \varphi_1(s_0)$ is invertible for s inside some subinterval $J_{s_0} \subset I$. Accordingly we may write $w \in W_0 \subset \mathbb{R}^n$ as (w_3, w_4) where $w_3 \in \mathbb{R}^m$ and $w_4 \in \mathbb{R}^{n-m}$. For $s \in J_{s_0}$, let us denote

$$\psi(s) := (\varphi_1(s) - \varphi_1(s_0))^{-1}(\varphi_2(s) - \varphi_2(s_0)) \in \mathrm{M}(m \times (n - m), \mathbb{R})$$

and $N(s) := [\mathrm{I}_m; \psi(s)] \in \mathrm{M}(m \times n, \mathbb{R})$. By our assumption, $\psi : J_{s_0} \to \mathrm{M}(m \times (n - m), \mathbb{R})$ is *generic*. Then for $w = (w_3, w_4) \in W_0 \subset \mathbb{R}^n$,

$$\begin{array}{rcl} M(s)w & = & M(s)(w_3, w_4) \\ & = & (\varphi_1(s) - \varphi_1(s_0))w_3 + (\varphi_2(s) - \varphi_2(s_0))w_4 \\ & = & (\varphi_1(s) - \varphi_1(s_0))(w_3 + \psi(s)w_4) \\ & = & (\varphi_1(s) - \varphi_1(s_0))N(s)(w_3, w_4). \end{array}$$

Since $\varphi_1(s) - \varphi_1(s_0)$ is invertible, we have that $\dim M(s)W_0 = \dim N(s)W_0$. Therefore,

$$\dim N(s)W_0 \leq r, \text{ for all } s \in J_{s_0}.$$

This implies that there exists some $r' \leq r$ and some subinterval $J'_{s_0} \subset J_{s_0}$ such that

$$\dim N(s)W_0 = r' \text{ for all } s \in J'_{s_0},$$

i.e.,

$$N(s) \in \mathcal{P}_{W_0,r'} \text{ for all } s \in J'_{s_0}.$$

But this contradicts our inductive assumption for case $(m, n-m)$. In fact, $W_0 \subset \mathbb{R}^{m+(n-m)}$,

$$N(s) = [\mathrm{I}_m; \psi(s)]$$

where the curve $\psi(s) \in \mathrm{M}(m \times (n-m), \mathbb{R})$ is *generic*. Thus to apply the inductive assumption for $(m, n-m)$, it suffices to check that $r' < \frac{m \dim W_0}{n}$. Since $r' \leq r$, we only need to show that $r < \frac{m \dim W_0}{n}$. The inequality is equivalent to

$$nr < m \dim W_0 = m(\dim W - r).$$

It is straightforward to check that it is the same as

$$r < \frac{m \dim W}{m+n},$$

which is our assumption. This allows us to apply the inductive assumption and conclude the contradiction.

This completes the proof. □

Therefore the *generic* condition implies the pencil condition given in [2].

Acknowledgements. The second author thanks The Ohio State University for hospitality during his visit when the project was initiated. Both authors thank MSRI where they collaborated on this work in Spring 2015. We thank Dmitry Kleinbock for helpful discussions on the generic condition and for drawing our attention to the work of Aka, Breuillard, Rosenzweig and de Saxcé [1] and [2]. Thanks are due to the anonymous referees for many valuable suggestions to help improve the exposition in this article.

References

[1] Menny Aka, Emmanuel Breuillard, Lior Rosenzweig, and Nicolas de Saxcé. On metric diophantine approximation in matrices and lie groups. *Comptes Rendus Mathematique*, 353:185–189, 2015.

[2] Menny Aka, Emmanuel Breuillard, Lior Rosenzweig, and Nicolas de Saxcé. Diophantine approximation on matrices and lie groups. *Geom. Funct. Anal.*, 28:1–57, 2018.

[3] R. C. Baker. Dirichlet's theorem on Diophantine approximation. *Math. Proc. Cambridge Philos. Soc.*, 83(1):37–59, 1978.

[4] Yann Bugeaud. Approximation by algebraic integers and hausdorff dimension. *Journal of the London Mathematical Society*, 65(3):547–559, 2002.

[5] SG Dani. On orbits of unipotent flows on homogeneous spaces. *Ergodic Theory and Dynamical Systems*, 4(01):25–34, 1984.

[6] SG Dani and GA Margulis. Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci*, 101(1):1–17, 1991.

[7] H. Davenport and W. Schmidt. Dirichlet's theorem on diophantine approximation. ii. *Acta Arithmetica*, 16(4):413–424, 1970.

[8] William Fulton and Joe Harris. *Representation Theory: A First Course*. Springer-Verlag New York, 2004.

[9] Dmitry Kleinbock, Gregory Margulis, and Junbo Wang. Metric diophantine approximation for systems of linear forms via dynamics. *International Journal of Number Theory*, 6(05):1139–1168, 2010.

[10] Dmitry Kleinbock and Barak Weiss. Dirichlet's theorem on diophantine approximation and homogeneous flows. *Journal of Modern Dynamics*, 2(1):43–62, 2008.

[11] Dmitry Y Kleinbock and Grigorij A Margulis. Flows on homogeneous spaces and diophantine approximation on manifolds. *Annals of mathematics*, pages 339–360, 1998.

[12] Shahar Mozes and Nimish Shah. On the space of ergodic invariant measures of unipotent flows. *Ergodic Theory Dynam. Systems*, 15(1):149–159, 1995.

[13] Marina Ratner. On raghunathan's measure conjecture. *Annals of Mathematics*, 134(2):545–607, 1991.

[14] Marina Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J*, 63(1):235–280, 1991.
[15] Nimish Shah. Expanding translates of curves and dirichlet-minkowski theorem on linear forms. *Journal of the American Mathematical Society*, 23(2):563–589, 2010.
[16] Nimish A Shah. Limit distributions of expanding translates of certain orbits on homogeneous spaces. *Proceedings Mathematical Sciences*, 106(2):105–125, 1996.
[17] Nimish A. Shah. Asymptotic evolution of smooth curves under geodesic flow on hyperbolic manifolds. *Duke Mathematical Journal*, 148(2):281–304, 2009.
[18] Nimish A Shah. Equidistribution of expanding translates of curves and dirichlet's theorem on diophantine approximation. *Inventiones mathematicae*, 177(3):509–532, 2009.
[19] Nimish A. Shah. Limiting distributions of curves under geodesic flow on hyperbolic manifolds. *Duke Mathematical Journal*, 148(2):251–279, 2009.
[20] Lei Yang. Equidistribution of expanding curves in homogeneous spaces and diophantine approximation for square matrices. *Proc. Amer. Math. Soc.*, 144(12):5291–5308, 2016.
[21] Lei Yang. Expanding curves in $\mathrm{T}^1(\mathbb{H}^n)$ under geodesic flow and equidistribution in homogeneous spaces. *Israel J. Math.*, 216(1):389–413, 2016.

Received 0902 2019; revised 0226 2020.