BIRKHOFF GENERICITY ON AFFINE SUBSPACES IN HOROSPHERES

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ABSTRACT. We study Birkhoff genericity for a simple uniformly expanding diagonal flow on $\mathrm{SL}_{n+1}(\mathbb{R})/\mathrm{SL}_{n+1}(\mathbb{Z})$, with initial points restricted to affine subspaces of the expanding horospherical orbit through the identity coset. We prove that almost every point on such an affine subspace is Birkhoff generic, except possibly in two situations: either the defining matrix of the affine subspace has Diophantine exponent at least n, or the affine subspace is arbitrarily well approximable by affine subspaces of dimension $r-1$ defined over a real number field of degree $m \geq 2$, with $n+1 = mr$. As applications, we obtain Dirichlet non-improvability and logarithmic density results for almost every point on these affine subspaces.

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1. Introduction

Let $X = G/\Gamma$ be a finite-volume homogeneous space and let $\{a_t\}_{t\in\mathbb{R}}$ be a one-parameter diagonal subgroup of G. A point $x \in X$ is said to be Birkhoff generic for the $\{a_t\}$-action with respect to the G-invariant probability measure μ_X if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(a_t x)\,\mathrm{d}t = \int_X f\,\mathrm{d}\mu_X, \qquad \forall f \in C_c(X).$$

Suppose that the a_t-action is ergodic with respect to μ_X. By the Birkhoff ergodic theorem, μ_X-almost every point is Birkhoff generic. Let $P_a^- = \{g \in G : \lim_{t\to\infty} a_t g a_{-t} \text{ exists in } G\}$ denote the stable subgroup for $\{a_t\}$. Then if x is generic then gx is also generic for any $g \in P_a^-$. Let $U_a^+ := \{g \in G : \lim_{t\to\infty} a_{-t} g a_t \to e\}$ denote the expanding horospherical subgroup for $\{a_t\}$. Then $P_a^- U_a^+$ is open in G. Therefore, for any $x \in X$, we have that gx is generic for almost every $g \in U_a^+$ with respect to the Haar measure on U_a^+. We are interested in finding out when the same conclusion holds for other measures on U_a^+, for example smooth measures on proper submanifolds of U_a^+.

The above question was motivated by another class of similar results, where one shows that translates of certain smooth measures on a submanifold of a horosphere by a diagonal action get equidistributed in the limit [**KM98**, **Sha09**, **Yan20**]. These type of results are like mixing of piece of a low-dimensional submanifold under translates. While in this article we are interested in ergodic averages along the diagonal orbit from various points of the submanifold.

Shi [**Shi20**] carefully studied this problem for subgroup action of semisimple groups H on G/Γ, and established Birkhoff genericity for points on orbits of horospheres of H for actions of $\mathbb{R}$-diagonalizable subgroups. For points on horospheres in the case of actions on the space of unimodular lattices, Kleinbock, Shi, and Weiss [**KSW17**] have also obtained precise error terms for ergodic averages. See also [**FSU18**, **Kha20**, **Zha23**] for progress on different cases of this problem.

More recently, a strong result was proved by Solan and Wieser [**SW25**] who established Birkhoff genericity for non-degenerate curves on horospheres of simple uniformly expanding diagonal group actions on the space of unimodular lattices.

In this paper we consider the Birkhoff genericity question for points on affine subspaces of the expanding horospherical subgroup. Our work complements the results in [**Shi20**, **KSW17**, **ES19**, **SW25**] and we refer the readers to the references therein for other applications, history and background in a more general context.

1.1. Main results.

Let $G = \mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$, $X = G/\Gamma$, the space of unimodular lattices in $\mathbb{R}^{n+1}$, μ_X denote the G-invariant probability measure on X, and $x_0 = e\Gamma/\Gamma \in X$. Let

$$t \mapsto a_t = \begin{pmatrix} e^{\frac{n}{n+1}t} & \\ & e^{-\frac{1}{n+1}t} I_n \end{pmatrix}$$

denote the uniformly expanding simple diagonal one-parameter subgroup. And let

$$U_a^+ = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} : (x_1, \dots, x_n) \in \mathbb{R}^n \right\},$$

denote the expanding horospherical subgroup of G with respect to a_1. From now on we will identify U_a^+ with $\mathbb{R}^n$. Let $e_1, \dots, e_n$ denote the basis of $U_a^+ \cong \mathbb{R}^n$. Let $\mathcal{A}$ be a (proper) affine subspace of U_a^+ of dimension $d < n$. To parametrize $\mathcal{A}$, by permuting the coordinates, without loss of generality we may assume that the projection of $\mathcal{A}$ to the coordinate plane spanned by $e_1, \dots, e_d$ is bijective. Then, there exists a unique matrix $A \in \mathrm{Mat}_{d+1,n-d}(\mathbb{R})$ such that

$$\mathcal{A} = \mathcal{L}_A := \{(\mathbf{x}, A\widetilde{\mathbf{x}})\colon \mathbf{x} \in \mathbb{R}^d\}, \text{ where } \widetilde{\mathbf{x}} = (1, \mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^d. \tag{1.1}$$

Here A is called the matrix defining $\mathcal{A}$. Let

$$u_A = \begin{pmatrix} I_{d+1} & A \\ & I_{n-d} \end{pmatrix} \text{ and } x_A = u_A x_0. \tag{1.2}$$

For $A \in \mathrm{Mat}_{d+1,n-d}(\mathbb{R})$, we define the Diophantine exponent $\omega(A)$ of A to be the supremum of $\omega > 0$ such that the inequality $\|A\mathbf{q} + \mathbf{p}\| \leq \|\mathbf{q}\|^{-\omega}$ has infinitely many nonzero integral solutions $(\mathbf{q}, \mathbf{p}) \in \mathbb{Z}^{n-d} \times \mathbb{Z}^{d+1}$. By Dirichlet's approximation theorem, $\omega(A) \geq \frac{n-d}{d+1}$ for every A, and the equality holds for almost every A.

We shall prove the following point-wise non-escape of mass result.

Theorem 1.1. *Suppose $\omega(A) < n$. Then for almost every $u \in \mathcal{A}$, every weak-$*$ limit point of $\{\frac{1}{T}\int_0^T a_t u \delta_{x_0} \mathrm{d}t\}_{T\to\infty}$ is a probability measure on X.*

Remark 1.1. We note that the condition on the Diophantine exponent is optimal in the following sense. Suppose $\omega(A) > n$. Then by the argument in the proof of [**SY24a**, Lemma 6.5], see also [**KdSSY24**, Theorem 1.1], escape of mass occurs for every $u \in \mathcal{A}$. More precisely, there exists $\varepsilon_0 > 0$ and $T_i \to \infty$ such that for every compact subset K of X and every $u \in \mathcal{A}$, the Lebesgue measure of the set $\{0 \leq t \leq T_i\colon a_t u x_0 \in K\}$ is at most $(1 - \varepsilon_0)T_i$ for all $i \gg 1$.

In the critical case of $\omega(A) = n$, we are not able to draw any conclusions.

In this article, a real number field is defined to be a subfield of $\mathbb{R}$ which is a finite extension of $\mathbb{Q}$; it is not necessarily totally real. Let $\mathrm{Gr}(k_1, k_2)$ denote the Grassmannian variety parametrizing k_1-dimensional linear subspaces of a k_2-dimensional vector space.

Definition 1.1 ($\mathbb{F}$-Liouville subspaces)**.** Let $\mathbb{F}$ be a real number field and $k \leq l < n$ are positive integers. We say a k-dimensional subspace $\mathcal{S}$ of $\mathbb{P}^n(\mathbb{R})$ is $(\mathbb{F}, l)$-*Liouville* if for every $\kappa \geq 1$, there exists an l-dimensional subspace $\mathcal{T}$ of $\mathbb{P}^n(\mathbb{R})$ which is defined over $\mathbb{F}$, such that

$$d(\mathcal{S}, \mathcal{T}) \leq H_{\mathcal{L}}(\mathcal{T})^{-\kappa}. \tag{1.3}$$

Here $d(\mathcal{S}, \mathcal{T}) := \max_{x\in\mathcal{S}} \min_{y\in\mathcal{T}} d(x, y)$, where $d(x, y)$ is the angle between the lines in $\mathbb{R}^{n+1}$ that x and y parametrize. The height function $H_{\mathcal{L}}$ is an exponential Weil height function $H_{\mathcal{L}} : \mathrm{Gr}(l+1, n+1)(\mathbb{F}) \to \mathbb{R}_{>0}$ associated with the ample generator $[\mathcal{L}]$ of the Picard group $\mathrm{Pic}(\mathrm{Gr}(l+1, n+1)) \cong \mathbb{Z}$. Note that the height function is determined by $[\mathcal{L}]$ up to multiplication by a constant, but changing the constants does not affect our definition of Liouville-ness.

We say an affine subspace of $\mathbb{A}^n$ is $(\mathbb{F}, l)$-Liouville if its projective closure in $\mathbb{P}^n$ is.

Remark 1.2. Let $\mathcal{L}_{k;\mathbb{F},l}$ denote the set of $(\mathbb{F},l)$-Liouville subspaces in $\mathrm{Gr}(k+1,n+1)$. Then Definition 1.1 gives natural coverings of $\mathcal{L}_{k;\mathbb{F},l}$ by thin neighborhoods of $\mathbb{F}$-Schubert cycles in $\mathrm{Gr}(k+1,n+1)$ defined by the condition $\mathcal{S}\subset\mathcal{T}$. These Schubert cycles are of dimension $(k+1)(l-k)$. From this one can show that the Hausdorff dimension of $\mathcal{L}_{k;\mathbb{F},l}$ is $(k+1)(l-k)$, which is strictly smaller than the dimension of $\mathrm{Gr}(k+1,n+1)$. Since this fact is not used in this article, we leave it to the reader to verify the details.

The following is the main result of this article.

Theorem 1.2. *Suppose $\omega(A)<n$. Then one of the following possibilities holds:*

(1) For almost every $u\in\mathcal{A}$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(a_tux_0)\mathrm{d}t=\int_X f\mathrm{d}\mu_X,\quad\forall f\in C_c(X).$$

(2) There exist a real number field $\mathbb{F}$ of degree $m\geq 2$ and a positive integer $r\geq d+1$ such that
- *$rm=n+1$,*
- *$\mathcal{A}$ is $(\mathbb{F},r-1)$-Liouville.*

In particular, if $n+1$ is prime, then possibility (2) does not occur.

Remark 1.3. Suppose that $\mathcal{A}$ is contained in a $(r-1)$-dimensional affine subspace defined over $\mathbb{F}$ such that $rm=n+1$; see Corollary 4.5. Then by [**SY24a**, Proof of Lemma 6.6], there exists a proper closed subset $\mathcal{K}$ of X such that $a_t\mathcal{A}x_0\subset\mathcal{K}$ for all $t\geq 0$. So for every $u\in\mathcal{A}$, the sequence of probability measures $\frac{1}{T_i}\int_0^{T_i}a_tu\delta_{x_0}\mathrm{d}t$ do not equidistribute in X for any sequence $T_i\to\infty$.

1.1.1. *Geometric formulation.* Let W be a linear subspace of $\mathbb{P}^n(\mathbb{R})$. As done in the 'geometric formulation' given in the introduction of [**SY24a**], we define the Diophantine exponent $\omega(W)$ of W to be the supremum of ω such that the following holds: there exist infinitely many hyperplanes $Q\subset\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ such that $d(W,Q)\leq H(Q)^{-\omega-1}$. Here $d(W,Q)=\sup_{[x]\in W}\inf_{[y]\in Q}\frac{\|x\wedge y\|}{\|x\|\|y\|}$, and H denotes the Weil height on $\mathrm{Gr}(n,n+1)(\mathbb{Q})\cong\mathbb{P}^n(\mathbb{Q})$ associated with the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

In the above notation, we have that $\omega(\mathcal{L}_A)=\omega(A)$; see [**SY24a**].

Theorem 1.3. *Let λ be a finite measure on G whose pushforward on $P_a^-\backslash G\cong\mathbf{P}^n(\mathbb{R})$ is absolutely continuous with respect to the Lebesgue measure on W. Suppose that $\omega(W)<n$ and W is not $(\mathbb{F},(r-1))$-Liouville for a real number field $\mathbb{F}$ of degree $m\geq 2$ such that $n+1=mr$. Then for λ-almost all $g\in G$, we have*

$$\frac{1}{T}\int_0^T f(a_tgx_0)\mathrm{d}t\to\int_X f\mathrm{d}\mu_X,\forall f\in C_c(X)$$

We leave it to the reader to verify that Theorem 1.3 can be deduced from Theorem 1.2.

1.2. **Dirichlet non-improvability on subspaces.** Initial motivation for our study comes from the Diophantine approximation. Denote by $\|\cdot\|$ the supremum norm on $\mathbb{R}^n$, where $n \geq 1$ (unless specified otherwise, all the norms on $\mathbb{R}^n$ will be taken to be the supremum norm). Let $0 < \delta \leq 1$. Following Davenport and Schmidt [DS70], let $\mathrm{DI}(\delta)$ denote the set of vectors $\mathbf{x} \in \mathbb{R}^n$ such that for all large $T \geq 1$,

$$\exists (p, \mathbf{q}) \in \mathbb{Z} \times (\mathbb{Z}^n \setminus \{0\}) \text{ such that } \begin{cases} |\mathbf{x} \cdot \mathbf{q} + p| \leq \delta T^{-n} \\ \|\mathbf{q}\| \leq \delta T. \end{cases} \tag{1.4}$$

Similarly, let $\mathrm{DI}'(\delta)$ denote the set of vectors $\mathbf{x} \in \mathbb{R}^n$ such that for all large $T \geq 1$,

$$\exists (\mathbf{p}, q) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\}) \text{ such that } \begin{cases} \|q\mathbf{x} + \mathbf{p}\| \leq \delta T^{-1} \\ |q| \leq \delta T^n. \end{cases} \tag{1.5}$$

By Dirichlet's approximation theorem, $\mathrm{DI}(1) = \mathbb{R}^n$ and $\mathrm{DI}'(1) = \mathbb{R}^n$.

Let $\mathrm{DI} = \cup_{0<\delta<1} \mathrm{DI}(\delta)$ and $\mathrm{DI}' = \cup_{0<\delta<1} \mathrm{DI}'(\delta)$, denote the sets of Dirichlet's improvable vectors. Davenport and Schmidt [DS70] proved that the sets DI and DI$'$ are Lebesgue null in $\mathbb{R}^n$, see also [KW08]. By the arguments as in [KW08, SW17] involving Dani-correspondence, it is straightforward to deduce the following result from Theorem 1.2.

Theorem 1.4. *Let $\mathcal{A}$ be a proper d-dimensional affine subspace of $\mathbb{R}^n$ that does not satisfy the condition in possibility (2) of Theorem 1.2, and suppose that $\omega(\mathcal{A}) < n$. Then $\mathrm{DI} \cap \mathcal{A}$ and $\mathrm{DI}' \cap \mathcal{A}$ are Lebesgue null in $\mathcal{A}$.*

The result weakens the conditions for Dirichlet non-improvability in [SY24a, Theorem 1.3] for the case of $d = 1$, $n \geq 3$ and $n + 1$ is even (for $\mathcal{T} = \mathbb{N}$ in its statement).

Following [SW25, Theorem 1.1] we formulate a density result on improvability of Dirichlet's approximation on affine subspaces of $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbb{R}^d$ and $\delta \in (0, 1)$, let

$$\overline{\mathcal{D}}_{\mathbf{x}}(\delta) = \limsup_{N\to\infty} \frac{1}{\log N} \sum_{\substack{T\in\{1,\dots,N\}: \\ (1.4) \text{ holds for } T}} \frac{1}{T},$$

$$\underline{\mathcal{D}}_{\mathbf{x}}(\delta) = \liminf_{N\to\infty} \frac{1}{\log N} \sum_{\substack{T\in\{1,\dots,N\}: \\ (1.4) \text{ holds for } T}} \frac{1}{T},$$

and similarly define $\overline{\mathcal{D}'}_{\mathbf{x}}(\delta)$ and $\underline{\mathcal{D}'}_{\mathbf{x}}(\delta)$, where T in the corresponding sum satisfies (1.5). Using the arguments of the proof of [SW25, Theorem 1.1], the following result is a direct consequence of Theorem 1.2.

Theorem 1.5. *There exist continuous strictly increasing functions f and g from $[0, 1]$ to $[0, 1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ with the following property: Let $\mathcal{A}$ be an affine subspace of $\mathbb{R}^n$ that does not satisfy possibility (2) of Theorem 1.2, and suppose that $\omega(\mathcal{A}) < n$. Then for every $0 < \delta < 1$ and Lebesgue almost every $\mathbf{x} \in \mathcal{A}$, we have*

$$\overline{\mathcal{D}}_{\mathbf{x}}(\delta) = \underline{\mathcal{D}}_{\mathbf{x}}(\delta) = f(\delta) \text{ and } \overline{\mathcal{D}'}_{\mathbf{x}}(\delta) = \underline{\mathcal{D}'}_{\mathbf{x}}(\delta) = g(\delta).$$

1.3. **Strategy of the proof and organization of the paper.** The general strategy of the proof in this article is based on extending the techniques developed in [Shi20], [PSS23], and [SY24b]. In Section 2 we describe and recall properties of the Margulis height function on G/Γ as defined in [BQ12] for the action of $H_d = \mathrm{SL}_{d+1}(\mathbb{R})$. We study the exponential growth rate of the height function along trajectories of a diagonal group $\{b_t\}$ commuting with H_d. Using the relation between the growth rate and Diophantine exponent obtained in [SY24b], we define a modified height function with contraction property for the $\{a_t\}$-action. This allows us to prove Theorem 1.1 in view of the methods developed in [Shi20]. In Section 3, we recall a result inspired by [CE15] from [Shi20] to obtain unipotent invariance of the limiting measures for almost all ergodic averages. These limiting measures are invariant under the simple diagonal group $\{a_t\}$ by definition. These observations allow one to use measure rigidity theorems of Ratner [Rat91] as done by Einsiedler and Shi [ES19]. They show that if such a limit measure is not G-invariant, then it must be positive on certain algebraic subvariety of G projected to G/Γ; called singular manifolds. To analyze how the ergodic averages assigns mass to the singular manifold, we carefully construct a height function to measure 'distance' from the singular submanifold. We ensure that this function satisfies nice properties, in particular it satisfies the contraction property with respect to the a_t-action. The contraction property is related to exponential growth of a new height function along the b_t-trajectories. This construction of the height function given in Section 3 is a novel part of this article. In Section 4 we interpret the high exponential growth rate in terms of the $\mathbb{F}$-Liouville property.

2. Non-escape of mass

In this section, we establish pointwise nonescape of mass on affine subspaces with small Diophantine exponents. We follow the strategy of [Shi20, Section 4]. Many of the results in this section have been proven in [Shi20] and [SY24b], and we provide some details for the readers' convenience.

2.1. **Notation.** Let $G = \mathrm{SL}_{n+1}(\mathbb{R})$ and $H_d = \begin{pmatrix} \mathrm{SL}_{d+1}(\mathbb{R}) & \\ & I_{n-d} \end{pmatrix}$, where I_k denotes the $k \times k$-identity matrix. Consider the one-parameter diagonal subgroups of G defined by the following: For all $t \in \mathbb{R}$,

$$a_t = \begin{pmatrix} e^{\frac{n}{n+1}t} & \\ & e^{-\frac{1}{n+1}t} I_n \end{pmatrix}, \tag{2.1}$$

$$b_t = \begin{pmatrix} e^{\frac{n-d}{(d+1)(n+1)}t} I_{d+1} & \\ & e^{-\frac{1}{n+1}t} I_{n-d} \end{pmatrix} \in Z_G(H_d), \text{ and} \tag{2.2}$$

$$c_t = \begin{pmatrix} e^{\frac{d}{d+1}t} & & \\ & e^{-\frac{1}{d+1}t} I_d & \\ & & I_{n-d} \end{pmatrix} \in H_d. \tag{2.3}$$

Then $a_t = b_t c_t$. With this normalization, the eigenvalues of $\mathrm{Ad}(a_t)$ on the Lie algebra of U_a^+ and the eigenvalues of $\mathrm{Ad}(c_t)$ on the Lie algebra of $U_a^+ \cap H_d$ are all equal to e^t.

Let $u : \mathbb{R}^d \to U := U_a^+ \cap H_d$ be the map defined as

$$u(s) = \begin{pmatrix} 1 & s & 0 \\ & I_d & \\ & & I_{n-d} \end{pmatrix} \in U = U_a^+ \cap H_d, \quad \forall s \in \mathbb{R}^d. \tag{2.4}$$

Given $A \in \mathrm{Mat}_{d+1,n-d}(\mathbb{R})$, we write A in the block form

$$A = \left(\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right), \text{ where } A_1 \in \mathrm{Mat}_{1,n-d}(\mathbb{R}) \text{ and } A_2 \in \mathrm{Mat}_{d,n-d}(\mathbb{R}).$$

Let $\mathcal{A} = \{(s, \tilde{s}A) \colon \mathbf{x} \in \mathbb{R}^d\}$, where $\tilde{s} = (1, s) \in \mathbb{R}^{d+1}$ for every $s \in \mathbb{R}^d$. We identify $\mathcal{A}$ with an affine subspace of $U_a^+ = \mathbb{R}^n$. Let

$$z_A = \left(\begin{smallmatrix} 1 & & -A_2 \\ & I_d & \\ & & I_{n-d} \end{smallmatrix}\right) \in Z_G(\{a_t\}). \tag{2.5}$$

Now for every $s \in \mathbb{R}^d$, we have

$$\left(\begin{smallmatrix} 1 & s & A\tilde{s} \\ & I_d & \\ & & I_{n-d} \end{smallmatrix}\right) = z_A u(s) u_A \in \mathcal{A} \subset U_a^+.$$

Let $x_A = u_A x_0$. Then, for any $t \in \mathbb{R}$ and $s \in \mathbb{R}^d$,

$$a_t \left(\begin{smallmatrix} 1 & s & A\tilde{s} \\ & I_d & \\ & & I_{n-d} \end{smallmatrix}\right) x_0 = z_A a_t u(s) x_A. \tag{2.6}$$

Therefore, since μ_X is z_A-invariant, the possibility (1) can be equivalently expressed as: For Lebesgue almost all $s \in \mathbb{R}^d$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(a_t u(s) x_A) \mathrm{d}t = \int_X f \mathrm{d}\mu_X, \quad \forall f \in C_c(X). \tag{2.7}$$

2.2. **Height function with respect to the point at infinity.** Restricting the standard action of G on $\mathbb{R}^{n+1}$ to H_d, we have the following decomposition of H_d-modules

$$\mathbb{R}^{n+1} = V_0^\perp \bigoplus V_0,$$

where $V_0^\perp$ is the $\mathbb{R}$-span of $\{e_0, \dots, e_d\}$ and V_0 is the $\mathbb{R}$-span of $\{e_{d+1}, \dots, e_n\}$. Here, $V_0^\perp$ is the standard representation of $H_d \cong \mathrm{SL}_{d+1}(\mathbb{R})$ and H_d acts trivially on V_0. For $0 \le k \le n+1$, taking the k-exterior products we get

$$\bigwedge^k \mathbb{R}^{n+1} = \bigoplus_{i=\max\{0,k-(n-d)\}}^{\min\{d+1,k\}} \bigwedge^i (V_0^\perp) \bigotimes \bigwedge^{k-i} V_0, \tag{2.8}$$

because if $\bigwedge^i (V_0^\perp) \neq 0$, then $0 \le i \le d+1$, and if $\bigwedge^{k-i} V_0 \neq 0$, then $0 \le k-i \le n-d$.

We recall the construction of a Margulis height function $\alpha : X \to [0, +\infty]$ from [**BQ12**]. We shall specify it in our setting, and optimize the constants. In view of (2.8), for $\max\{0, k-(n-d)\} \le i \le \min\{d+1, k\}$, let π_i denote the projection

$$\pi_i : \bigwedge^k \mathbb{R}^{n+1} \longrightarrow \bigwedge^i (V_0^\perp) \bigotimes \bigwedge^{k-i} V_0. \tag{2.9}$$

Since V_0 and $V_0^\perp$ are $N_G(H_d)$-invariant, each π_i is $N_G(H_d)$-equivariant.

Now

$$\left(\bigwedge^0 (V_0^\perp) \bigotimes \bigwedge^k V_0\right) \bigoplus \left(\bigwedge^{d+1} (V_0^\perp) \bigotimes \bigwedge^{k-(d+1)} V_0\right)$$

is the space of H_d-fixed k-vectors. Let $\pi_{\mathrm{fix}} = \pi_0 \oplus \pi_{d+1}$.

We take $\delta_k = (n+1-k)k$ for $0 \le k \le n+1$. Let $\varepsilon > 0$ and $0 < k < n+1$. For every $v \in \bigwedge^k \mathbb{R}^{n+1}$, we let

$$\varphi_\varepsilon(v) = \begin{cases} \min_{1\le i\le d} \varepsilon^{\frac{d+1}{d-i+1}\delta_k} \|\pi_i(v)\|^{-\frac{d+1}{d-i+1}}, & \text{if } \|\pi_{\mathrm{fix}}(v)\| < \varepsilon^{\delta_k}, \\ 1, & \text{otherwise.} \end{cases} \tag{2.10}$$

Given $\theta > 0$ and $\varepsilon > 0$, for $y \in X$ we define

$$\alpha_\varepsilon^\theta(y) = \max_v \varphi_\varepsilon^\theta(v) \in [1, +\infty], \tag{2.11}$$

where v varies over all nonzero y-integral decomposable vectors in $\cup_{k=1}^n \bigwedge^k \mathbb{R}^{n+1}$; that is if $y = g\mathbb{Z}^{n+1}$ for some $g \in \mathrm{SL}_{n+1}(\mathbb{R})$, then $v = g(v_1 \wedge \cdots \wedge v_k)$ for some $1 \le k \le n$ and linearly independent $v_1, \dots, v_k \in \mathbb{Z}^{n+1}$.

To simplify notation, we write $\alpha_\varepsilon := \alpha_\varepsilon^1$. Note that $\alpha_\varepsilon^\theta = (\alpha_\varepsilon)^\theta$.

Remark 2.1. (1) We note that, if $\alpha_\varepsilon^\theta(x) < \infty$, then $\alpha_\varepsilon^\theta(gx) < \infty$ for all $g \in H_d$. In fact, $\alpha_\varepsilon^\theta$ is Lipschitz with respect to the action of H_d; see [**Shi20**, Lemma 4.1].
(2) For any compact set $K \subset X$, there exists $\varepsilon \in (0,1)$ such that $\sup_{y\in K} \alpha_\varepsilon^\theta(y) < \infty$.

2.3. **Exponential growth rate of height function along b_t-trajectory.** For each $x \in X$, we define the exponential growth rate of the trajectory $\{b_t x\}_{t\ge 0}$ to be

$$\rho_{\varepsilon,\theta}(x) := \limsup_t \frac{\log \alpha_\varepsilon^\theta(b_t x)}{t} \ge 0. \tag{2.12}$$

Remark 2.2. Let $A \in \mathrm{Mat}_{d+1,n-d}(\mathbb{R})$. Let $x_A = u_A x_0$ be as defined in (1.2). By [**SY24b**, Proposition 4.1], if $\omega(A) < n$, then $\rho_{\varepsilon,\theta}(x_A) < \theta$.

Let P_b^- denote the stable parabolic subgroup associated to $\{b_t\}$; that is,

$$P_b^- = \{\omega \in G : \lim_{t\to\infty} b_t \omega b_t^{-1} \in G \text{ exists}\}. \tag{2.13}$$

We note that $N_G(H_d) = Z_G(\{b_t\}) \subset P_b^-$. Also, $Z_G(\{a_t\}\{u(s)\}_{s\in\mathbb{R}^d}) \subset P_b^-$, see Lemma 3.9.

Lemma 2.1. *Let $x \in X$. If $\rho_{\varepsilon,\theta}(x) \le \theta$, then*

$$\rho_{\varepsilon,\theta}(\omega x) = \rho_{\varepsilon,\theta}(x),\ \forall \omega \in P_b^-.$$

More generally, let $\Omega \subset P_b^-$ be a compact set. We define

$$\rho_{\varepsilon,\theta,\Omega}(x) = \limsup_{t\to\infty} \frac{\sup_{\omega\in\Omega} \log \alpha_\varepsilon^\theta(b_t \omega x)}{t}. \tag{2.14}$$

Then $\rho_{\varepsilon,\theta}(x) = \rho_{\varepsilon,\theta,\{e\}}(x)$. We will prove the following uniform version of Lemma 2.1.

Proposition 2.2. *Suppose $\rho_{\varepsilon,\theta}(x) \le \theta$. Then $\rho_{\varepsilon,\theta,\Omega}(x) = \rho_{\varepsilon,\theta}(x)$.*

These results are needed only in Section 4, so the proof may be skipped in the first reading.

Proof. It is enough to prove that $\rho_{\varepsilon,\theta,\Omega}(x) \le \rho_{\varepsilon,\theta}(x)$. So, we assume $\rho_{\varepsilon,\theta,\Omega}(x) > 0$ and let $0 < \rho_0 < \min\{\theta, \rho_{\varepsilon,\theta,\Omega}(x)\}$. So it is enough to prove that $\rho_{\varepsilon,\theta}(x) \ge \rho_0$.

The arguments given here are based on [**SY24b**, Section 4]. Write $x = gx_0$ for some $g \in G$. Then, there exist sequences $t_m \to \infty$, and $\{\omega_m\} \subset \Omega$, such that

$$\rho_0 < \rho_{\varepsilon,\theta,\Omega}(x) = \lim_{m\to\infty} \frac{\log \alpha_\varepsilon^\theta(b_{t_m}\omega_m x)}{t_m}.$$

In view of the definitions (2.10) of φ_ε and (2.11) of $\alpha_\varepsilon^\theta$, after passing to a subsequence, there exists a $1 \le k \le n$, and a sequence of nonzero decomposable k-vectors $v_m \in \bigwedge^k \mathbb{Z}^{n+1}$ such that

$$\alpha_\varepsilon^\theta(b_{t_m}\omega_m x) = (\varphi_\varepsilon(b_{t_m}\omega_m g v_m))^\theta.$$

We recall that $\pi_l : \bigwedge^k \mathbb{R}^{n+1} \to W_l := \bigwedge^l V_0^\perp \oplus \bigwedge^{k-l} V_0$ for all $0 \le l \le d+1$, see (2.9). Since $\rho_0 > 0$, by (2.10),

$$\|\pi_{\mathrm{fix}}(b_{t_m}\omega_m g v_m)\| < \varepsilon^{\delta_k}, \tag{2.15}$$

where $\pi_{\mathrm{fix}} = \pi_0 + \pi_{d+1}$ and $\delta_k = (n+1-k)k$, and

$$e^{\theta^{-1}\rho_0 t_m} \le \varphi_\varepsilon(b_{t_m}\omega_m g v_m) = \min_{1\le l\le d} \varepsilon^{\frac{d+1}{d+1-l}\delta_k} \|\pi_l(b_{t_m}\omega_m g v_m)\|^{-\frac{d+1}{d+1-l}},$$

equivalently,

$$\|\pi_l(b_{t_m}\omega_m g v_m)\| \le \varepsilon^{\delta_k} e^{-(1-\frac{l}{d+1})\theta^{-1}\rho_0 t_m}, \quad \forall 1 \le l \le d. \tag{2.16}$$

Since $\Omega \subset P_b^-$, we have

$$C := \sup_{\omega\in\Omega}\sup_{t\ge 0} \left\| b_t \omega^{-1} b_t^{-1} \right\| < \infty. \tag{2.17}$$

We note that by (2.2), b_t acts on W_l, the image of π_l, as a scalar multiplication by

$$e^{\left(\frac{l}{d+1}-\frac{k}{n+1}\right)t}, \ \forall 0 \le l \le d+1. \tag{2.18}$$

In particular,

$$\omega W_l \subset W_l \oplus \cdots \oplus W_0, \ \forall \omega \in P_b^-. \tag{2.19}$$

For each $m \in \mathbb{N}$, let

$$E_m := \|\pi_0(b_{t_m}\omega_m g v_m)\|.$$

Then by passing to a subsequence, we may assume that

$$\text{either } E_m \le \varepsilon^{\delta_k} e^{-\theta^{-1}\rho_0 t_m}, \text{ for all } m \in \mathbb{N} \tag{2.20}$$

$$\text{or } E_m > \varepsilon^{\delta_k} e^{-\theta^{-1}\rho_0 t_m}, \text{ for all } m \in \mathbb{N}. \tag{2.21}$$

Case 1. Suppose that (2.20) holds. Let $m \in \mathbb{N}$. Let $t_m' = t_m - \theta^{-1}\rho_0 t_m$. Then, by (2.16) and (2.18), we get

$$\left\|\pi_l(b_{t_m'}\omega_m g v_m)\right\| \le \varepsilon^{\delta_k} e^{-\left(1-\frac{k}{n+1}\right)\theta^{-1}\rho_0 t_m}, \quad \forall 0 \le l \le d+1.$$

Hence,

$$\left\|b_{t_m'}\omega_m g v_m\right\| \le \varepsilon^{\delta_k} e^{-\left(1-\frac{k}{n+1}\right)\theta^{-1}\rho_0 t_m}. \tag{2.22}$$

As $\theta^{-1}\rho_0 < 1$, we have $t'_m \geq 0$. Hence by (2.17) applied to $\bigwedge^k \mathbb{R}^{n+1}$, we get

$$\left\|b_{t'_m} g v_m\right\| = \left\|(b_{t'_m}\omega_m^{-1} b_{t'_m}^{-1}) b_{t'_m}\omega_m g v_m\right\| \leq C^k \varepsilon^{\delta_k} e^{-(1-\frac{k}{n+1})\theta^{-1}\rho_0 t_m}.$$

Now going back from t'_m to $t_m = t'_m + \theta^{-1}\rho_0 t_m$, in view of (2.18), we get

$$\begin{aligned}
\|\pi_0(b_{t_m} g v_m)\| &\leq C^k \varepsilon^{\delta_k} e^{-\theta^{-1}\rho_0 t_m},\\
\|\pi_{d+1}(b_{t_m} g v_m)\| &\leq C^k \varepsilon^{\delta_k},\\
\|\pi_l(b_{t_m} g v_m)\| &\leq C^k \varepsilon^{\delta_k} e^{-(1-\frac{l}{d+1})\theta^{-1}\rho_0 t_m},\ \forall 1 \leq l \leq d.
\end{aligned}$$

Pick $s_m = -\sqrt{t_m}$ and $t''_m = t_m + s_m$. Then,

$$\begin{aligned}
\left\|\pi_0(b_{t''_m} g v_m)\right\| &\leq C^k \varepsilon^{\delta_k} e^{-\frac{k}{n+1}s_m} e^{-\theta^{-1}\rho_0 t_m} \to 0, \text{ as } s_m/t_m \to 0,\\
\left\|\pi_{d+1}(b_{t''_m} g v_m)\right\| &\leq C^k e^{(1-\frac{k}{n+1})s_m} \varepsilon^{\delta_k} \to 0, \text{ as } s_m \to -\infty,\\
\left\|\pi_l(b_{t''_m} g v_m)\right\| &\leq (C^k e^{(\frac{l}{d+1}-\frac{k}{n+1})s_m}) \varepsilon^{\delta_k} e^{-\frac{d+1-l}{d+1}\theta^{-1}\rho_0 t_m},\ \forall 1 \leq l \leq d.
\end{aligned}$$

So,

$$\lim_{m\to\infty} \left\|\pi_{\mathrm{fix}}(b_{t''_m} g v_m)\right\| = 0. \tag{2.23}$$

Therefore, for all $m \gg 1$, we have

$$\varphi_\varepsilon^\theta(b_{t''_m} g v_m) = \min_{1\leq l\leq d+1} \varepsilon^{\frac{d+1}{d+1-l}\delta_k} \left\|\pi_l(b_{t''_m} g v_m)\right\|^{-\frac{d+1}{d+1-l}} \geq C e^{cs_m} e^{\rho_0 t_m},$$

for some $C > 0$ and $c \in \mathbb{R}$ independent of m. So, since $\lim_{m\to\infty} s_m/t_m \to 0$, we get

$$\rho_{\varepsilon,\theta}(x) \geq \limsup_{m\to\infty} \frac{\log \varphi_\varepsilon^\theta(b_{t''_m} g v_m)}{t''_m} \geq \rho_0.$$

This completes the proof in Case 1.

Case 2. Suppose that (2.21) holds. This is the most interesting case. We note that in this case $k \leq n-d$, because if $k > n-d$, then the image of π_0 is $\{0\}$.

Let $m \in \mathbb{N}$. Since v_m is a decomposable vector, using Plücker relations as in [**SY24b**, (4.16)], we get the following: For all $1 \leq l \leq d$,

$$\|\pi_{l+1}(b_{t_m}\omega_m g v_m)\|\|\pi_0(t_m\omega_m g v_m)\| \leq k\|\pi_l(b_{t_m}\omega_m g v_m)\|\|\pi_1(b_{t_m}\omega_m g v_m)\|. \tag{2.24}$$

Inductively, for all $1 \leq l \leq d+1$, we get

$$\begin{aligned}
\|\pi_l(b_{t_m}\omega_m g v_m)\| &\leq k^{l-1} E_m^{-(l-1)} \|\pi_1(b_{t_m}\omega_m g v_m)\|^l\\
&\leq k^{l-1} E_m^{-(l-1)} \varepsilon^{l\delta_k} e^{-\frac{ld}{d+1}\theta^{-1}\rho_0 t_m}.
\end{aligned} \tag{2.25}$$

Let $t'_m = t_m + (d\theta^{-1}\rho_0 t_m + (d+1)\log E_m)$. So for all $0 \leq l \leq d+1$, by we get

$$\left\|\pi_l(b_{t'_m}\omega_m g v_m)\right\| \leq k^d \varepsilon^{l\delta_k} E_m^{\left(1-\frac{k(d+1)}{n+1}\right)} e^{-\frac{kd}{n+1}\theta^{-1}\rho_0 t_m}. \tag{2.26}$$

Since $\log E_m \geq -\theta^{-1}\rho_0 t_m$ and $\theta^{-1}\rho_0 < 1$, we have $t'_m \geq 0$. Therefore, by (2.17) and (2.19), assuming $0 < \varepsilon < 1$, for any $0 \leq l \leq d+1$,

$$\begin{aligned}\left\|\pi_l(b_{t'_m} g v_m)\right\| &\leq \sup_{j \geq l} C^k \left\|\pi_j(b_{t'_m}\omega_m g v_m)\right\| \\ &\leq C^k k^d \varepsilon^{l\delta_k} E_m^{\left(1-\frac{k(d+1)}{n+1}\right)} e^{-\frac{kd}{n+1}\theta^{-1}\rho_0 t_m}.\end{aligned}$$

Therefore, going back from t'_m to t_m using (2.18), we get for all $0 \leq l \leq d+1$,

$$\|\pi_l(b_{t_m} g v_m)\| \leq C^k k^d \varepsilon^{l\delta_k} E_m^{-(l-1)} e^{-\frac{ld}{d+1}\theta^{-1}\rho_0 t_m}.$$

Now by (2.15), we have $E_m < \varepsilon^{\delta_k}$. And by (2.21), $E_m^{-(l-1)} \leq \varepsilon^{-(l-1)\delta_k} e^{(l-1)\theta^{-1}\rho_0 t_m}$, and $(l-1) - \frac{ld}{d+1} = -(1 - \frac{l}{d+1})$ for $1 \leq l \leq d+1$. Therefore,

$$\|\pi_l(b_{t_m} g v_m)\| \leq \begin{cases} C^k k^d E_m & \text{if } l = 0, \\ C^k k^d \varepsilon^{(d+1)\delta_k} (E_m^{-1} e^{-\theta^{-1}\rho_0 t_m})^d \leq C^k k^d \varepsilon^{\delta_k} & \text{if } l = d+1 \\ C^k k^d \varepsilon^{\delta_k} e^{-(1-\frac{l}{d+1})\theta^{-1}\rho_0 t_m} & \text{if } 1 \leq l \leq d. \end{cases} \tag{2.27}$$

Claim. After passing to a subsequence of t_m, there exists $s_m \in \mathbb{R}$ such that $s_m/t_m \to 0$, and if we put $t''_m = t_m + s_m$, then

$$\lim_{m\to\infty} \left\|\pi_{\mathrm{fix}}(b_{t''_m} g v_m)\right\| = 0. \tag{2.28}$$

To prove this claim, without loss of generality, after passing to a subsequence we may assume that

$$\text{either} \quad \inf_{m\in\mathbb{N}} \|\pi_0(b_{t_m} g v_m)\| > 0, \tag{2.29}$$

$$\text{or} \quad \inf_{m\in\mathbb{N}} \|\pi_{d+1}(b_{t_m} g v_m)\| > 0, \tag{2.30}$$

indeed if each infimum is 0, then choose $s_m = 0$.

First suppose that (2.29) holds. Then, from (2.27) we obtain that $\inf_{m\in\mathbb{N}} E_m > 0$. Therefore, $\lim_{m\to\infty} E_m e^{\theta^{-1}\rho_0 t_m} = \infty$. Let

$$s_m = \min\left\{\sqrt{t_m}, \frac{d(n+1)}{2(n+1-k)} \log E_m e^{\theta^{-1}\rho_0 t_m}\right\}.$$

Then using (2.18) and (2.27), we obtain (2.28).

Now suppose that (2.30) holds. Then, from (2.27) we deduce that

$$\inf_{m\in\mathbb{N}} E_m^{-1} e^{-\theta^{-1}\rho_0 t_m} > 0.$$

Therefore, $\lim_{m\to\infty} E_m^{-1} = \infty$. Let

$$s_m = -\min\left\{\frac{2(n+1)}{k} \log E_m^{-1}, \sqrt{t_m}\right\}.$$

Then (2.28) follows from (2.18) and (2.27). This completes the proof of the Claim.

Hence, by (2.18) and (2.27), we get

$$\left\|\pi_{\mathrm{fix}}(b_{t''_m} g v_m)\right\| \to 0 \text{ as } m \to \infty, \text{ and} \tag{2.31}$$

$$\left\|\pi_l(b_{t''_m} g v_m)\right\| \le C^k k^d \varepsilon^{\delta_k} \varepsilon^{\delta_k} e^{(\frac{l}{d+1} - \frac{k}{n+1}) s_m} e^{-\frac{d+1-l}{d+1}\theta^{-1}\rho_0 t_m}, \ \forall 1 \le l \le d. \tag{2.32}$$

Therefore,

$$\varphi^\theta_\varepsilon(b_{t''_m} g v_m) = \min_{1\le l\le d} \varepsilon^{\frac{d+1}{d+1-l}\delta_k} \left\|\pi_l(b_{t''_m} g v_m)\right\|^{-\frac{d+1}{d+1-l}} \ge C e^{c s_m} e^{\rho_0 t_m},$$

for some $C > 0$ and $c \ge 0$ which are independent of m. So, as $s_m/t_m \to 0$, we get

$$\rho_{\varepsilon,\theta}(x) \ge \limsup_{m\to\infty} \frac{\log \varphi^\theta_\varepsilon(b_{t''_m} g v_m)}{t''_m} \ge \rho_0.$$

This completes the proof in Case 2. □

The following consequence of the above proof is not used in our arguments, but clarifies whether ε plays any role in the value of $\rho_{\varepsilon,\theta}(x)$.

Lemma 2.3. *Let $x \in X$, $\theta > 0$, and $0 < \varepsilon_1 \le \varepsilon$. Then $\rho_{\varepsilon_1,\theta}(x) \le \rho_{\varepsilon,\theta}(x)$.*
Moreover, if $\rho_{\varepsilon,\theta}(x) \le \theta$, then $\rho_{\varepsilon_1,\theta}(x) = \rho_{\varepsilon,\theta}(x)$.

Proof. We write $x = gx_0$ for some $g \in G$. To prove the first inequality, without loss of generality, we may assume that $\rho_{\varepsilon_1,\theta}(x) > 0$. Then there exist a sequence $t_m \to \infty$, $k \in \{1, \dots, n\}$, and integral k-vectors v_m for all m such that $\|\pi_{\mathrm{fix}}(b_{t_m} g v_m)\| < \varepsilon_1^{\delta_k}$, and

$$\begin{aligned}
\rho_{\rho_{\varepsilon_1,\theta}(x)} &= \lim_{m\to\infty} \frac{\log \varphi^\theta_{\varepsilon_1}(b_{t_m} g v_m)}{t_m} \\
&= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \varepsilon_1^{\frac{d+1}{d-i+1}\delta_k} \|\pi_i(b_{t_m} g v_m)\|^{-\frac{d+1}{d-i+1}}}{t_m} \\
&= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \|\pi_i(b_{t_m} g v_m)\|^{-\frac{d+1}{d-i+1}}}{t_m}, \text{ as } t_m \to \infty, \\
&= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \varepsilon^{\frac{d+1}{d-i+1}\delta_k} \|\pi_i(b_{t_m} g v_m)\|^{-\frac{d+1}{d-i+1}}}{t_m} \\
&= \lim_{m\to\infty} \frac{\log \varphi^\theta_{\varepsilon}(b_{t_m} g v_m)}{t_m} \le \rho_{\varepsilon,\theta}(x),
\end{aligned}$$

where the last equality holds because $\|\pi_{\mathrm{fix}}(b_{t_m} g v_m)\| < \varepsilon_1^{\delta_k} < \varepsilon^{\delta_k}$.

To prove the inequality in the opposite direction, we further assume that $0 < \rho_{\varepsilon,\theta}(x) < \theta$. In the proof of Proposition 2.2, in (2.23) for Case 1 and (2.31) for Case 2,

we showed that the lim sup in the definition of $\rho_{\varepsilon,\theta}(gx_0)$ can be achieved along a sequence $t''_m \to \infty$ such that $\left\|\pi_{\mathrm{fix}}(b_{t''_m} g v_m)\right\| \to 0$, and

$$\begin{aligned}\rho_{\varepsilon,\theta}(gx_0) &= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \varepsilon^{\frac{d+1}{d-i+1}\delta_k} \left\|\pi_i(b_{t''_m} g v_m)\right\|^{-\frac{d+1}{d-i+1}}}{t''_m} \\ &= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \left\|\pi_i(b_{t''_m} g v_m)\right\|^{-\frac{d+1}{d-i+1}}}{t''_m} \\ &= \lim_{m\to\infty} \frac{\theta \log \min_{1\le i\le d} \left\|\pi_i(b_{t''_m} g v_m)\right\|^{-\frac{d+1}{d-i+1}}}{t''_m}, \text{ as } t''_m \to \infty, \\ &= \lim_{m\to\infty} \frac{\theta \log \varphi_{\varepsilon_1}(b_{t''_m} g v_m)}{t''_m} \le \rho_{\varepsilon_1,\theta}(x),\end{aligned}$$

where the last equality holds because $\left\|\pi_{\mathrm{fix}}(b_{t''_m} g v_m)\right\| < \varepsilon_1^{\delta_k}$ for all large m.

□

2.4. **Properties of the height function and non-divergence.** We recall the following lemma from [**SY24b**, Lemma 3.4]. Let I denote the interval $[-1/2, 1/2]$.

Lemma 2.4. *Let $0 < \theta < \frac{d}{d+1}$ and $\varepsilon > 0$. There exists $C \ge 1$ such that for any $t \ge 0$ there exists $b \ge 0$ such that for any $x \in X$ we have*

$$\int_{I^d} \alpha_\varepsilon^\theta(c_t u(s) x) \mathrm{d}s \le C e^{-\theta t} \alpha_\varepsilon^\theta(x) + b. \tag{2.33}$$

We also need the log Lipschitz property.

Lemma 2.5. *Let $\theta > 0$. There exists $C_1 \ge 1$ such that for every $s \in I^d$ and $t \ge 0$,*

$$C_1^{-1} e^{-\theta t} \alpha_\varepsilon^\theta(x) \le \alpha_\varepsilon^\theta(c_t u(s) x) \le C_1 e^{d\theta t} \alpha_\varepsilon^\theta(x). \tag{2.34}$$

Proof. By (2.11), it suffices to prove that for every k and every $v \in \bigwedge^k \mathbb{R}^{n+1}$,

$$e^{-\theta t} \varphi_\varepsilon^\theta(v) \ll \varphi_\varepsilon^\theta(c_t u(s) v) \ll e^{d\theta t} \varphi_\varepsilon^\theta(v), \tag{2.35}$$

where the implied constants are absolute. By (2.10), since $\pi_{\mathrm{fix}}(c_t u(s) v) = \pi_{\mathrm{fix}}(v)$, it remains to check that for every $1 \le i \le d$,

$$e^{-t} \|\pi_i(v)\|^{-\frac{d+1}{d-i+1}} \ll \|\pi_i(c_t u(s) v)\|^{-\frac{d+1}{d-i+1}} \ll e^{dt} \|\pi_i(v)\|^{-\frac{d+1}{d-i+1}}, \tag{2.36}$$

where the implied constants are absolute.

We now check (2.36). For each $1 \le i \le d$, since π_i is H_d-equivariant, in view of (2.3) and (2.9), we have

$$e^{-\frac{i}{d+1}t} \|\pi_i(v)\| \ll \|\pi_i(c_t u(s) v)\| \ll e^{\frac{d-(i-1)}{d+1}t} \|\pi_i(v)\|. \tag{2.37}$$

Taking the $(-\frac{d+1}{d-i+1})$-th power, we get

$$e^{-t} \|\pi_i(v)\|^{-\frac{d+1}{d-i+1}} \ll \|\pi_i(c_t u(s) v)\|^{-\frac{d+1}{d-i+1}} \ll e^{\frac{i}{d-i+1}t} \|\pi_i(v)\|^{-\frac{d+1}{d-i+1}}. \tag{2.38}$$

Since $\frac{i}{d-i+1} \le d$ for every $1 \le i \le d$, (2.36) follows from (2.38). □

Fix

$$0 < \delta < \theta < \frac{d}{d+1}. \tag{2.39}$$

Let $t_0 \geq 0$. For any $x \in X$, we define

$$\widetilde{\alpha}(x) = \int_{t_0}^{\infty} e^{-\delta t} \alpha_{\varepsilon}^{\theta}(b_t x) \mathrm{d}t. \tag{2.40}$$

In the rest of the section, we write $\alpha = \alpha_{\varepsilon}^{\theta}$ to simplify notation.

Lemma 2.6. *(1) $\widetilde{\alpha}\colon X \to [1, +\infty]$ is a lower semicontinuous function.*

(2) There exists $C \geq 1$ such that for every $x \in X$ and every $t_1 \geq 0$, there exists $b'(t_1) > 0$ such that

$$\int_{I^d} \widetilde{\alpha}(a_{t_1} u(s) x) \mathrm{d}s \leq C e^{-(\theta-\delta)t_1} \widetilde{\alpha}(x) + b'(t_1). \tag{2.41}$$

(3) $\widetilde{\alpha}$ is Lipschitz with respect to the action of the semigroup $\{b_t\}_{t\geq 0} H_d$. More precisely, for every compact subset $\mathcal{O}$ of $\{b_t\}_{t\geq 0} H_d$, there exists $C \geq 1$ such that

$$\widetilde{\alpha}(gx) \leq C \widetilde{\alpha}(x), \quad \forall g \in \mathcal{O}, \forall x \in X.$$

(4) For any $M > 0$, $\widetilde{\alpha}^{-1}([0, M])$ is compact.

(5) Suppose that $\rho_{\varepsilon,\theta}(x) < \delta$ and

$$\sup_{t \geq t_0} \frac{\log \alpha_{\varepsilon,\theta}(b_t x)}{t} < \infty. \tag{2.42}$$

Then $\tilde{\alpha}(x) < \infty$.

Proof of (1). Suppose there is a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $x_i \to x$ and $\widetilde{\alpha}(x_i) \leq M$ for each i. Since α is lower semicontinuous (see e.g. [**Shi20**, Lemma 4.1]), we have $\liminf_i \alpha(b_t x_i) \geq \alpha(b_t x)$ for every $t \geq t_0$. Hence by Fatou's lemma, we have

$$\begin{aligned}
\widetilde{\alpha}(x) &= \int_{t_0}^{\infty} e^{-\delta t} \alpha(b_t x) \mathrm{d}t \\
&\leq \int_{t_0}^{\infty} \liminf_i e^{-\delta t} \alpha(b_t x_i) \mathrm{d}t \\
&\leq \liminf_i \int_{t_0}^{\infty} e^{-\delta t} \alpha(b_t x_i) \mathrm{d}t \\
&= \liminf_i \widetilde{\alpha}(x_i).
\end{aligned}$$

This proves that $\widetilde{\alpha}$ is lower semicontinuous.

Proof of (2). Let $b' = \delta^{-1}b$. We have

$$\begin{aligned}\int_{I^d} \widetilde{\alpha}(a_{t_1}u(s)x)\,\mathrm{d}s &= \int_{I^d} \widetilde{\alpha}(c_{t_1}u(s)b_{t_1}x)\,\mathrm{d}s\\ &= \int_{I^d}\int_{t_0}^{\infty} e^{-\delta t}\alpha(b_t c_{t_1}u(s)b_{t_1}x)\,\mathrm{d}t\,\mathrm{d}s\\ &= \int_{t_0}^{\infty} e^{-\delta t}\left(\int_{I^d}\alpha(c_{t_1}u(s)b_{t+t_1}x)\,\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

Therefore, by (2.33),

$$\begin{aligned}\int_{I^d} \widetilde{\alpha}(a_{t_1}u(s)x)\,\mathrm{d}s &\le \int_{t_0}^{\infty} e^{-\delta t}(Ce^{-\theta t_1}\alpha(b_{t+t_1}x)+b)\,\mathrm{d}t\\ &= Ce^{-\theta t_1}\int_{t_0+t_1}^{\infty} e^{\delta t_1}e^{-\delta t'}\alpha(b_{t'}x)\,\mathrm{d}t' + b', \quad t' = t+t_1\\ &\le Ce^{-(\theta-\delta)t_1}\int_{t_0}^{\infty} e^{-\delta t'}\alpha(b_{t'}x)\,\mathrm{d}t' + b'\\ &= Ce^{-(\theta-\delta)t_1}\widetilde{\alpha}(x) + b'.\end{aligned}$$

Proof of (3). By [**Shi20**, Lemma 4.1], for any compact subset $\mathcal{O}$ of H_d, there exists $C_1 \ge 1$ such that $\alpha(hx) \le C_1\alpha(x)$ for every $h \in \mathcal{O}$ and $x \in X$. Since b_t commutes with H_d, we have

$$\begin{aligned}\widetilde{\alpha}(hx) &= \int_{t_0}^{\infty} e^{-\delta t}\alpha(b_t hx)\mathrm{d}t\\ &= \int_{t_0}^{\infty} e^{-\delta t}\alpha(hb_t x)\mathrm{d}t\\ &\le C_1\int_{t_0}^{\infty} e^{-\delta t}\alpha(b_t x)\mathrm{d}t\\ &= C_1\widetilde{\alpha}(x).\end{aligned}$$

Now it suffices to show that for every $T > 0$ there exists $C_2 \ge 1$ such that for every $0 \le t_1 \le T$ we have $\widetilde{\alpha}(b_{t_1}x) \le C_2\widetilde{\alpha}(x)$. Indeed,

$$\begin{aligned}\widetilde{\alpha}(b_{t_1}x) &= \int_{t_0}^{\infty} e^{-\delta t}\alpha(b_t b_{t_1}x)\mathrm{d}t\\ &= e^{\delta t_1}\int_{t_0+t_1}^{\infty} e^{-\delta t'}\alpha(b_{t'}x)\mathrm{d}t'\\ &\le e^{\delta T}\widetilde{\alpha}(x).\end{aligned}$$

Proof of (4). Since $\widetilde{\alpha}$ is lower semi-continuous by (1), $\widetilde{\alpha}^{-1}([0,M])$ is a closed subset of X. It follows from the construction of $\widetilde{\alpha}$ that

$$\widetilde{\alpha}(y) \ge \int_{t_0}^{t_0+1} e^{-\delta t}\alpha(b_t y)\mathrm{d}t \ge e^{-\delta}\min_{0\le t\le 1}\alpha(b_t y),\ \forall y \in X.$$

So, $\widetilde{\alpha}^{-1}([0,M]) \subset b([-1,0])\alpha^{-1}([1,e^{\delta}M])$. Since $\alpha^{-1}([1,e^{\delta}M])$ is compact in X, it follows that $\widetilde{\alpha}^{-1}([0,M])$ is pre-compact in X.

Proof of (5). By the definition (2.12) of $\rho := \rho_{\varepsilon,\theta}(x)$, for every $0 < \varepsilon' < \delta - \rho$, there exists $T \geq t_0$ such that for every $t \geq T$, we have $\alpha(b_t x) \leq e^{(\rho+\varepsilon')t}$. By (2.42), $\sup_{t\in[t_0,T]} \alpha_{\varepsilon}^{\theta}(b_t x) < \infty$. Therefore,

$$\widetilde{\alpha}(x) = \int_{t_0}^{\infty} e^{-\delta t}\alpha(b_t x)\mathrm{d}t \leq \int_{t_0}^{T} e^{-\delta t}\alpha(b_t x)\mathrm{d}t + \int_{T}^{\infty} e^{(-\delta+\rho+\varepsilon')t}\mathrm{d}t < \infty.$$

This completes the proof of all parts of Lemma 2.6. □

2.4.1. *Obtaining non-divergence.* Let $x \in X$. For any measurable subset K of X and $T > 0$, let

$$\mathcal{C}_K^T(s) = \frac{1}{T}\int_0^T \mathbf{1}_K(a_t u(s)x)\mathrm{d}t,$$

where $\mathbf{1}_K$ is the characteristic function of K.

The following result is the analogue of [**Shi20**, Proposition 2.3].

Proposition 2.7. *Let $\theta < \frac{d}{d+1}$. Suppose that $\rho_{\varepsilon,\theta}(x) < \theta$. Then for every $0 < \epsilon < 1$, there exist a compact subset K of X, $0 < a < 1$ and $C \geq 1$ such that*

$$\left|\{s \in I^d \colon \mathcal{C}_K^T(s) \leq 1-\epsilon\}\right| \leq Ca^T, \ \forall\, T > 0. \tag{2.43}$$

Proof. In (2.39) we pick δ such that $\rho_{\varepsilon,\theta}(x) < \delta < \theta$. Moreover, we can pick $t_0 > 0$ such that (2.42) holds. Hence $\tilde{\alpha}(x) < \infty$.

Note that Lemma 4.9, Lemma 4.10, Corollary 4.11, Lemma 4.12 and Lemma 4.13 in [**Shi20**] still hold if we replace the function α there with the function $\widetilde{\alpha}$ defined in (2.40), thanks to Lemma 2.6. Hence the proposition follows from [**Shi20**, Lemma 4.12] and [**Shi20**, Lemma 4.13] with the above modification. □

Proof of Theorem 1.1. In view of (2.6), it suffices to prove that for almost every $s \in I^d$, every weak-$*$ limit of $\{\frac{1}{T}\int_0^T a_t u(s)\delta_{x_A}\mathrm{d}t\}_{T\to\infty}$ is a probability measure on X. Indeed, we can cover $\mathbb{R}^d$ by boxes of side length 1 centered at integral points. The box centered at $\mathbf{p} \in \mathbb{Z}^d$ is parametrized by $\left(\begin{smallmatrix} 1 & \mathbf{p} \\ & 1 \end{smallmatrix}\right)A$, which has the same Diophantine exponent as A.

Let $0 < \theta < \frac{d}{d+1}$. Since the Diophantine exponent $\omega(A) < n$, by [**SY24b**, Proposition 4.1] we have that $\rho_{\varepsilon,\theta}(x_A) < \theta$. Let $0 < \epsilon < 1$ be given. By Proposition 2.7 there exist a compact subset K of X, $a < 1$, and $C \geq 1$ such that (2.43) holds for all $T \in \mathbb{N}$. Hence by Borel-Cantelli lemma, we have $\liminf_{T\to\infty \text{ in } \mathbb{N}} \mathcal{C}_K^T(s) \geq 1-\epsilon$ holds for almost every $s \in I^d$. Consider the compact set $K_1 = \{a_t K : t \in [0,1]\}$. Then, for almost every $s \in I^d$, any weak-$*$ limit ν of $\{\frac{1}{T}\int_0^T a_t u(s)\delta_{x_A}\,dt\}_{T\to\infty}$ satisfies $\nu(K_1) \geq 1-\epsilon$. □

Remark 2.3. Alternatively, using Lemma 2.6, one can obtain an analogue of Proposition 2.7 as done in [**KKLM17**, Theorem 1.5] or [**Kha20**, Proposition 4.8], and then deduce Theorem 1.1 as noted in [**KKLM17**, Remark 2.1] or [**Kha20**, Theorem 4.3(b)].

3. Unipotent invariance and avoidance of singular submanifolds

3.1. **Homogeneity of ergodic invariant measures.** By Theorem 1.1 and [**Shi20**, Proposition 2.2], for almost every $s \in \mathbb{R}^d$, any weak-$*$ limit point, say ν_s, of the family of probability measures $\{\frac{1}{T}\int_0^T a_t u(s)\delta_{x_A}\mathrm{d}t\}_{T\to\infty}$ is a probability measure on X invariant under the unipotent subgroup $U = U_a^+ \cap H_d$. We note that for the conclusion of [**Shi20**, Proposition 2.2] to hold, it is sufficient that $U \subset U_a^+$, but the U-fixed point subspace need not be a_1-*expanding* for finite-dimensional representations of G. Moreover, by construction, ν_s is invariant under the action of $S = \{a_t\}_{t\in\mathbb{R}}$. Let $F = SU$.

Using Ratner's description of ergodic invariant measures for unipotent flows [**Rat91**], in [**ES19**] it is shown that every ergodic invariant measure for the F-action on X is homogeneous; here it is important that S is simple, meaning a_t has only one eigenvalue > 1 and all others are < 1 for all $t > 0$. In fact, the arguments in [**ES19**, Proof of Theorem 1.1, p.427] show that each F-invariant ergodic probability measure is concentrated on a closed orbit gLx_0 and it is invariant under gHg^{-1}, where $H = \mathbf{H}(\mathbb{R})^0$, $L = N_G(H)^0 = Z_G(H)^0 H$, $\mathbf{H}$ is a semisimple algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ such that $Z_{\mathbf{G}}(\mathbf{H})$ is a $\mathbb{Q}$-anisotropic torus, and $g \in G$ is such that $F \subset gLg^{-1}$. In particular, $Z_G(H)x_0$ is a compact orbit of a $\mathbb{C}$-diagonalizable subgroup of G.

For any proper closed subgroup L of G such that the orbit Lx_0 is closed, we define

$$N(F, L) := \{g \in G\colon g^{-1}Fg \subset L\}, \tag{3.1}$$

which is an algebraic variety, and we call the set $N(F, L)x_0$ a singular submanifold. We observe that

$$N(F, L) = N_G(F)N(F, L)N_G(L) \text{ and } N(F, L) = N(F, L^0), \tag{3.2}$$

the second equality holds because F is connected.

Being defined over $\mathbb{Q}$, there are only countably many possibilities for $\mathbf{H}$. Due to the F-ergodic decomposition of ν_s, if ν_s is not G-invariant, then there exists a proper subgroup $\mathbf{H}$ of $\mathbf{G}$ as above such that $\nu_s(N(F, N_G(H))x_0) > 0$. Therefore, to prove Theorem 1.2, in view of (2.7), it is enough to prove the following.

Proposition 3.1. *Suppose that possibility 2 of Theorem 1.2 does not hold. Then for any proper semisimple $\mathbf{Q}$-subgroup $\mathbf{H}$ of $\mathbf{G}$ such that $Z_{\mathbf{G}}(\mathbf{H})$ is a $\mathbf{Q}$-anisotropic torus, we have $\nu_s(N(F, N_G(H))x_0) = 0$ for almost all s, where ν_s denotes any weak-$*$ limit point of $\{\frac{1}{T}\int_0^T a_t u(s)\delta_{x_A}\mathrm{d}t\}_{T\to\infty}$ in $\mathcal{P}(X)$.*

The following sections are devoted to the proof of this proposition.

3.2. **Larger singular submanifolds with simpler structure.** Let $\mathbf{H}$ be a proper semisimple subgroup of $\mathbf{G}$ defined over $\mathbf{Q}$ such that $Z_{\mathbf{G}}(\mathbf{H})$ is $\mathbf{Q}$-anisotropic, and $N_G(H)$ contains a conjugate of $F = SU$; that is, $N(F, N_G(H)) \neq \emptyset$, see (3.1), where $H = \mathbf{H}(\mathbb{R})^0$. Consider the representation of $\mathbf{H}$ obtained from the restriction of the standard representation of $\mathbf{G} = \mathbf{SL_{n+1}} = \mathrm{SL}(\mathbf{V})$, where $\mathbf{V}$ is an $(n+1)$-dimensional vector space over $\mathbb{Q}$. Then, the $\mathbf{H}$-action on $\mathbf{V}$ has only one $\mathbb{Q}$-isotypic component. Now, using [**Tit71**, Theorem 7.2] and the fact that a conjugation of the simple one-parameter $\mathbb{R}$-diagonalizable subgroup $S = \{a_t\}$ is contained in $Z_G(H)H$, in [**ES19**, Proof of Proposition 2.1] it is shown that there exists a number field $\mathbb{F}$ contained in $\mathbb{R}$,

a vector space $\mathbf{W}$ over $\mathbb{F}$, and a representation $\rho : \mathbf{H} \to \mathrm{SL}(\mathbf{W})$ defined over $\mathbb{F}$ which is irreducible over $\mathbb{C}$ such that

$$\mathbf{V} \cong \mathrm{Res}_{\mathbb{F}/\mathbb{Q}} \mathbf{W}.$$

In particular, $\mathbf{H}$ acts irreducibly over $\mathbb{Q}$ and the centralizer $Z_{\mathbf{G}}(\mathbf{H})$ is abelian, and hence $Z_{\mathbf{G}}(\mathbf{H})$ is a $\mathbb{Q}$-anisotropic torus.

Proposition 3.2. *Suppose L acts irreducibly on $\mathbf{V}$ over $\mathbb{C}$ and L contains a conjugate of $F = SU$. Then $L = \mathrm{SL}_{n+1}(\mathbb{R})$. In particular, $\mathbf{H} = \mathbf{SL_{n+1}}$.*

Proof. Since L contains a conjugate of U and L is reductive, by a theorem due to Jacobson–Morozov and Kostant, a conjugate of $H_2 \cong \mathrm{SL}_2(\mathbb{R})$ is contained in L. Therefore by [SY24a, Theorem A.7], either $L = \mathrm{SL}_{n+1}(\mathbb{R})$ or $n+1$ is even and $L = \mathrm{Sp}(\mathbb{R}^{n+1}, \omega)$ for a symplectic form ω on $\mathbb{R}^{n+1}$. Now eigenvalues of a symplectic matrix occur in reciprocal pairs, so a conjugate of $S = \{a_t\}_{t\in\mathbb{R}}$ is not contained in $\mathrm{Sp}(\mathbb{R}^{n+1}, \omega)$ for $n > 1$. Since a conjugate of S is contained in L, we conclude that $L = \mathrm{SL}_{n+1}(\mathbb{R})$; we note that $\mathrm{Sp}(\mathbb{R}^2, \omega) = \mathrm{SL}_2(\mathbb{R})$. □

We have that $\mathbf{H} \neq \mathbf{G}$, so $N_G(H) \neq G$. And $N_G(H)$ contains a conjugate of F, So, by Proposition 3.2, $N_G(H)$, and hence H, does not act irreducibly on V over $\mathbb{C}$. Therefore, $\mathbb{F} \neq \mathbb{Q}$.

Let $m = [\mathbb{F} : \mathbb{Q}]$. Now $m = s_1 + 2s_2$, where $\{\sigma_i : 1 \le i \le s_1\}$ are the real and $\{\sigma_{s_1+i}, \overline{\sigma_{s_1+i}} : 1 \le i \le s_2\}$ are the complex embeddings of $\mathbb{F}$. Let $\{\alpha_1, \dots, \alpha_m\}$ be a basis of $\mathbb{F}$ over $\mathbb{Q}$. Now,

$$V = \prod_{i=1}^{s_1+s_2} V_i, \text{ where } V_i = \begin{cases} \mathbf{W}^{\sigma_i}(\mathbb{R}), \text{ if } 1 \le i \le s_1 \\ \mathbf{W}^{\sigma_i}(\mathbb{C}), \text{ if } s_1+1 \le i \le s_1+s_2, \text{ as a } \mathbb{R}\text{-vector space,} \end{cases} \tag{3.3}$$

and

$$H = \prod_{i=1}^{s_1+s_2} H_i, \text{ where } H_i = \begin{cases} \rho^{\sigma_i}(\mathbf{H})(\mathbb{R}) \text{ for } 1 \le i \le s_1 \\ \rho^{\sigma_i}(\mathbf{H})(\mathbb{C}) \text{ for } s_1+1 \le i \le s_1+s_2, \text{ as a real Lie group,} \end{cases} \tag{3.4}$$

and H_i acts irreducibly on V_i for each i; see [ES19, p. 424, (2.2)]. Now

$$Z_G(H) = G \cap \prod_{i=1}^{s_1+s_2} Z_i, \text{ where } Z_i = \begin{cases} \mathbb{R}^* I_{V_i} \text{ for } 1 \le i \le s_1 \\ \mathbb{C}^* I_{V_i} \text{ for } s_1+1 \le i \le s_1+s_2. \end{cases} \tag{3.5}$$

Since $\mathbb{F} \subset \mathbb{R}$, $s_1 \ge 1$ and since $\mathbb{F} \neq \mathbb{Q}$, we have $m \ge 2$. Therefore, $\mathbb{R}$-rank of $Z_G(H)$ is $s_1 + s_2 - 1 \ge 1$. Let $\mathbf{M}$ be the centralizer of $Z_G(\mathbf{H})$ in $\mathbf{G}$. Then

$$M = G \cap \prod_{i=1}^{s_1+s_2} J_i, \text{ where } J_i = \begin{cases} \mathrm{GL}_{\mathbb{R}}(\mathbf{W}^{\sigma_i}(\mathbb{R})) \text{ for } 1 \le i \le s_1 \\ \mathrm{GL}_{\mathbb{C}}(\mathbf{W}^{\sigma_i}(\mathbb{C})) \text{ for } s_1+1 \le i \le s_1+s_2. \end{cases} \tag{3.6}$$

Let $r = \dim_{\mathbb{F}} \mathbf{W}$. Then, $\dim V_i = r$ for $1 \le i \le s_1$ and $\dim V_i = 2r$ for $s_1+1 \le i \le s_1+s_2$. So, $n+1 = rm$.

Now $\mathbf{M}$ is a reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$ and its center equals $Z_{\mathbf{G}}(\mathbf{H})$. Since $Z_{\mathbf{G}}(\mathbf{H})$ is $\mathbb{Q}$-anisotropic, $\mathbf{M}$ admits no nontrivial $\mathbb{Q}$-characters. So, Mx_0 admits a finite M-invariant measure, and in particular, Mx_0 is closed in X. Also, $N_G(H)^0 = HZ_G(H)^0 \subset$

M. So, by (3.2),

$$N(F,M) \supset N(F,N_G(H)^0) = N(F,N_G(H)) \neq \emptyset. \tag{3.7}$$

3.2.1. *Claim:* $r \geq d+1$. As we will explain, this is a consequence of the fact that the dimension of the strictly expanding weight space of $\mathrm{Ad}(a_t)$ on the Lie algebra of F is d. Due to (3.7), pick $g \in G$ be such that $gFg^{-1} \subset M$; so ga_tg^{-1} preserves V_i for each i. Therefore, the eigen space of ga_tg^{-1} corresponding to the simple eigenvalue $e^{\frac{n}{n+1}t}$ is contained in exactly one V_i corresponding to a real embedding of $\mathbb{F}$, as any complex embedding would have repeated real eigenvalues; and hence on each of the V_j for $j \neq i$, ga_tg^{-1} acts as a scalar. Therefore, the strictly expanding eigen space of $\mathrm{Ad}(ga_tg^{-1})$ on the Lie algebra of M is contained in the Lie algebra of $\mathrm{GL}(V_i)$ and hence its dimension is $r-1$. Since M contains gFg^{-1}, and the dimension of the expanding eigen space of $\mathrm{Ad}(ga_tg^{-1})$ on the Lie algebra of gFg^{-1} is d, we get $r-1 \geq d$. This proves the claim.

3.2.2. *Choice of a conjugate of M.* Let v_0 be a regular $\mathbb{Q}$-point of the Lie algebra of $Z_G(H)$. Then $\mathbf{M}$ is the isotropy group of v_0 for the adjoint representation of $\mathbf{G}$. Let $g_1 \in G$ be a change of basis matrix corresponding to the decomposition 3.3, where for $i \geq s_1+s_2$, the multiplication by $z \in \mathbb{C}$ is given by the matrix $\varphi(z) = \begin{pmatrix} \mathrm{Re}(z)I_r & \mathrm{Im}(z)I_r \\ -\mathrm{Im}(z)I_r & \mathrm{Re}(z)I_r \end{pmatrix}$. Due to the construction of restriction of scalars (see, e.g. [**Zim84**, Proposition 6.1.3]), there exists $x \in \mathbb{F}$ such that $\mathrm{Ad}\, g_1(v_0)$ is of the form

$$\begin{pmatrix} \sigma_1(x)I_r & & & & & \\ & \ddots & & & & \\ & & \sigma_{s_1}(x)I_r & & & \\ & & & \varphi(\sigma_{s_1+1}(x)) & & \\ & & & & \ddots & \\ & & & & & \varphi(\sigma_{s_1+s_2}(x)) \end{pmatrix}. \tag{3.8}$$

Due to (3.3) and (3.6), the $\sigma_i(x)$ are all distinct for $1 \leq i \leq s_1+s_2$. In particular, $x \notin \mathbb{Q}$.

Remark 3.1. We have chosen g_1 such that for $1 \leq i \leq s_1$, the i-th block of $(r \times r)$-rows of g_1 are defined over $\sigma_i(\mathbb{F})$.

We set $M_1 = g_1 M g_1^{-1}$. Then M_1 is the centralizer of $\mathrm{Ad}(g_1)v_0$. As a consequence of Claim 3.2.1, $H_d \subset M_1$. Moreover the full diagonal subgroup of G is contained in M_1. In particular, $F \subset M_1$, so $g_1 \in N(F,M)$.

Proposition 3.3. *There exist $p_1, \ldots, p_{s_1} \in N_G(M_1)$ such that*

$$N(F,M) = \bigcup_{i=1}^{s_1} Z_G(F) M_1 p_i g_1.$$

In fact, for each $1 \leq i \leq s_1$, we let $p_i \in \mathbf{G}(\mathbb{Z})$ to be a permutation on coordinates such that under conjugation it sends the i-th $(r \times r)$-block of M_1 to the first $(r \times r)$-block of M_1.

Proof. Let $g \in N(F,M)$. Then $g^{-1}Fg \subset M$. So $(g_1g^{-1})F(gg_1^{-1}) \subset M_1$. Now the eigenvalue $e^{\frac{n}{n+1}t}$ of $(g_1g^{-1})a_t(gg_1^{-1})$ must occur in the i-th $(r \times r)$-blocks of M_1 where

$1 \le i \le s_1$. Let $p = p_i \in N_G(M_1)$ be as chosen in the statement of the proposition. Therefore, the eigenvalue of $(pg_1g^{-1})a_t(pg_1g^{-1})^{-1}$ is in the first $r \times r$-block of M_1. Further, we can find $m_1 \in M_1$ such that $(m_1pg_1g^{-1})a_t(m_1pg_1g^{-1})^{-1} = a_t$. Moreover,

$$(m_1pg_1g^{-1})U(m_1pg_1g^{-1})^{-1} \subset m_1p(g_1g^{-1})F(gg_1^{-1})p^{-1}m_1^{-1} \subset M_1.$$

Now, $(m_1pg_1g^{-1})U(m_1pg_1g^{-1})^{-1}$ is contained in the expanding horospherical subgroup of a_t in M_1, so it is contained in the first row of the first $r \times r$-block of M_1. So we can pick, m_2 in the first block of M_1 such that m_2 commutes with a_t and

$$m_2(m_1pg_1g^{-1})u(m_1pg_1g^{-1})^{-1}m_2^{-1} = u,\ \forall u \in U.$$

Let $z = (m_2m_1pg_1g^{-1})^{-1}$. Then $z \in Z_G(F)$ and $g = zm_2m_1pg_1 \in zM_1pg_1$. There are only s_1 possibilities for p as above; they are representatives of $Z_G(F)\backslash N_G(M_1)/M_1$ in $N_G(M_1)$. □

3.3. **Construction of a height function with respect to the singular submanifold.** Fix any $0 < \theta < \frac{d}{d+1}$. We pick θ' such that

$$\frac{d}{d+1}\theta' < \theta < \frac{d}{d+1}. \tag{3.9}$$

For any $l \ge 1$, we define

$$X_l = \{x \in X \mid \alpha_\varepsilon^\theta(x) \le l\}.$$

Since $\alpha_\varepsilon^\theta$ is lower semicontinuous and proper, X_l is a compact subset of X. Let

$$X' = \{x \in X \mid \alpha_\varepsilon^\theta(x) < +\infty\}.$$

As we have noted in the paragraph following (2.11), $H_dX' = X'$.

3.3.1. *Around a chosen section of the singular submanifold.* In view of Proposition 3.3, pick $p = p_i \in N_G(M_1)$ for some $1 \le i \le s_1$, and let

$$y = pg_1x_0,\ Y = M_1y = pg_1Mx_0, \tag{3.10}$$

which is a closed orbit of M_1 in X.

Let $\mathfrak{s}$ denote the $\mathrm{Ad}(H_d)$-invariant subspace of $\mathrm{Lie}(Z_G(H_d))$ which is complementary to $\mathrm{Lie}(M_1)$. Then $\mathfrak{g} = \mathrm{Lie}(G)$ can be expressed as a direct sum of $\mathrm{Ad}(H_d)$-invariant subspaces

$$\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{p} \oplus \mathfrak{s} \oplus \mathrm{Lie}(M_1), \tag{3.11}$$

where

$$\mathfrak{q} = \left\{\begin{pmatrix} B_1 \\ \end{pmatrix} : B_1 \in \mathrm{Mat}_{d+1,n+1-r}\right\}, \qquad \mathfrak{p} = \left\{\begin{pmatrix} B_2 & \end{pmatrix} : B_2 \in \mathrm{Mat}_{n+1-r,d+1}\right\}. \tag{3.12}$$

Remark 3.2. Note that $\mathfrak{q}$, as a $H_d \cong \mathrm{SL}_{d+1}(\mathbb{R})$-module, is a direct sum of $(n+1-r)$-copies of the standard representation $\mathbb{R}^{d+1}$, and $\mathfrak{p}$ is a direct sum of $(n+1-r)$-copies of the contragradient of the standard representation, which is isomorphic to $\wedge^d\mathbb{R}^{d+1}$.

In particular, the eigenvalues of c_t-action on $\mathfrak{q}$ are $e^{\frac{d}{d+1}t}$ and $e^{-\frac{1}{d+1}t}$; and on $\mathfrak{p}$ they are $e^{\frac{1}{d+1}t}$ and $e^{-\frac{d}{d+1}t}$.

As a consequence we have the following:

Lemma 3.4. *For every $v \in \mathfrak{q}$, $w \in \mathfrak{p}$, $t \geq 0$ and $s \in \mathbb{R}^d$,*

$$e^{-\frac{1}{d+1}t}\left\|u(s)^{-1}\right\|^{-1}\|v\| \ll \|c_t u(s)v\| \ll e^{\frac{d}{d+1}t}\|u(s)\|\|v\|, \tag{3.13}$$

$$e^{-\frac{d}{d+1}t}\left\|u(s)^{-1}\right\|^{-1}\|w\| \ll \|c_t u(s)w\| \ll e^{\frac{1}{d+1}t}\|u(s)\|\|w\|. \tag{3.14}$$

Remark 3.3. For any $r > 0$, let B_r denote the norm ball of radius r in $\mathfrak{g}$ and the map

$$\psi : (\mathfrak{q} \cap B_r) \times (\mathfrak{p} \cap B_r) \times (\mathfrak{s} \cap B_r) \times (\mathrm{Lie}(M_1) \cap B_r) \to G,$$

given by $(v, w, z, y) \mapsto \exp(-v)\exp(-w)\exp(z)\exp(y)$, and let Ψ_r denotes its image. Then fix $0 < r_0 < 1$ such that ψ_{r_0} is a diffeomorphism onto Ψ_{r_0}.

We will need an estimate for return to a fixed compact set in terms of $\alpha_\varepsilon^\theta(x)$.

Lemma 3.5. *Let $\sigma_0 > 1$ be given. Then there exist $t_1 \geq 1$ and $l_1 \geq 1$ such that for any $l \geq l_1$, and any $x \in X'$ the following holds: There exist*

$$\tau \leq t_1 + \sigma_0\theta^{-1}\log^+(2l^{-1}\alpha_\varepsilon^\theta(x)) \text{ and } s \in [-1,1]^d$$

such that

$$c_\tau u(s)x \in X_l.$$

Here $\log^+(y) := \max\{\log y, 0\}$ for every $y > 0$.

Proof. Let $C > 1$ be such that the conclusion of Lemma 2.4 holds. Take $t_1 \geq 1$ sufficiently large such that

$$\frac{\theta}{\theta - t_1^{-1}\log C} \leq \sigma_0.$$

We will write $t = t_1$ for simplicity. By Lemma 2.4, there exists $b > 0$ such that

$$\int_{I^d} \alpha_\varepsilon^\theta(c_t u(s)x)\mathrm{d}s \leq Ce^{-\theta t}\alpha_\varepsilon^\theta(x) + b.$$

Iterating n times,

$$\begin{aligned}\int_{\mathbb{R}^d} \alpha_\varepsilon^\theta(c_{nt}u(s)x)\mathrm{d}\nu^{(n)}(s) &\leq C^n e^{-n\theta t}\alpha_\varepsilon^\theta(x) + \frac{1 - C^n e^{-n\theta t}}{1 - Ce^{-\theta t}}b \\ &< C^n e^{-n\theta t}\alpha_\varepsilon^\theta(x) + \frac{b}{1 - Ce^{-\theta t}},\end{aligned} \tag{3.15}$$

where $\nu^{(n)}$ is a probability measure on $\mathbb{R}^d$ supported on $(1 - e^{-t})^{-1}I^d$. Indeed, $\nu^{(1)}$ is the Lebesgue measure restricted to $I^d = [-1/2, 1/2]^d$, and for each $n \geq 2$,

$$\begin{aligned}&\int_{I^d}\left(\int_{\mathbb{R}^d} \alpha_\varepsilon^\theta(c_{(n-1)t}u(s_1)c_t u(s)x)\, d\nu^{(n-1)}(s_1)\right) d\nu^{(1)}(s) \\ &= \int_{I^d}\left(\int_{\mathbb{R}^d} \alpha_\varepsilon^\theta(c_{nt}u(e^{-t}s_1 + s)x)\, d\nu^{(n-1)}(s_1)\right) d\nu^{(1)}(s) \\ &= \int_{\mathbb{R}^d} \alpha_\varepsilon^\theta(c_{nt}u(s)x)\, d\nu^{(n)}(s),\end{aligned}$$

where $\nu^{(n)}$ is the convolution of $(e^{-t}I_d)_*\nu^{(n-1)}$ and $\nu^{(1)}$, where $e_{-t}I_d$ denotes the multiplication by e^{-t} on $\mathbb{R}^d$.

Remark 3.4. For later purpose we note that $\mathrm{d}\nu^{(n)} = f_n \mathrm{d}s$, where $f_n \geq 0$, f_n is symmetric about each coordinate axis, $\int f_n = 1$, and if $t_1 \geq 2$, then $f_n \geq 4^{-d}\mathbf{1}_{I^d}$; that is, $\nu^{(1)} \leq 4^d \nu^{(n)}$ for all n.

Take $l_1 = \frac{2b}{1-Ce^{-\theta t}}$. Let $l \geq l_1$ and choose smallest $n \in \mathbb{N}$ such that

$$C^n e^{-n\theta t} \alpha_\varepsilon^\theta(x) \leq l/2.$$

So, we choose $n = \left\lceil \frac{\log^+(2l^{-1}\alpha_\varepsilon^\theta(x))}{\theta t - \log C} \right\rceil$. Then, the right hand side of (3.15) is at most $l/2 + l_1/2 \leq l$. Hence, by (3.15), there exist

$$s \in (1-e^{-t})^{-1} I^d \subset [-1,1]^d$$

such that $\alpha_\varepsilon^\theta(c_{nt} u(s) x) \leq l$. Now, $t = t_1$ and

$$\begin{aligned} nt &\leq t + \frac{t}{\theta t - \log C} \cdot \log^+(2l^{-1}\alpha_\varepsilon^\theta(x)) \\ &= t_1 + \theta^{-1} \cdot \frac{\theta}{\theta - t_1^{-1}\log C} \cdot \log^+(2l^{-1}\alpha_\varepsilon^\theta(x)) \\ &\leq t_1 + \theta^{-1}\sigma_0 \log^+(2l^{-1}\alpha_\varepsilon^\theta(x)). \end{aligned}$$

We conclude the proof of the lemma by choosing $\tau = nt_1$. □

Lemma 3.6. *Let $\sigma > 1$. We can pick an open neighborhood O of the 0 in $\mathfrak{s}$ and an $0 < \varepsilon_0 < r_0^d$ such that for any $x \in X$, if we set*

$$r_x := \varepsilon_0 \left[\alpha_\varepsilon^{\frac{d}{d+1}\sigma}(x) \right]^{-1}, \tag{3.16}$$

then for every $x \in X'$, there exists at most one $v \in \mathfrak{q}$ and $w \in \mathfrak{p}$ such that

$$\exp(w)\exp(v)x \in \exp(O)Y \quad \text{and} \quad \|v\| < r_x, \|w\| < (r_x)^{\frac{1}{d}}. \tag{3.17}$$

Proof. This follows from Lemma 3.4, Lemma 3.5 and [**PSS23**, Lemma 6.6]; c.f. the claim before [**PSS23**, eq.(6.15)]. We give a detailed proof below.

Pick σ_0 such that $1 < \sigma_0 < \sigma$, and let t_1 and l_1 as in Lemma 3.5. Pick any $l \geq l_1$. Now X_l is compact, and hence the set $\tilde{X}_l = \overline{\Psi_1} X_l$ is compact, where Ψ_r is as in Remark 3.3.

Since $M_1 y$ is closed in X, there exists $0 < r_1 < r_0$ such that for any $g \in \Psi_{r_1}$, if $g(\tilde{X}_l \cap M_1 y) \cap M_1 y \neq \emptyset$, then $g \in M_1$.

Let $0 < r < r_0$ be such that $\Psi_r^{-1}\Psi_r \cap \mathrm{Stab}(y) = \{e\}$ for all $y \in Y$ and $(\Psi_r^{-1}\Psi_r) \subset \Psi_{r_1}$ and $\Psi_r^{-1} \subset \Psi_1$.

Let $O = \mathfrak{s} \cap B_r$. Then for any $x' \in X_l$, there exists at most one $v' \in \mathfrak{q}$ and $w' \in \mathfrak{p}$ such that

$$\exp(w')\exp(v')x' \in \exp(O)Y \quad \text{and} \quad \|v'\| < r, \|w'\| < r. \tag{3.18}$$

Indeed, suppose $\exp(w_i)\exp(v_i)x' = \exp(z_i)m_i y$ for some $v_i \in \mathfrak{q} \cap B_r$, $w_i \in \mathfrak{p} \cap B_r$, and $z_i \in \mathfrak{s} \cap B_r$ and $m_i \in M_1$ for $i = 1, 2$. Then $m_i y \in \Psi_r^{-1} x' \subset \Psi_1 X_l$. Let

$$g = (\exp(-v_2)\exp(-w_2)\exp(z_2))^{-1}\exp(-v_1)\exp(-w_1)\exp(z_1) \in \Psi_r^{-1}\Psi_r \subset \Psi_{r_1}.$$

Then, $g(m_1 y) = m_2 y$, and hence $g \in M_1$ by our choice of r_1 above. So,

$$g \in M_1 \cap \Psi_{r_1} \subset \exp(\mathrm{Lie}(M_1) \cap B_{r_1}).$$

Then

$$\exp(-v_2)\exp(-w_2)\exp(z_2)g = \exp(-v_1)\exp(-w_1)\exp(z_1).$$

Therefore, by Remark 3.3, we get $v_1 = v_2$ and $w_1 = w_2$. This proves there is at most one solution to (3.18).

Let $x \in X'$. By Lemma 3.5, there exists $\tau \le t_1 + \sigma_0\theta^{-1}\log(2l^{-1}\alpha_\varepsilon^\theta(x))$ and $s \in \mathbb{R}^d$ with $\|s\|_\infty \le 1$ such that

$$c_\tau u(s)x \in X_l. \tag{3.19}$$

Then by Lemma 3.4, for all $v \in \mathfrak{q}$, we have

$$\begin{aligned}\|\mathrm{Ad}(c_\tau u(s))v\| &\le e^{\frac{d}{d+1}\tau}\cdot\|u(s)\|\|v\| \\ &\le e^{\frac{d}{d+1}t_1}(2\ell^{-1})^{\frac{d}{d+1}\sigma_0\theta^{-1}}\alpha_\varepsilon^{\frac{d}{d+1}\sigma_0}(x)\cdot\|u(s)\|\|v\| \\ &\ll \alpha_\varepsilon^{\frac{d}{d+1}\sigma}(x)\|v\|,\end{aligned} \tag{3.20}$$

and for all $w \in \mathfrak{p}$, we have

$$\|\mathrm{Ad}(c_\tau u(s))w\| \le e^{\frac{1}{d+1}\tau}\|u(s)\|\|w\| \ll \alpha_\varepsilon^{\frac{1}{d+1}\sigma}(x)\|w\|. \tag{3.21}$$

For $\varepsilon_0 > 0$, let r_x be as in (3.16). Now suppose $v \in \mathfrak{q}$ and $w \in \mathfrak{p}$ satisfy (3.17). Then,

$$\begin{aligned}&c_\tau u(s)\exp(w)\exp(v)x \\ &= \exp(\mathrm{Ad}(c_\tau u(s))w)\exp(\mathrm{Ad}(c_\tau u(s))v)(c_\tau u(s))x \in \exp(O)Y,\end{aligned} \tag{3.22}$$

because $c_\tau u(s)\exp(O)Y = \exp(O)Y$. By (3.16), (3.17), (3.20) and (3.21),

$$\|\mathrm{Ad}(c_\tau u(s))v\| \ll \varepsilon_0 < r \text{ and } \|\mathrm{Ad}(c_\tau u(s))w\| \ll \varepsilon_0^{\frac{1}{d}} < r,$$

where we pick $\varepsilon_0 > 0$ sufficiently small depending only on the implicit constants in (3.20) and (3.21) and r. Then, by (3.18), in view of (3.19) and (3.22), we conclude that the v and w are unique. □

Choices of parameters. We have picked θ and θ' to satisfy (3.9). Now we pick $\sigma > 1$ such that

$$\frac{d}{d+1}\theta' < \sigma\frac{d}{d+1}\theta' < \theta < \frac{d}{d+1}. \tag{3.23}$$

For this choice of $\sigma > 1$, we pick σ_0 such that $1 < \sigma_0 < \sigma$, and let t_1 and l_1 as in Lemma 3.5. Further we pick any $l \ge l_1$, which we shall specify later, we obtain O and r_x such that the conclusion of Lemma 3.6 holds. We set

$$\mathcal{N} = \exp(O)Y = \exp(O)M_1 y. \tag{3.24}$$

3.3.2. *Height function for $\mathcal{N}$ with $c_t u(I^d)$-contraction.* Define $\beta^{\theta'} : X \to [1,\infty]$ by

$$\beta^{\theta'}(x) = \begin{cases}\min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} & \text{if } \exists!\,(v,w) \in \mathfrak{q}\times\mathfrak{p} \text{ satisfying (3.17)} \\ r_x^{-\theta'} & \text{otherwise.}\end{cases} \tag{3.25}$$

Remark 3.5. For any $x \in X$,

$$\beta^{\theta'}(x) = \max\Big(\{r_x^{-\theta'}\} \bigcup \big\{\min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} : v \in \mathfrak{q}, w \in \mathfrak{p}, \exp(w)\exp(v)x \in \mathcal{N}\big\}\Big).$$

To see this, let $x \in X$, and $(v, w) \in \mathfrak{q} \times \mathfrak{p}$. Then, (3.17) is equivalent to

$$\exp(w)\exp(v)x \in \mathcal{N} \text{ and } r_x^{-\theta'} < \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\}. \tag{3.26}$$

Therefore, $r_x^{-\theta'} \leq \beta^{\theta'}(x)$.

Moreover, if $\exp(w)\exp(v)x \in \mathcal{N}$, then we claim that

$$\beta^{\theta'}(x) \geq \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\}. \tag{3.27}$$

To show this by contradiction, suppose (3.27) fails to hold, then (3.26) holds. Hence (3.17) hold. In particular, $r_x^{-\theta'} < \infty$, so $x \in X'$. And by Lemma 3.6, there exists at most one pair $(v, w) \in \mathfrak{q} \times \mathfrak{p}$ satisfying (3.17). So (3.27) holds with equality, which is a contradiction.

Lemma 3.7. *There exists $C_2 \geq 1$ such that given any $x \in X'$, $t \geq 0$ and $s \in I^d$,*

$$\beta^{\theta'}(c_t u(s)x) \leq C_2 e^{\frac{\sigma d^2\theta'}{d+1}t}\beta^{\theta'}(x). \tag{3.28}$$

Proof. By Lemma 3.4, there exists $C_3 \geq 1$ such that for all $t \geq 0$, $s \in I^d$, $v \in \mathfrak{q}$ and $w \in \mathfrak{p}$, we have

$$C_3^{-1}e^{-\frac{d\theta'}{d+1}t}\|v\|^{-\theta'} \leq \|c_t u(s)v\|^{-\theta'} \leq C_3 e^{\frac{\theta'}{d+1}t}\|v\|^{-\theta'}, \tag{3.29}$$

and

$$C_3^{-1}e^{-\frac{d\theta'}{d+1}t}\|w\|^{-d\theta'} \leq \|c_t u(s)w\|^{-d\theta'} \leq C_3 e^{\frac{d^2\theta'}{d+1}t}\|w\|^{-d\theta'}. \tag{3.30}$$

We observe that, by (3.16) and then by Lemma 2.5,

$$r_{c_t u(s)x}^{-\theta'} \ll \left(\alpha_\varepsilon^{\frac{d}{d+1}\sigma}(c_t u(s)x)\right)^{\theta'} \ll \left(e^{d\frac{d}{d+1}\sigma t}\alpha_\varepsilon^{\frac{d}{d+1}\sigma}(x)\right)^{\theta'} \ll e^{\frac{\sigma d^2\theta'}{d+1}t} r_x^{-\theta'}. \tag{3.31}$$

Suppose first that $\beta^{\theta'}(c_t u(s)x) = r_{c_t u(s)x}^{-\theta'}$. By (3.31) and Remark 3.5, we have

$$\beta^{\theta'}(c_t u(s)x) = r_{c_t u(s)x}^{-\theta'} \ll e^{\frac{\sigma d^2\theta'}{d+1}t} r_x^{-\theta'} \leq e^{\frac{\sigma d^2\theta'}{d+1}t}\beta^{\theta'}(x). \tag{3.32}$$

Otherwise, we have that

$$\beta^{\theta'}(c_t u(s)x) = \min\{\|v'\|^{-\theta'}, \|w'\|^{-d\theta'}\} > r_{c_t u(s)x}^{-\theta'}$$

for unique $v' \in \mathfrak{q}$ and $w' \in \mathfrak{p}$ such that $\exp(w')\exp(v')c_t u(s)x \in \mathcal{N}$. Let

$$v = \mathrm{Ad}(c_t u(s))^{-1}v' \in \mathfrak{q} \text{ and } w = \mathrm{Ad}(c_t u(s))^{-1}w' \in \mathfrak{p}.$$

Then, $c_t u(s)\exp(w)\exp(v)x \in \mathcal{N}$. Since $H_d\mathcal{N} = \mathcal{N}$, we have $\exp(w)\exp(v)x \in \mathcal{N}$. Then by (3.29), (3.30), and Remark 3.5, we have

$$\begin{aligned}\beta^{\theta'}(c_t u(s)x) &= \min\{\|v'\|^{-\theta'}, \|w'\|^{-d\theta'}\}\\ &\ll \min\{e^{\frac{\theta'}{d+1}t}\|v\|^{-\theta'}, e^{\frac{d^2\theta'}{d+1}t}\|w\|^{-d\theta'}\}\\ &\le e^{\frac{d^2\theta'}{d+1}t}\min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\}\\ &\le e^{\frac{d^2\theta'}{d+1}t}\beta^{\theta'}(x),\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33), we have

$$\beta^{\theta'}(c_t u(s)x) \ll e^{\frac{\sigma d^2\theta'}{d+1}t}\beta^{\theta'}(x),$$

where the implied constant is independent of $t \ge 0$ and $s \in I^d$. □

Let l_1 be as in Lemma 3.5. Let $l \ge l_1$. We note that our choices of $\mathcal{N}$ and the function $r_x^{\theta'}$ as in Lemma 3.6 depend on l.

Let $\beta_{\mathcal{N}} : X \to [1,\infty]$ be defined by

$$\beta_{\mathcal{N}}(x) = \alpha_\varepsilon^\theta(x) + \beta^{\theta'}(x). \tag{3.34}$$

Proposition 3.8. *The function $\beta_{\mathcal{N}}$ satisfies the following properties:*

(1) $\beta_{\mathcal{N}}$ is lower semicontinuous.

(2) For any $x \in X'$, $\beta_{\mathcal{N}}(x) = \infty$ if and only if $x \in \mathcal{N}$.

(3) $\beta_{\mathcal{N}}$ is Lipschitz with respect to the H_d-action; that is, for any compact set $\mathcal{C} \subset H_d$, there exists $C > 0$ such that $\beta_{\mathcal{N}}(gx) \le C\beta_{\mathcal{N}}(x)$ for all $g \in \mathcal{C}$ and $x \in X$.

(4) There exists $E \ge 1$ such that for any $t \ge 0$ there exists $B_1(t) \ge 0$ such that for any $x \in X$ we have

$$\int_{I^d} \beta_{\mathcal{N}}(c_t u(s)x)\mathrm{d}s \le Ee^{-\frac{d}{d+1}\theta' t}\beta_{\mathcal{N}}(x) + B_1(t). \tag{3.35}$$

(5) For any $l' \ge 1$, the set $\beta_{\mathcal{N}}^{-1}([1,l'])$ is compact.

Proof. (Cf. Proof of [**PSS23**, Theorem 6.4])

Proof of (1): By [**Shi20**, Lemma 4.1] we know that α^θ is lower semicontinuous. So by (3.16), we have that $x \mapsto r_x^{-\theta'}$ is lower semicontinuous.

It remains to prove lower semicontinuity of $\beta^{\theta'}$. To prove this, let $x_i \to x$.

First suppose that $\beta^{\theta'}(x) = r_x^{-\theta'}$. So,

$$\liminf_{i\to\infty}\beta^{\theta'}(x_i) \ge \liminf_{i\to\infty} r_{x_i}^{-\theta'} \ge r_x^{-\theta'} = \beta^{\theta'}(x);$$

where the first inequality holds because of Remark 3.5 and the second one holds due to lower semi-continuity of $y \mapsto r_y^{-\theta'}$.

Now suppose that $\beta^{\theta'}(x) > r_x^{-\theta'}$. Then there exists $(v,w) \in \mathfrak{q}\times\mathfrak{p}$ such that

$$\exp(w)\exp(v)x \in \mathcal{N} \text{ and } \beta^{\theta'}(x) = \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} > r_x^{-\theta'}.$$

In particular, $\|v\| < r_x < r_0$ and $\|w\| < r_x^{1/d} < r_0$.

Let $g_i \to e$ such that $x_i = g_i x$. Then

$$x_i \in g_i \exp(-v)\exp(-w)\mathcal{N} = g_i \exp(-v)\exp(-w)\exp(z)M_1 y.$$

where $z \in \mathfrak{s} \cap B_r$. Therefore, by Remark 3.3, for all large i, there exist $v_i \to v$ in $\mathfrak{q}$, $w_i \to w$ in $\mathfrak{p}$, and $z_i \to z$ in $\mathfrak{z}$ such that

$$g_i \exp(-v)\exp(-w)\exp(z)M_1 = \exp(-v_i)\exp(-w_i)\exp(z_i)M_1,$$

and hence $x_i \in \exp(-v_i)\exp(-w_i)\exp(O)M_1 y$.

Therefore, by Remark 3.5, we have $\beta^{\theta'}(x_i) \geq \min\{\|v_i\|^{-\theta'}, \|w_i\|^{-d\theta'}\}$. Hence,

$$\liminf_{i\to\infty} \beta^{\theta'}(x_i) \geq \liminf_{i\to\infty} \min\{\|v_i\|^{-\theta'}, \|w_i\|^{-d\theta'}\} = \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} = \beta^{\theta'}(x).$$

Combining both cases, we proved that $\beta^{\theta'}$ is lower semicontinuous.

Proof of (2): By the definition of X', for $x \in X'$ we have $\alpha^{\theta}(x) < \infty$, and it follows that $r_x < \infty$. Therefore, $\beta_{\mathcal{N}}(x) = \infty$ if and only if $\exp(w)\exp(v)x \in \mathcal{N}$ and $v = w = 0$, that is, $x \in \mathcal{N}$.

Proof of (3): By [**Shi20**, Lemma 4.1], X_l is compact and α^{θ} is Lipschitz with respect to the action of H_d. It follows that $x \mapsto r_x^{-\theta'}$ is also Lipschitz with respect to the action of H_d. Since the action of H_d on $\mathfrak{q} \times \mathfrak{p}$ by conjugation is Lipschitz, and $\mathcal{N}$ is H_d-invariant, from definition it is straightforward to verify that $\beta^{\theta'}$ is also Lipschitz with respect to the action of H_d. Hence $\beta_{\mathcal{N}}$ is Lipschitz with respect to the action of H_d.

Proof of (4): Let $C_1 \geq 1$, $C_2 \geq 1$, $C_3 \geq 1$, and $\varepsilon_0 > 0$ be the constants in (2.34)(3.28)(3.29)(3.16). Let $x \in X$ and $t \geq 0$ be given.

Firstly, suppose that there exists $v \in \mathfrak{q}$ and $w \in \mathfrak{p}$ such that $\exp(w)\exp(v)x \in \mathcal{N}$ and

$$\min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} > C_3 e^{\left(\frac{d\theta'}{d+1}\right)t} \cdot C_1 e^{d\left(\frac{\sigma d}{d+1}\theta'\right)t} \cdot \varepsilon_0^{-\theta'} \alpha^{\left(\frac{\sigma d}{d+1}\theta'\right)}(x). \tag{3.36}$$

Note that

$$C_3 e^{\left(\frac{d\theta'}{d+1}\right)t} \cdot C_1 e^{d\left(\frac{\sigma d}{d+1}\theta'\right)t} \cdot \varepsilon_0^{-\theta'} \alpha^{\left(\frac{\sigma d}{d+1}\theta'\right)}(x) \geq \varepsilon_0^{-\theta'} \alpha^{\frac{\sigma d\theta'}{d+1}}(x) = r_x^{-\theta'}. \tag{3.37}$$

Hence by the definition of $\beta^{\theta'}$, (3.36), (3.37), and Remark 3.5, we have that

$$\beta^{\theta'}(x) = \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\}. \tag{3.38}$$

Then for every $s \in I^d$, we have

$$\begin{aligned}
&\min\{\|c_t u(s) v\|^{-\theta'}, \|c_t u(s) w\|^{-d\theta'}\} \\
&\geq C_3^{-1} e^{-\frac{d\theta'}{d+1}t} \min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\}, \text{ by (3.29) and (3.30)}, \\
&> C_1 e^{d\left(\frac{\sigma d}{d+1}\theta'\right)t} \cdot \varepsilon_0^{-\theta'} \alpha^{\left(\frac{\sigma d}{d+1}\theta'\right)}(x), \text{ by (3.36)}, \\
&\geq \varepsilon_0^{-\theta'} \alpha^{\left(\frac{\sigma d}{d+1}\theta'\right)}(c_t u(s) x), \text{ by (2.34)}, \\
&= r_{c_t u(s)x}^{-\theta'}, \text{ by (3.16)}.
\end{aligned}$$

Therefore,

$$\beta^{\theta'}(c_t u(s) x) = \min\{\|c_t u(s) v\|^{-\theta'}, \|c_t u(s) w\|^{-d\theta'}\}. \tag{3.39}$$

Since $v \in \mathfrak{q}$ and $w \in \mathfrak{p}$, by Remark 3.2 and [**SY24b**, Lemma 3.1], we have

$$\begin{aligned} \int_{I^d} \|c_t u(s)v\|^{-\theta'} \mathrm{d}s &\ll e^{-\frac{d}{d+1}\theta' t}\|v\|^{-\theta'} \quad \text{and} \\ \int_{I^d} \|c_t u(s)w\|^{-d\theta'} \mathrm{d}s &\ll e^{-\frac{1}{d+1}(d\theta')t}\|w\|^{-d\theta'}. \end{aligned} \tag{3.40}$$

By (3.38), (3.39), and (3.40) we have

$$\begin{aligned} \int_{I^d} \beta^{\theta'}(c_t u(s)x)\mathrm{d}s &= \int_{I_d} \min\{\|c_t u(s)v\|^{-\theta'}, \|c_t u(s)w\|^{-d\theta'}\}\mathrm{d}s \\ &\le \min\left\{\int_{I_d} \|c_t u(s)v\|^{-\theta'}\mathrm{d}s, \int_{I_d} \|c_t u(s)w\|^{-d\theta'}\mathrm{d}s\right\} \\ &\ll e^{-\frac{d\theta'}{d+1}t}\min\{\|v\|^{-\theta'}, \|w\|^{-d\theta'}\} = e^{-\frac{d\theta'}{d+1}t}\beta^{\theta'}(x). \end{aligned} \tag{3.41}$$

On the other hand, by Lemma 2.4,

$$\int_{I^d} \alpha^{\theta}_{\varepsilon}(c_t u(s)x)\mathrm{d}s \le Ce^{-\theta t}\alpha^{\theta}_{\varepsilon}(x) + b. \tag{3.42}$$

Recall from (3.9) that $\frac{d\theta'}{d+1} < \theta$. Now (3.35) follows from (3.34), (3.41), and (3.42).

Secondly, suppose that

$$\beta^{\theta'}(x) \le C_2^{-1}e^{-\frac{\sigma d^2}{d+1}\theta' t} \cdot C_1^{-1}e^{-\frac{d}{d+1}\theta' t} \cdot \alpha^{\theta}(x). \tag{3.43}$$

Then, for all $s \in I^d$ we have

$$\begin{aligned} \beta^{\theta'}(c_t u(s)x) &\le C_2 e^{\frac{\sigma d^2\theta'}{d+1}t}\beta^{\theta'}(x), \text{ by (3.28)}, \\ &\le C_1^{-1}e^{-\frac{d\theta'}{d+1}t}\alpha^{\theta}(x), \text{ by (3.43)}, \\ &\le e^{(\theta-\frac{d\theta'}{d+1})t}\alpha^{\theta}(c_t u(s)x), \text{ by (2.34)}. \end{aligned} \tag{3.44}$$

So,

$$\begin{aligned} \int_{I^d} \beta_{\mathcal{N}}(c_t u(s)x)\mathrm{d}s &= \int_{I^d} \alpha^{\theta}(c_t u(s)x)\mathrm{d}s + \int_{I^d} \beta^{\theta'}(c_t u(s)x)\mathrm{d}s, \text{ by (3.34)}, \\ &\le (e^{(\theta-\frac{d\theta'}{d+1})t} + 1)\int_{I^d} \alpha^{\theta}(c_t u(s)x)\mathrm{d}s, \text{ by (3.44)}, \\ &\ll e^{(\theta-\frac{d\theta'}{d+1})t}(e^{-\theta t}\alpha^{\theta}(x) + b), \text{ by (2.33)}, \\ &< e^{-\frac{d\theta'}{d+1}t}\beta_{\mathcal{N}}(x) + be^{(\theta-\frac{d\theta'}{d+1})t}, \text{ by (3.34)}. \end{aligned}$$

Hence (3.35) holds.

Finally, suppose the negation of (3.36) and the negation of (3.43) hold. Then,

$$C_1^{-1}C_2^{-1}e^{-\frac{\sigma d^2+d}{d+1}\theta' t}\alpha^{\theta}(x) < \beta^{\theta'}(x) < \varepsilon_0^{-\theta'}C_1C_3e^{\frac{\sigma d^2+d}{d+1}\theta' t}\alpha^{\frac{\sigma d\theta'}{d+1}}(x). \tag{3.45}$$

By (3.23) we have $\theta > \frac{\sigma d\theta'}{d+1}$. By the definition, we have $\alpha^\theta(x) \geq 1$. Hence (3.45) implies that there exists $C_4 > 0$ such that

$$\alpha^\theta(x) < C_4 e^{2\frac{\sigma d^2+d}{d+1}\theta'(1-\frac{\sigma d\theta'}{(d+1)\theta})^{-1}t}.$$

Therefore, by (2.34), (3.28) and (3.45), there exist $C_5 \geq 1$ and $A > 0$, both independent of t, such that

$$\int_{I^d} \beta_{\mathcal{N}}(c_t u(s)x)\mathrm{d}s \leq C_5 e^{At},$$

and this implies (3.35).

Thus, (3.35) holds in all the three possible cases.

Proof of (5): This follows because the inverse image of $[1, l']$ under $\alpha^\theta_\varepsilon$ is compact, $\beta_{\mathcal{N}} \geq \alpha^\theta_\varepsilon$, and $\beta_{\mathcal{N}}$ is lower semicontinuous. □

3.3.3. *Structure of $Z_G(F)$.* We have that $P_b^- = Z_G(\{b_t\})U_b^-$, see (2.13), where

$$U_b^- = \{u \in G : \lim_{t\to\infty} b_t u b_{-t} \to e\}, \tag{3.46}$$

and U_b^- is an abelian group. Moreover, $Z_G(\{b_t\}) = H_d Z_G(H_d)$.

Lemma 3.9. *$Z_G(F) = Z_G(H_d)(U_b^- \cap Z_G(F))$. Moreover,*

$$Z_G(F) \cap U_b^- = (Z_G(F) \cap \exp(\mathfrak{p}))(Z_G(F) \cap U_b^- \cap M_1).$$

Proof. We note that $F = \{a_t\}U$, $a_t = c_t b_t$ for all $t \in \mathbb{R}$. Let $t \geq 0$. Since U is the horospherical subgroup of H_d for c_t, for any finite dimensional representation of H_d, if a vector is fixed by U then it is contained in the sum of expanding and zero eigen subspaces of c_t; for example, see [**Sha96**, Lemma 5.2]. Therefore, if $g \in G$ is fixed by U and $a_t = c_t b_t$ under conjugation, then g is contained in the sum of contracting and zero eigen subspace for the action of b_t by conjugation on the space of $(n+1)\times(n+1)$-real matrices. Hence by definition, $g \in P_b^-$.

Now $\{b_t\} \subset Z_G(F)$, so $Z_G(F) = (Z_G(F) \cap Z_G(\{b_t\}))(Z_G(F) \cap U_b^-)$. Now $Z(H_d) \subset Z_G(F)$ and $H_d \cap Z_G(F) \subset Z(H_d)$. Therefore $Z_G(F) \cap Z_G(\{b_t\}) = Z_G(H_d)$. □

3.3.4. *Height function for $\Omega\mathcal{N}$ with $a_t u(I^d)$-contraction.* Let Ω be a bounded open neighborhood of 0 in

$$\mathfrak{p}_1 := \mathrm{Lie}(Z_G(F)) \cap \mathfrak{p} \subset \mathrm{Lie}(U_b^-).$$

For any $x \in X$, we define

$$\rho_{\beta_{\mathcal{N}},\Omega}(x) = \limsup_{t\to\infty} \frac{\log\big(\sup_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega)x)\big)}{t}, \tag{3.47}$$

which measures exponential growth rate of the $\beta_{\mathcal{N}}$ along the b_t-trajectory of Ωx.

Let $d\omega$ denote the Lebesgue integral on $\mathfrak{p}_1$ normalized such that Ω has volume 1. Let

$$0 < \delta < \frac{d}{d+1}\theta', \tag{3.48}$$

and $t_0 \geq 0$. For any $x \in X$, define

$$\widetilde{\beta}_\Omega(x) = \int_{t_0}^\infty \int_{\omega\in\Omega} e^{-\delta t} \beta_{\mathcal{N}}(b_t \exp(\omega)x)\mathrm{d}\omega\mathrm{d}t. \tag{3.49}$$

Proposition 3.10. *Let $x \in X$. Suppose that $\rho_{\beta_{\mathcal{N}},\Omega}(x) < \delta$ and $t_0 \geq 0$ is such that $\sup_{t\in[t_0,\infty)} \frac{\log\left(\sup_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega)x)\right)}{t} < \infty$. Then $\widetilde{\beta}_\Omega(x) < \infty$.*

Proof. Pick ρ such that $\rho_{\beta_{\mathcal{N}},\Omega}(x) < \rho < \delta$. Then, there exists $t_1 \geq t_0$ such that $\beta_{\mathcal{N}}(b_t \exp(\omega)x) \leq e^{\rho t}$ for all $\omega \in \Omega$ and $t \geq t_1$. Now,

$$\widetilde{\beta}_\Omega(x) = \int_{t_0}^{t_1} \int_{\omega\in\Omega} e^{-\delta t} \beta_{\mathcal{N}}(b_t \exp(\omega)x) \mathrm{d}\omega \mathrm{d}t + \int_{t_1}^{\infty} \int_{\omega\in\Omega} e^{-\delta t} \beta_{\mathcal{N}}(b_t \exp(\omega)x) \mathrm{d}\omega \mathrm{d}t.$$

By our assumption on t_0 the integral over $[t_0, t_1]$ is finite, and the second integral is bounded by $\int_{t_0}^{\infty} e^{-(\delta-\rho)t}\, \mathrm{d}t < \infty$. □

Proposition 3.11. *(1) $\widetilde{\beta}_\Omega$ is lower semicontinuous.*

(2) There exists $C > 0$ such that for any $t \geq 0$ there exists $B'(t) > 0$ such that

$$\int_{I^d} \widetilde{\beta}_\Omega(a_t u(s)x) \mathrm{d}s \leq C e^{-(\frac{d}{d+1}\theta' - \delta)t} \widetilde{\beta}_\Omega(x) + B'(t), \quad \forall x \in X. \tag{3.50}$$

(3) $\widetilde{\beta}_\Omega$ is Lipschitz with respect to the action of the semigroup $\{a_t\}_{t\geq 0} U$.

(4) If $x \in \exp(\Omega)\mathcal{N}$, then $\widetilde{\beta}_\Omega(x) = \infty$.

(5) For any $M > 0$, $\widetilde{\beta}_\Omega^{-1}([1, M])$ is compact.

We will prove each part of the proposition separately. In the proof, we will use the fact that $\Omega \subset Z_G(F)$ commutes with $a_t u(s)$ without mentioning it.

Proof of (1). Since $\beta_{\mathcal{N}}$ is lower semicontinuous by Proposition 3.8, for any sequence $x_i \to x$ in X, we have

$$\begin{aligned}
\widetilde{\beta}_\Omega(x) &= \int_{t_0}^{\infty} e^{-\delta t} \int_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega)x) \mathrm{d}\omega \mathrm{d}t \\
&\leq \int_{t_0}^{\infty} \int_{\omega\in\Omega} \liminf_{i\to\infty} e^{-\delta t} \beta_{\mathcal{N}}(b_t \exp(\omega)x_i) \mathrm{d}\omega \mathrm{d}t \\
&\leq \liminf_{i\to\infty} \int_{t_0}^{\infty} \int_{\omega\in\Omega} e^{-\delta t} \beta_{\mathcal{N}}(b_t \exp(\omega)x_i) \mathrm{d}\omega \mathrm{d}t \quad \text{(by Fatou's lemma)} \\
&= \liminf_{i\to\infty} \widetilde{\beta}_\Omega(x_i).
\end{aligned}$$

Hence $\widetilde{\beta}_\Omega$ is lower semicontinuous. □

Proof of (2). Put

$$\lambda = \frac{d}{d+1}\theta'.$$

We shall prove that, for every $\tau \geq 0$, there is $B'(\tau) > 0$ such that

$$\int_{I^d} \widetilde{\beta}_\Omega(a_\tau u(s)x)\, ds \leq E e^{-(\lambda-\delta)\tau} \widetilde{\beta}_\Omega(x) + B'(\tau),$$

where E is the constant in Proposition 3.8(4).

All integrands below are non-negative, so Tonelli's theorem justifies the changes in the order of integration. Let $x \in X$ and $\tau \geq 0$. By definition,

$$\int_{I^d} \widetilde{\beta}_\Omega(a_\tau u(s)x)\, ds = \int_{I^d} \int_{t_0}^{\infty} \int_\Omega e^{-\delta t} \beta_N(b_t \exp(\omega) a_\tau u(s)x)\; d\omega\, dt\, ds.$$

Since $\omega \in \mathfrak{p}_1 \subset \mathrm{Lie}(Z_G(F))$ and $a_\tau u(s) \in F$, we have

$$\exp(\omega) a_\tau u(s) = a_\tau u(s) \exp(\omega).$$

Moreover $a_\tau = b_\tau c_\tau$, and b_t commutes with $c_\tau u(s) \in H_d$. Therefore

$$\begin{aligned} b_t \exp(\omega) a_\tau u(s) &= b_t a_\tau u(s) \exp(\omega) \\ &= b_t b_\tau c_\tau u(s) \exp(\omega) \\ &= c_\tau u(s) b_{t+\tau} \exp(\omega). \end{aligned}$$

Hence

$$\int_{I^d} \widetilde{\beta}_\Omega(a_\tau u(s)x)\, ds = \int_{t_0}^{\infty} \int_\Omega e^{-\delta t} \left(\int_{I^d} \beta_N(c_\tau u(s) b_{t+\tau} \exp(\omega)x)\; ds \right) d\omega\, dt.$$

Applying Proposition 3.8(4) with

$$y = b_{t+\tau} \exp(\omega)x$$

gives

$$\int_{I^d} \beta_N(c_\tau u(s) b_{t+\tau} \exp(\omega)x)\; ds \leq E e^{-\lambda\tau} \beta_N(b_{t+\tau} \exp(\omega)x) + B_1(\tau).$$

Therefore,

$$\begin{aligned} \int_{I^d} \widetilde{\beta}_\Omega(a_\tau u(s)x)\, ds \leq E e^{-\lambda\tau} &\int_{t_0}^{\infty} \int_\Omega e^{-\delta t} \beta_N(b_{t+\tau} \exp(\omega)x)\; d\omega\, dt \\ &+ B_1(\tau) \int_{t_0}^{\infty} \int_\Omega e^{-\delta t}\, d\omega\, dt. \end{aligned}$$

Since $d\omega$ is normalized so that Ω has volume 1, the second term is

$$B_1(\tau) \int_{t_0}^{\infty} e^{-\delta t}\, dt = \delta^{-1} e^{-\delta t_0} B_1(\tau).$$

For the first term, change variables $t' = t + \tau$. Then

$$\begin{aligned} \int_{t_0}^{\infty} \int_\Omega e^{-\delta t} \beta_N(b_{t+\tau} \exp(\omega)x)\; d\omega\, dt &= e^{\delta\tau} \int_{t_0+\tau}^{\infty} \int_\Omega e^{-\delta t'} \beta_N(b_{t'} \exp(\omega)x)\; d\omega\, dt' \\ &\leq e^{\delta\tau} \widetilde{\beta}_\Omega(x). \end{aligned}$$

Combining the last two estimates, we obtain

$$\int_{I^d} \widetilde{\beta}_\Omega(a_\tau u(s)x)\, ds \leq E e^{-(\lambda-\delta)\tau} \widetilde{\beta}_\Omega(x) + \delta^{-1} e^{-\delta t_0} B_1(\tau).$$

Thus the desired estimate holds with

$$C = E, \qquad B'(\tau) = \delta^{-1} e^{-\delta t_0} B_1(\tau).$$

If one wants $B'(\tau) > 0$ strictly, replace this by $B'(\tau) = 1 + \delta^{-1} e^{-\delta t_0} B_1(\tau)$.

□

Proof of (3). Let $T \geq 0$ and $\mathcal{C}$ be a compact subset of U. By (3) of Proposition 3.8, there exists $C_1 \geq 1$ such that

$$\beta_{\mathcal{N}}(c_t u x) \leq C_1 \beta_{\mathcal{N}}(x),\ \forall t \in [0,T],\ u \in \mathcal{C},\ \forall x \in X.$$

Let $t_1 \in [0,T]$ and $u \in \mathcal{C}$, and $h = a_{t_1} u \in F$. Then,

$$\begin{aligned}
\widetilde{\beta}_\Omega(hx) &= \int_{\omega\in\Omega}\int_{t_0}^{\infty} e^{-\delta t}\beta_{\mathcal{N}}(b_t \exp(\omega) hx)\mathrm{d}t\mathrm{d}\omega \\
&= \int_{\omega\in\Omega}\int_{t_0}^{\infty} e^{-\delta t}\beta_{\mathcal{N}}(h b_t \exp(\omega) x)\mathrm{d}t\mathrm{d}\omega, \text{ as } \{b_t\}\exp(\Omega) \subset Z_G(F), \\
&= \int_{\omega\in\Omega}\int_{t_0}^{\infty} e^{-\delta t}\beta_{\mathcal{N}}(c_{t_1} u b_{t+t_1} \exp(\omega) x)\mathrm{d}t\mathrm{d}\omega, \text{ as } a_{t_1} = c_{t_1} b_{t_1}, \\
&\leq C_1 \int_{\omega\in\Omega}\int_{t_0}^{\infty} e^{-\delta t}\beta_{\mathcal{N}}(b_{t+t_1} \exp(\omega) x)\mathrm{d}t\mathrm{d}\omega \\
&= C_1 e^{\delta t_1} \int_{t_0+t_1}^{\infty}\int_{\omega\in\Omega} e^{-\delta t'}\beta_{\mathcal{N}}(b_{t'} \exp(\omega) x)\mathrm{d}\omega\mathrm{d}t' \\
&\leq C_1 e^{\delta T}\widetilde{\beta}_\Omega(x).
\end{aligned}$$

□

Proof of (4). Let $x \in \exp(\Omega)\mathcal{N}$. Let $\omega_1 \in \Omega$ such that $x \in \exp(\omega_1)\mathcal{N}$.

For any $t \geq 0$, and $w \in \mathfrak{p}$, in view of (2.2),

$$b_t \exp(w)\exp(-\omega_1)x = \exp(\mathrm{Ad}(b_t)(w)) b_t \exp(-\omega_1)x = \exp(e^{-\frac{1}{d+1}t} w) b_t \exp(-\omega_1)x.$$

Now, by (3.8), $\{b_t\} \subset M_1$ and $Z_G(H_d) \subset Z_G(\{b_t\})$. Hence $\mathcal{N}$ is $\{b_t\}$-invariant. So, $b_t \exp(-\omega_1)x \in \mathcal{N}$, and $e^{-\frac{1}{d+1}t} w \in \mathfrak{p}$. Therefore, by Remark 3.5, we have

$$\beta^{\theta'}(b_t \exp(w)\exp(-\omega_1)x) \geq \left\| e^{-\frac{1}{d+1}t} w \right\|^{-d\theta'} \geq e^{\frac{d\theta'}{d+1}t} \|w\|^{-d\theta'}. \tag{3.51}$$

Therefore,

$$\begin{aligned}
\int_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega) x)\mathrm{d}\omega &\geq \int_{\omega\in\Omega} \beta^{\theta'}(b_t \exp(\omega) x)\mathrm{d}\omega \\
&\geq \int_{\omega\in\Omega} \beta^{\theta'}(b_t \exp(\omega+\omega_1)\exp(-\omega_1) x)\mathrm{d}\omega \\
&\geq C^{-d\theta'} e^{\frac{d\theta'}{d+1}t}, \text{ by (3.51)},
\end{aligned}$$

where $C = \sup\{\|\omega + \omega_1\| : \omega \in \Omega\} < \infty$. Therefore,

$$\begin{aligned}
\widetilde{\beta}_\Omega(x) &\geq \int_{t_0}^{\infty} e^{-\delta t}\left(\int_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega) x)\mathrm{d}\omega\right)\mathrm{d}t \\
&\geq C^{-d\theta'} \int_{t_0}^{\infty} e^{(\frac{d\theta'}{d+1}-\delta)t}\mathrm{d}t = \infty,
\end{aligned}$$

as by (3.48), $\delta < \frac{d\theta'}{d+1}$. □

Proof of (5). Let $M \geq 0$. Then $\widetilde{\beta}_\Omega^{-1}([0,M])$ is closed, because $\widetilde{\beta}_\Omega$ is lower semicontinuous and nonnegative. For pre-compactness, note that for any $y \in X$:

$$\widetilde{\beta}_\Omega(y) \geq \int_{t_0}^{t_0+1} \int_{\omega\in\Omega} e^{-\delta t}\beta_{\mathcal{N}}(b_t \exp(\omega)y)\mathrm{d}\omega\mathrm{d}t \geq e^{-\delta} \min_{t_0\leq t\leq t_0+1} \min_{\omega\in\Omega} \beta_{\mathcal{N}}(b_t \exp(\omega)y).$$

By (5) of Proposition 3.8, $\beta_{\mathcal{N}}^{-1}([0,e^\delta M])$ is a compact subset of X, and $\{b_t : t \in [t_0,t_0+1]\}$ and $\overline{\exp(\Omega)}$ are compact subsets of G. So, the set

$$\{y \in X : \min_{t_0\leq t\leq t_0+1} \min_{\omega\in\Omega} e^{-\delta}\beta_{\mathcal{N}}(b_t \exp(\omega)y) \leq M\}$$

is compact, and contains $\widetilde{\beta}_\Omega^{-1}([0,M])$. □

4. Obstruction to avoiding a singular submanifold

Let $x \in X$. For any measurable subset K of X and $T > 0$, define

$$\mathcal{C}_K^T(s) = \frac{1}{T}\int_0^T \mathbf{1}_K(a_t u(s)x)\mathrm{d}t,$$

where $\mathbf{1}_K$ is the characteristic function of K.

Let the notation be as in Section 3.3. Let l_1 be as in Lemma 3.5 and we fix $l \geq l_1$. Let O be a bounded open neighborhood of 0 in $\mathfrak{s} \subset \mathrm{Lie}(Z_G(H_d))$ in Lemma 3.6. Let Ω be any bounded neighborhood of 0 in $\mathfrak{p}_1 = \mathrm{Lie}(Z_G(F)) \cap \mathfrak{p}$. Then $\exp(\Omega)$ is a relatively compact subset of P_b^-.

In this section, for $g \in G$ and $v \in \mathfrak{g}$, sometimes we simply write gv instead of $(\mathrm{Ad}\, g)v$ to shorten the expressions.

Proposition 4.1. *Suppose $\rho_{\beta_{\mathcal{N}},\Omega}(x) < \frac{d}{d+1}\theta'$. Then, for any $0 < \epsilon < 1$, there exists a compact set K in X with $K \cap \exp(\Omega)\mathcal{N} = \emptyset$ and real numbers $0 < a < 1, C \geq 1$ such that*

$$\left|\{s \in I^d : \mathcal{C}_K^T(s) \leq 1-\epsilon\}\right| \leq Ca^T. \tag{4.1}$$

Proof. Choose δ in (3.48) such that $\rho_{\beta_{\mathcal{N}},\Omega}(x) < \delta < \frac{d}{d+1}\theta'$. We can pick $t_0 \geq 0$ be such that $\sup_{t\geq t_0} \frac{\log\sup_{\omega\in\Omega}\beta_{\mathcal{N}}(b_t\exp(\omega)x)}{t} < \infty$. For these choices of δ and t_0, we define $\widetilde{\beta}_\Omega$ as in (3.49). By Proposition 3.10, we have $\widetilde{\beta}_\Omega(x) < +\infty$.

And by (4) of Proposition 3.11 we have $\widetilde{\beta}_\Omega(y) = \infty$ for all $y \in \exp(\Omega)\mathcal{N}$. Then the proof of the desired conclusion is analogous to the proof of Proposition 2.7, where we replace $\widetilde{\alpha}$ by $\widetilde{\beta}_\Omega$ and use properties of $\widetilde{\beta}_\Omega$ noted in Proposition 3.11. □

Corollary 4.2. *Let $x \in X$. Let $z \in Z_G(F)$ and suppose that $\rho_{\beta_{\mathcal{N}},\Omega}(zx) < \frac{d}{d+1}\theta'$. Then given a sequence $T_i \to \infty$, for almost all $s \in I^d$, any subsequence of the sequence $\left\{\frac{1}{T_i}\int_0^{T_i} a_t u(s)\delta_x \mathrm{d}t\right\}_{i\in\mathbb{N}}$ has a limit point, say ν in $\mathcal{P}(X)$, such that $\nu(z\exp(\Omega)\mathcal{N}) = 0$, where $\mathcal{P}(X)$ denotes the space of probability measures on X endowed with the weak-$*$ topology.*

Proof. It follows from Proposition 4.1 and Borel-Cantelli lemma that there exists a null set $E \subset I^d$, such that for any $s \in I^d \backslash E$, and any subsequence of $\left\{\frac{1}{T_i}\int_0^{T_i} a_t u(s)\delta_{zx}\mathrm{d}t\right\}_{i\in\mathbb{N}}$ has a limit point, say ν, in $\mathcal{P}(X)$ such that $\nu(\exp(\Omega)\mathcal{N}) = 0$. The proof is analogous

to the deduction of Theorem 1.1 from Proposition 2.7. Now for any $s \in I^d \setminus E$, since z commutes with $\{a_t\}u(s)$ and $\delta_{zx} = z_*\delta_x$, we have

$$\frac{1}{T_i}\int_0^{T_i} a_t u(s)\delta_x \mathrm{d}t = (z^{-1})_* \left(\frac{1}{T_i}\int_0^{T_i} a_t u(s)\delta_{zx}\mathrm{d}t\right), \forall i \in \mathbb{N}.$$

Therefore any subsequence of this sequence has a limit point of the form $(z^{-1})_*\nu$, where $\nu \in \mathcal{P}(X)$ such that $\nu(\exp(\Omega)\mathcal{N}) = 0$. Therefore $(z^{-1})_*\nu(z\exp(\Omega)\mathcal{N}) = \nu(\exp(\Omega)\mathcal{N}) = 0$. □

Remark 4.1. As noted in Remark 2.3, one can also use the techniques developed in [**KKLM17, Kha20**] to prove Corollary 4.2 using Proposition 3.11.

Corollary 4.3. *Let $x \in X$ and suppose that $\rho_{\beta_{\mathcal{N}},\Omega}(zx) < \frac{d}{d+1}\theta'$ for all $z \in Z_G(F)$. Then given a sequence $T_i \to \infty$, for almost all $s \in I^d$, any subsequence of the sequence $\left\{\frac{1}{T_i}\int_0^{T_i} a_t u(s)\delta_x \mathrm{d}t\right\}_{i\in\mathbb{N}}$ has a limit point, say ν in $\mathcal{P}(X)$, such that $\nu(N(F,M)x_0) = 0$.*

Proof. We have that $Z_G(F)M_1 = \cup_{z\in Z_G(F)} z\exp(\Omega)\exp(O)M_1$. Due to Lemma 3.9, $\mathrm{Lie}(Z_G(F)) = \mathfrak{p}_1 \oplus \mathfrak{s} \oplus \mathrm{Lie}(Z_G(F)\cap M_1)$. Therefore, $z\exp(\Omega)\exp(O)M_1$ is an open subset of $Z_G(F)M_1$ for each $z \in Z_G(F)$. By second countability, we pick countably many $z_i \in Z_G(F)$ such that $Z_G(F)M_1 = \cup_{i=1}^{\infty} z_i\exp(\Omega)\exp(O)M_1$.

We picked any $p \in N_G(M_1)$, and we set $y = pg_1$, $Y = M_1 y$, and $\mathcal{N} = \exp(O)Y$. So we have $Z_G(F)Y = \cup_{i=1}^{\infty} z_i\exp(\Omega)\mathcal{N}$. Due to Corollary 4.2, we deduce that for almost all $s \in I^d$, $\nu(Z_G(F)Y) = 0$ for any of the limiting measures as in the proposition. We note that $Z_G(F)Y = Z_G(F)M_1pg_1\Gamma/\Gamma \subset X$.

Further, by Proposition 3.3 we have $N(F,M) = \cup_{i=1}^{s_1} Z_G(F)M_1p_ig_1$, where $p_i \in N_G(M_1)$. Therefore, the conclusion of the proposition follows from the above observations. □

4.1. **Consequence of large exponent.** Let $\mathbf{G} = \mathrm{SL}_{n+1}$, and let α_i be the root $\mathrm{diag}(t_j) \mapsto t_i t_{i+1}^{-1}$ for $1 \le i \le n$. Then $\Delta = \{-\alpha_i\}_{1\le i\le n}$ form a set of simple roots. Let $\mathbf{P}_i$ be the maximal parabolic subgroup of $\mathbf{G}$ associated to $\Delta \setminus \{-\alpha_i\}$ for $1 \le i \le n$, and let $P_i = \mathbf{P}_i(\mathbb{R})$. In block matrix form,

$$P_i = \left\{ \begin{pmatrix} A_{i\times i} & 0_{i\times(n+1-i)} \\ C_{(n+1-i)\times i} & D_{(n+1-i)\times(n+1-i)} \end{pmatrix} \right\}. \tag{4.2}$$

Recall that $\mathcal{A}$ is the d-dimensional affine subspace of $U_a^+ \cong \mathbb{R}^n$ parametrized by A. We suppose that $\omega(A) < n$. We fix any $\theta \in (0, \frac{d}{d+1})$. Then, by [**SY24b**, Proposition 4.1], we have $0 \le \rho_{\varepsilon,\theta}(x_A) < \theta$. We choose θ' as in (3.9) and $\sigma > 1$ as in (3.23), and then ρ_1 such that

$$\rho_{\varepsilon,\theta}(A) < \rho_1 < \frac{d\theta'}{d+1} < \sigma\frac{d\theta'}{d+1} < \theta < \frac{d}{d+1}. \tag{4.3}$$

Proposition 4.4. *Suppose that $\rho_{\beta_{\mathcal{N}},\Omega}(zx_A) \ge \frac{d}{d+1}\theta'$ for some bounded open neighborhood Ω of 0 in $\mathfrak{p}_1$ and for some $z \in Z_G(F)$. Then $\mathcal{A}$ satisfies possibility (2) of Theorem 1.2.*

Proof. By (3.34) and (3.47), there exists $t_i \to \infty$, $\omega_i \in \exp(\Omega)z \subset Z_G(F)$, and $\xi_i \uparrow \frac{d}{d+1}$ such that

$$\beta_{\mathcal{N}}(b_{t_i}\omega_i x_A) = \alpha_\varepsilon^\theta(b_{t_i}\omega_i x_A) + \beta^{\theta'}(b_{t_i}\omega_i x_A) > e^{\xi_i\theta' t_i}.$$

Therefore, by Proposition 2.2, $\rho_{\varepsilon,\theta,\exp(\Omega)z}(x_A) = \rho_{\varepsilon,\theta}(x_A) < \rho_1$. Therefore, after passing to a subsequence, since $\rho_1 < \xi_i\theta'$,

$$\alpha_\varepsilon^\theta(b_{t_i}\omega_i x_A) \le e^{\rho_1 t_i} < e^{\xi_i\theta' t_i}. \tag{4.4}$$

Hence, by (4.3) we deduce that

$$\beta^{\theta'}(b_{t_i}\omega_i x_A) > e^{\xi_i\theta' t_i}. \tag{4.5}$$

Now, recalling (3.16), by (4.4) we have

$$r_{b_{t_i}\omega_i x_A}^{-\theta'} = (\varepsilon_0 \alpha_\varepsilon^{\sigma\frac{d}{d+1}}(b_{t_i}\omega_i x_A))^{\theta'} \le \alpha_\varepsilon^\theta(b_{t_i}\omega_i x_A)^{\frac{\sigma d\theta'}{(d+1)\theta}} < e^{\frac{\sigma\rho_1}{\theta}\frac{d\theta'}{d+1}t_i}. \tag{4.6}$$

By (4.3), we have $\sigma\rho_1 < \sigma\frac{d\theta'}{d+1} < \theta$, so $\frac{\sigma\rho_1}{\theta} < 1$. Therefore, by (4.5) and (4.6), since $\xi_i \uparrow \frac{d}{d+1}$, after passing to a subsequence, we get

$$\beta^{\theta'}(b_{t_i}\omega_i x_A) > r_{b_{t_i}\omega_i x_A}^{-\theta'}, \ \forall i.$$

Therefore, by the definition (3.25) of $\beta^{\theta'}$, for every sufficiently large i, there exists $(v_i, w_i) \in \mathfrak{q} \times \mathfrak{p}$ such that

$$\exp(w_i)\exp(v_i)b_{t_i}\omega_i x_A \in \mathcal{N} \text{ and } \beta^{\theta'}(b_{t_i}\omega_i x_A) \le \min\{\|v_i\|^{-\theta'}, \|w_i\|^{-d\theta'}\}, \tag{4.7}$$

Recall that by (3.10) and (3.24),

$$\mathcal{N} = \exp(O)Y = \exp(O)pg_1Mx_0 = \exp(O)M_1pg_1x_0 \text{ and } x_A = g_Ax_0.$$

Hence there exists $\gamma_i \in \Gamma = \mathrm{Stab}_G(x_0)$ such that

$$b_{t_i}\omega_i g_A\gamma_i \in \exp(-v_i)\exp(-w_i)\exp(O)M_1pg_1. \tag{4.8}$$

Hence, since $\{b_t\}$ commutes with elements of $Z_G(H_d)$ and $\{b_t\} \subset M_1$, we have

$$\omega_i g_A\gamma_i \in \exp(-\mathrm{Ad}(b_{-t_i})v_i)\exp(-\mathrm{Ad}(b_{-t_i})w_i)\exp(O)M_1pg_1. \tag{4.9}$$

By (4.5) and (4.7),

$$\|v_i\| \ll e^{-\xi_i t_i} \text{ and } \|w_i\| \ll e^{-\frac{\xi_i}{d}t_i}.$$

And, since $v_i \in \mathfrak{q}$ and $w_i \in \mathfrak{p}$, in view of (2.2) and (3.12), we get

$$\begin{aligned} \|\mathrm{Ad}(b_{-t_i})v_i\| &\ll e^{-\frac{1}{d+1}t_i}\|v_i\| \ll e^{-(1-\epsilon_i)t_i} \\ \|\mathrm{Ad}(b_{-t_i})w_i\| &\ll e^{\frac{1}{d+1}t_i}\|w_i\| \ll e^{(\epsilon_i/d)t_i}, \end{aligned} \tag{4.10}$$

where $\epsilon_i = \frac{d}{d+1} - \xi_i \downarrow 0$. Let $Y_i = \mathrm{Ad}(g_A^{-1}\omega_i^{-1}b_{-t_i})(v_i)$ for each i, and it follows that

$$\|Y_i\| \ll e^{-(1-\epsilon_i)t_i}. \tag{4.11}$$

We have $\omega_i \in Z_G(F) \subset P_{d+1}$ and $\exp(O) \subset Z_G(H_d) \subset P_{d+1}$. Moreover due to the initial $(r \times r)$-block form of M_1 and that $r \ge d+1$, $P_{d+1}M_1 = P_{d+1}P_r$. Therefore,

$$g_A\exp(Y_i) \in \omega_i^{-1}\exp(-\mathrm{Ad}(b_{-t_i})w_i)\exp(O)M_1pg_1\gamma_i^{-1} \subset P_{d+1}P_rpg_1\gamma_i^{-1}, \quad \forall i. \tag{4.12}$$

Moreover, we recall that $M_1 p g_1 = p g_1 M$ and $M = \mathrm{Stab}_G(v_0)$. So, for each i, we have

$$l_i \in g_A^{-1}\omega_i^{-1}\exp(-\mathrm{Ad}(b_{-t_i})v_i)\exp(-\mathrm{Ad}(b_{-t_i})w_i)\exp(O)pg_1 \in G \tag{4.13}$$

and $m_i \in M$ such that

$$\gamma_i = l_i m_i \text{ and } \|l_i\| \ll e^{(\epsilon_i/d)t_i}. \tag{4.14}$$

In view of reduction theory for the lattice $M \cap \Gamma$ acting on the left on M, by passing to a subsequence, there exists a generalized Siegel domain $\mathfrak{S}$ in M and $\lambda_i \in M \cap \Gamma$ such that $m_i^{-1} \in \lambda_i \mathfrak{S}$ for all i. Then, $\gamma_i\lambda_i \in l_i\mathfrak{S}^{-1}$. Then, by [**DO22**, Theorem 1.1] there exist $A_1, C_1 > 0$ such that

$$\|\gamma_i\lambda_i\| \le C_1\|l_i\|^{A_1}. \tag{4.15}$$

To see how this follows from the quoted result, consider the action of $m \in M$ on the vector space $\mathrm{Mat}(n+1,\mathbb{R})$ by right multiplication by m^{-1}. And note that $\Gamma \subset \mathrm{Mat}(n+1,\mathbb{Z})$ and the action of l_i on $\mathrm{Mat}(n+1,\mathbb{R})$ by left multiplication commutes with the above right action of M.

Noting that

$$M_1 p g_1 \lambda_i = p g_1 M \lambda_i = p g_1 M = M_1 p g_1,$$

replacing γ_i by $\gamma_i\lambda_i$ in (4.8), without loss of generality, we may assume that

$$\|\gamma_i\| \ll e^{\epsilon_i' t_i}, \text{ where } \epsilon_i' \to 0 \text{ as } i \to \infty. \tag{4.16}$$

We recall the notation from Definition 1.1. Let $\overline{\mathcal{A}}$ denote the projective closure of $\mathcal{A}$ in $\mathbb{P}^n$. Then it follows that $\overline{\mathcal{A}}$ is parametrized by $P_{d+1}g_A \in P_{d+1}\backslash G \cong \mathrm{Gr}(d+1, n+1)$. Let $\mathcal{T}_i$ denote the $(r-1)$-dimensional subspace of $\mathbb{P}^n$ parametrized by $P_r p g_1 \gamma_i^{-1}$. Since right multiplication by γ_i^{-1} does not change the field of definition, we have $\mathcal{T}_i$ is defined over $\sigma(\mathbb{F})$, where $\sigma : \mathbb{F} \hookrightarrow \mathbb{R}$ is a real embedding determined by p. Indeed, in view of the choice of $p = p_j$ for some $1 \le j \le s_1$ as in Proposition 3.3 and by Remark 3.1 we have that the first r-rows of $p_j g_1$ have entries in $\sigma_j(\mathbb{F})$. By replacing $\mathbb{F}$ with $\sigma(\mathbb{F})$, without loss of generality we may assume that $\mathcal{T}_i$ is defined over $\mathbb{F}$. Moreover, there exist constants $C > 0$ and $A > 0$ such that

$$H_{\mathcal{L}}(\mathcal{T}_i) \le C\|\gamma_i\|^A \ll e^{A\epsilon_i' t_i}, \tag{4.17}$$

which can be verified by considering the Plücker embedding $\mathrm{Gr}(r, n+1) \hookrightarrow \mathbb{P}(\wedge^r V)$ and noting that $\mathcal{L}$ is the pull-back of $\mathcal{O}(1)$.

By (4.12), we have $P_1 P_{d+1} g_A \exp(Y_i) \subset P_1 P_r p g_1 \gamma_i^{-1}$, or in other words, we have the inclusion $\overline{\mathcal{A}}\exp(Y_i) \subset \mathcal{T}_i$ in $\mathbb{P}^n \cong P_1\backslash G$. Therefore, by (4.11),

$$d(\overline{\mathcal{A}}, \mathcal{T}_i) \le d(\overline{\mathcal{A}}, \overline{\mathcal{A}}\exp(Y_i)) \ll \|Y_i\| \ll e^{-(1-\epsilon_i)t_i}. \tag{4.18}$$

Now $\epsilon_i \to 0$ and $\epsilon_i' \to 0$ as $i \to \infty$. Therefore, by (4.17), given any $\kappa > 1$,

$$d(\overline{\mathcal{A}}, \mathcal{T}_i) \le H_{\mathcal{L}}(\mathcal{T}_i)^{-\kappa}, \quad \forall i \gg 1.$$

Hence by Definition 1.1 we conclude that $\overline{\mathcal{A}}$ is $(\mathbb{F}, r-1)$-Liouville, and thus $\mathcal{A}$ is $(\mathbb{F}, r-1)$-Liouville.

□

Though not needed for our proof, we note the following.

Corollary 4.5. *Suppose* $\rho_{\beta_{\mathcal{N}},\Omega}(zx_A) > \frac{d}{d+1}\theta'$ *in Proposition 4.4. Then* $\mathcal{A}$ *is contained in a* $(r-1)$*-dimensional affine subspace defined over* $\mathbb{F}$*.*

Proof. We can choose $\frac{\rho_{\beta_{\mathcal{N}},\Omega}(zx_A)}{\theta'} > \xi_i = \xi > \frac{d}{d+1}$ and we can obtain (4.9) and (4.10) with $\varepsilon_i = \frac{d}{d+1} - \xi < 0$. Therefore, after passing to a subsequence we get $\gamma_i = \gamma_1$ for all i. Therefore $\mathcal{T}_i = \mathcal{T}_1$ for all i. Since $Y_i \to 0$, we get $\overline{\mathcal{A}} \subset \mathcal{T}_1$. □

4.2. **Proof of the main theorem.**

Proof of Theorem 1.2. Under the assumption $\omega(A) < n$, suppose possibility (2) of Theorem 1.2 does not occur, then by Proposition 4.4, for every $z \in Z_G(F)$ and any relatively compact open Ω, we have $\rho_{\beta_{\mathcal{N}},\Omega}(zx_A) < \frac{d}{d+1}\theta'$. Then due to (3.7), the Corollary 4.3 completes the proof of Proposition 3.1. So by the analysis in the beginning of Section 3.1 we can obtain (2.7), which is equivalent to (1) of Theorem 1.2. □

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