COUNTING VISIBLE CIRCLES ON THE SPHERE AND KLEINIAN GROUPS

HEE OH AND NIMISH SHAH

Abstract. For a circle packing $\mathcal{P}$ on the sphere invariant under a non-elementary Kleinian group satisfying certain finiteness conditions, we compute the asymptotic of the number of circles in $\mathcal{P}$ of spherical curvature at most T which are contained in any given region.

1. Introduction

In the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ with the Riemannian metric induced from $\mathbb{R}^3$, the distance between two points is simply the angle between the rays connecting them to the origin o. Let $\mathcal{P}$ be a circle packing on the sphere $\mathbb{S}^2$, i.e., a union of circles which may intersect with each other.

In the beautiful book *Indra's pearls*, Mumford, Series and Wright ask the question (see [13, 5.4 in P. 155])

How many visible circles are there?

The *visual size* of a circle C in $\mathbb{S}^2$ can be measured by its spherical radius $0 < \theta(C) \le \pi/2$, that is, the half of the visual angle of C from the origin $o = (0, 0, 0)$. We label the circles by their spherical curvatures given by

$$\mathrm{Curv}_S(C) := \cot\theta(C).$$

We suppose that $\mathcal{P}$ is locally finite in the sense that for any $T > 1$, there are only finitely many circles in $\mathcal{P}$ of spherical curvature at most T. We then set for any subset $E \subset \mathbb{S}^2$,

$$N_T(\mathcal{P}, E) = \{C \in \mathcal{P} : C \cap E \neq \emptyset,\ \ \mathrm{Curv}_S(C) < T\} < \infty.$$

In order to present our main result on the asymptotic for $N_T(\mathcal{P}, E)$, we consider the Poincare ball model $\mathbb{B} = \{x_1^2 + x_2^2 + x_3^2 < 1\}$ of the hyperbolic 3-space with the metric given by $\frac{2\sqrt{dx_1^2+dx_2^2+dx_3^2}}{1-(x_1^2+x_2^2+x_3^2)}$. The geometric boundary of $\mathbb{B}$ naturally identifies with $\mathbb{S}^2$.

Let G denote the group of orientation preserving isometries of $\mathbb{B}$. Let $\Gamma < G$ be a non-elementary (=non virtually-abelian) Kleinian group.

We denote by $\Lambda(\Gamma) \subset \mathbb{S}^2$ the limit set of Γ, that is, the set of accumulation points of an orbit of Γ in $\mathbb{B} \cup \mathbb{S}^2$. Denote by δ_Γ the cirtical exponent of Γ and by $\{\nu_x : x \in \mathbb{B}\}$ a Γ-invariant conformal density of dimension δ_Γ on $\Lambda(\Gamma)$, which exists by the work of Patterson [17] and Sullivan [20]. We denote

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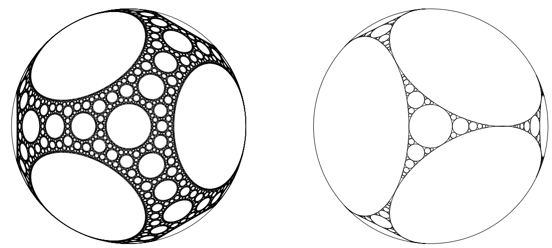

FIGURE 1. Sierpinski curve and Apollonian gasket (by C. McMullen)

by $m_\Gamma^{\rm BMS}$ the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{B})$ associated to the density $\{\nu_x\}$ (Def. 2.2).

For a vector $u \in \mathrm{T}^1(\mathbb{B})$, denote by $u^+ \in \mathbb{S}^2$ the forward end point of the geodesic determined by u, and by $\pi(u) \in \mathbb{B}$ the basepoint of u. For $x_1, x_2 \in \mathbb{B}$ and $\xi \in \mathbb{S}^2$, $\beta_\xi(x_1, x_2)$ denotes the signed distance between horospheres based at ξ and passing through x_1 and x_2.

Definition 1.1 (The Γ-skinning size of $\mathcal{P}$)**.** For a circle packing $\mathcal{P}$ on $\mathbb{S}^2$ invariant under Γ, we define $0 \le \mathrm{sk}_\Gamma(\mathcal{P}) \le \infty$ as follows:

$$\mathrm{sk}_\Gamma(\mathcal{P}) := \sum_{i\in I} \int_{s\in \mathrm{Stab}_\Gamma(C_i^\dagger)\backslash C_i^\dagger} e^{\delta_\Gamma \beta_{s^+}(x,\pi(s))} d\nu_x(s^+)$$

where $x \in \mathbb{B}$, $\{C_i : i \in I\}$ is a set of representatives of Γ-orbits in $\mathcal{P}$ and $C_i^\dagger \subset \mathrm{T}^1(\mathbb{B})$ is the set of unit normal vectors to the convex hull of C_i.

By the conformal property of $\{\nu_x\}$, the definition of $\mathrm{sk}_\Gamma(\mathcal{P})$ is independent of the choice of x and the choice of representatives $\{C_i\}$.

Theorem 1.2. *Let Γ be a non-elementary Kleinian group with $|m_\Gamma^{\rm BMS}| < \infty$. Let $\mathcal{P}$ be a locally finite Γ-invariant circle packing on the sphere $\mathbb{S}^2$ with finitely many Γ-orbits. Suppose that $\mathrm{sk}_\Gamma(\mathcal{P}) < \infty$. Then for any Borel subset $E \subset \mathbb{S}^2$ with $\nu_o(\partial(E)) = 0$,*

$$N_T(\mathcal{P}, E) \sim \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\rm BMS}|} \cdot \nu_o(E) \cdot (2T)^{\delta_\Gamma} \quad \text{as } T \to \infty$$

where $o = (0, 0, 0)$. If $\mathcal{P}$ is infinite, $\mathrm{sk}_\Gamma(\mathcal{P}) > 0$.

Remark 1.3. Under the assumption of $|m_\Gamma^{\rm BMS}| < \infty$, ν_o is atom-free by [18, Sec.1.5], and hence the above theorem works for any Borel subset E whose boundary intersecting $\Lambda(\Gamma)$ in finitely many points. If Γ is Zariski dense in G, then any proper real subvariety of $\mathbb{S}^2$ has zero ν_o-measure [6, Cor. 1.4] and hence Theorem 1.2 holds for any Borel subset of $\mathbb{S}^2$ whose boundary is contained in a countable union of real algebraic curves.

A Kleinian group Γ is called *geometrically finite* if Γ admits a finite sided fundamental domain in $\mathbb{B}$. For such, it is known that $|m_\Gamma^{\mathrm{BMS}}| < \infty$ [21] and δ_Γ is equal to the Hausdorff dimension of $\Lambda(\Gamma)$[20].

Definition 1.4. *A union of infinitely many pairwise tangent circles, with radii both going to* 0 *and* ∞*, glued at a tangent point will be called a bouquet.*

Theorem 1.5. [15] *Let Γ be a non-elementary geometrically finite Kleinian group and $\mathcal{P}$ be a locally finite Γ-invariant circle packing on the sphere $\mathbb{S}^2$ with finitely many Γ-orbits. In the case of $\delta_\Gamma \le 1$, we further assume that $\mathcal{P}$ does not contain the bouquet of circles glued at a parabolic fixed point of Γ. Then*

$$\mathrm{sk}_\Gamma(\mathcal{P}) < \infty.$$

Example 1.6.
(1) If X is a finite volume hyperbolic 3 manifold with totally geodesic boundary, its fundamental group $\Gamma := \pi_1(X)$ is geometrically finite and X is homeomorphic to $\Gamma\backslash\mathbb{B} \cup \Omega(\Gamma)$ [8]. The universal cover $\tilde{X}$ developed in $\mathbb{B}$ has geodesic boundary components which are Euclidean hemispheres normal to $\mathbb{S}^2$. Then $\Omega(\Gamma)$ is the union of a countably many disjoint open disks corresponding to the geodesic boundary components of $\tilde{X}$. The Ahlfors finiteness theorem [1] implies that the circle packing $\mathcal{P}$ on $\mathbb{S}^2$ consisting of the geodesic boundary components of $\tilde{X}$ is locally finite and has finitely many Γ-orbits. Hence by (4) of Theorem 1.5, it satisfies $\mathrm{sk}_\Gamma(\mathcal{P}) < \infty$.
(2) Starting with four mutually tangent circles on the sphere $\mathbb{S}^2$, one can inscribe into each of the curvilinear triangle a unique circle by an old theorem of Apollonius of Perga (c. BC 200). Continuing to inscribe the circles this way, one obtains an Apollonian circle packing on $\mathbb{S}^2$ (see Fig. 1). Apollonian circle packings are examples of circle packing obtained in the way described in (1) (cf. [5] and [10].). In the case when $\pi_1(X)$ is convex co-compact, then no disks in $\Omega(\Gamma)$ are tangent to each other and $\Lambda(\Gamma)$ is known to be homeomorphic to a Sierpinski curve [4] (see Fig. 1).
(3) Take $k \ge 1$ pairs of mutually disjoint closed disks $\{(D_i, D_i') : 1 \le i \le k\}$ in $\mathbb{S}^2$ and choose $\gamma_i \in G$ which maps the interior of D_i to the exterior of D_i' and vice versa. The group, say, Γ, generated by $\{\gamma_i\}$ is called a Schottky group of genus k (cf. [11, Sec. 2.7]). The Γ-orbit of the disks nest down onto the limit set $\Lambda(\Gamma)$ which is totally disconnected. If we set $\mathcal{P} := \cup_{1\le i\le k}\Gamma(C_i) \cup \Gamma(C_i')$ where C_i and C_i' are the boundaries of D_i and D_i' respectively, then $\mathcal{P}$ is locally finite, as the nesting disks will become smaller and smaller (cf. [13, 4.5]), and called Schottky dance (see [13, Fig. 4.11]). The common exterior of hemispheres above the initial disks D_i and D_i' is a fundamental domain for Γ in $\mathbb{B}$ and hence Γ is geometrically finite. Since $\mathcal{P}$ consists of disjoint circles, Theorem 1.5 applies.

Since for $o = (0,0,0)$, $\sin\theta(C) = \frac{1}{\cosh d(\hat{C},o)}$ for the convex hull $\hat{C}$ of C (cf. [22, P.24]), we deduce

$$\mathrm{Curv}_S(C) = \sinh d(\hat{C},o).$$

Hence Theorem 1.2 follows from the following:

Theorem 1.7. *Keeping the same assumption as in Theorem 1.2, we have, for any $o \in \mathbb{B}$,*

$$\#\{C \in \mathcal{P} : C \cap E \neq \emptyset,\ d(\hat{C},o) < t\} \sim \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \nu_o(E) \cdot e^{\delta_\Gamma \cdot t} \quad \textit{as } t \to \infty.$$

The main result in this paper was announced in [14] and an analogous problem of counting circles in a circle packing of the *plane* was studied in [9] and [15].

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2. Preliminaries and expansion of a hyperbolic surface

In this section, we set up notations as well as recall a result from [16] on the weighted equidistribution of expansions of a hyperbolic surface by the geodesic flow.

Denote by G the group of orientation preserving isometries of $\mathbb{B}$ and fix a circle $C_0 \subset \mathbb{S}^2$. Denote by $\hat{C}_0 \subset \mathbb{B}$ the convex hull of C_0. Fix $p_0 \in \hat{C}_0$ and $o \in \mathbb{B}$. As G acts transitively on $\mathbb{B}$, there exists $g_0 \in G$ such that

$$o = g_0(p_0).$$

Denote by K the stabilizer subgroup of p_0 in G and by H the stabilizer subgroup of $\hat{C}_0$ in G. We note that H is locally isomorphic to $\mathrm{SO}(2,1)$ and has two connected components, one of which is the group of orientation preserving isometries of $\hat{C}_0$. There exist commuting involutions σ and θ of G such that the Lie subalgebras $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{k} = \mathrm{Lie}(K)$ are the $+1$ eigenspaces of $d\sigma$ and $d\theta$ respectively. With respect to the symmetric bilinear form on $\mathfrak{g} = \mathrm{Lie}(G)$ given by

$$B_\theta(X,Y) = \mathrm{Tr}(ad(X) \circ ad(\theta(Y))),$$

we have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are the -1 eigenspaces of $d\sigma$ and $d\theta$ respectively. Let $\mathfrak{a}$ be a one dimensional subalgebra of $\mathfrak{p} \cap \mathfrak{q}$, $A := \exp(\mathfrak{a})$, and M the centralizer of A in K. The map $K \times \mathfrak{p} \to G$ given by $(k,X) \mapsto k \exp X$ is a diffeomorphism and for the canonical projection $\pi : G \to G/K = \mathbb{B}$, the differential $d\pi : \mathfrak{p} \to \mathrm{T}_{p_0}(G/K) = \mathrm{T}_{p_0}(\mathbb{B})$ is an isomorphism.

Choosing an element $X_0 \in \mathfrak{a}$ of norm one, we can identify the unit tangent bundle $\mathrm{T}^1(\mathbb{B})$ with $G.X_0 = G/M$: here $g.X_0$ is given by $d\lambda(g)(X_0)$ where $\lambda(g) : G \to G$ is the left translation $\lambda(g)(g') = gg'$ and $d\lambda$ is its derivative at p_0.

Setting $A^+ = \{\exp(tX_0) : t \ge 0\}$ and $A^- = \{\exp(tX_0) : t \le 0\}$, we have the following generalized Cartan decompositions (cf. [19]):

$$G = KA^-K = HA^+K.$$

in the sense that every element of $g \in G$ can be written as $g = k_1 a_s k_2 = h a_t k$, $s \le 0$, $t \ge 0$, $h \in H, , k_1, k_2, k \in K$. Moreover, $k_1 a_s k_2 = k_1' a_{s'} k_2'$ implies $s = s'$, $k_1 = k_1' m_1$, and $k_2 = m_1^{-1} k_2'$ for some $m_1 \in M$, and $h a_t k = h' a_{t'} k'$ implies that $t = t'$, $h = h' m_2$, and $k = m_2^{-1} k'$ for some $m_2 \in H \cap K$.

The set $K.X_0 = K/M$ represents the set of unit tangent vectors at p_0, and as X_0 is orthogonal to $\mathfrak{h} \cap \mathfrak{p} = \mathrm{T}_{p_0}(\hat{C}_0)$, $H.X_0 = H/M$ corresponds to the set of unit normal vectors to the convex hull $\hat{C}_0 = H/H \cap K$, which will be denoted by $C_0^\dagger$. Moreover if $a_t = \exp(tX_0)$, the set $(H/M)a_t = (Ha_tM)/M$ represents the orthogonal translate of $\hat{C}_0$ by distance $|t|$. We refer to [16] for the above discussion.

Let $\Gamma < G$ be a non-elementary discrete subgroup of G in the rest of this section.

Proposition 2.1 ([15])**.** (1) *If $\Gamma(C_0)$ is infinite, then $[\Gamma : H \cap \Gamma] = \infty$.*
(2) *$\Gamma(C_0)$ is locally finite if and only if the natural projection map $\Gamma \cap H \backslash \hat{C}_0 \to \Gamma \backslash \mathbb{B}$ is proper.*

We denote by $\{\nu_x : x \in \mathbb{B}\}$ a Γ-invariant conformal density for Γ of dimension δ_Γ: for any $x, y \in \mathbb{B}$, $\xi \in \mathbb{S}^2$ and $\gamma \in \Gamma$,

$$\gamma_* \nu_x = \nu_{\gamma x}; \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-\delta_\Gamma \beta_\xi(y,x)}.$$

Here $\gamma_*\nu_x(R) = \nu_x(\gamma^{-1}(R))$ and the Busemann function $\beta_\xi(y_1, y_2)$ is given by $\lim_{t\to\infty} d(y_1, \xi_t) - d(y_2, \xi_t)$ for a geodesic ray ξ_t toward ξ.

For $u \in \mathrm{T}^1(\mathbb{B})$, we define $u^+ \in \mathbb{S}^2$ (resp. $u^- \in \mathbb{S}^2$) the forward (resp. backward) endpoint of the geodesic determined by u and $\pi(u) \in \mathbb{B}$ the basepoint. The map

$$u \mapsto (u^+, u^-, \beta_{u^-}(\pi(u), o))$$

yields a homeomorphism between $\mathrm{T}^1(\mathbb{B})$ with $(\mathbb{S}^2 \times \mathbb{S}^2 - \{(\xi, \xi) : \xi \in \mathbb{S}^2\}) \times \mathbb{R}$.

Definition 2.2 (The Bowen-Margulis-Sullivan measure)**.** The Bowen-Margulis-Sullivan measure m_Γ^{BMS} ([2], [12], [21]) associated to $\{\nu_x\}$ is the measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{B})$ induced by the following Γ-invariant measure on $\mathrm{T}^1(\mathbb{B})$: for $x \in \mathbb{B}$,

$$d\tilde{m}^{\mathrm{BMS}}(u) = e^{\delta_\Gamma \beta_{u^+}(x,\pi(u))}\, e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))}\, d\nu_x(u^+) d\nu_x(u^-) dt.$$

It follows from the conformality of $\{\nu_x\}$ that this definition is independent of the choice of x. The finiteness of $|m_\Gamma^{\mathrm{BMS}}|$ implies that $\{\nu_x\}$ is uniquely determined up to a constant multiple [18, Cor. 1.8].

We denote by $\{m_x : x \in \mathbb{B}\}$ a G-invariant conformal density of $\mathbb{S}^2$ of dimension 2, which is unique up to homothety. Each m_x defines a measure on $\mathbb{S}^2$ which is invariant under the maximal compact subgroup $\mathrm{Stab}_G(x)$.

Definition 2.3 (The Burger-Roblin measure)**.** The Burger-Roblin measure m_Γ^{BR} ([3], [18]) associated to $\{\nu_x\}$ and $\{m_x\}$ is the measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{B})$ induced by the following Γ-invariant measure on $\mathrm{T}^1(\mathbb{B})$:

$$d\tilde{m}^{\mathrm{BR}}(u) = e^{2\beta_{u^+}(x,\pi(u))}\, e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))}\, dm_x(u^+)d\nu_x(u^-)dt$$

for $x \in \mathbb{B}$. By the conformal properties of $\{\nu_x\}$ and $\{m_x\}$, this definition is independent of the choice of $x \in \mathbb{B}$.

On $H/M = C_0^\dagger$, we consider the following two measures:

(2.4)
$$d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s) = e^{2\beta_{s^+}(x,\pi(s))} dm_x(s) \quad \text{and} \quad d\mu^{\mathrm{PS}}_{C_0^\dagger}(s) := e^{\delta_\Gamma \beta_{s^+}(x,\pi(s))} d\nu_x(s^+)$$

for $x \in \mathbb{B}$. These definitions are independent of the choice of x and $\mu^{\mathrm{Leb}}_{C_0^\dagger}$ (resp. $\mu^{\mathrm{PS}}_{C_0^\dagger}$) is left-invariant by H (resp. $H \cap \Gamma$). Hence we may consider the measures $\mu^{\mathrm{Leb}}_{C_0^\dagger}$ and $\mu^{\mathrm{PS}}_{C_0^\dagger}$ on the quotient $(H\cap\Gamma)\backslash C_0^\dagger$.

Theorem 2.5. [16] *We assume that the natural projection map* $\Gamma\cap H\backslash \hat{C}_0 \to \Gamma\backslash\mathbb{B}$ *is proper. Suppose that* $|m_\Gamma^{\mathrm{BMS}}| < \infty$ *and that* $\mathrm{sk}_\Gamma(C_0) < \infty$*. For* $\psi \in C_c(\Gamma\backslash G/M)$*, we have*

$$e^{(2-\delta_\Gamma)t} \int_{s\in(\Gamma\cap H)\backslash C_0^\dagger} \psi(sa_t) d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s) \sim \frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{|m_\Gamma^{\mathrm{BMS}}|} m_\Gamma^{\mathrm{BR}}(\psi) \quad \textit{as } t\to\infty.$$

Moreover $|\mu^{\mathrm{PS}}_{C_0^\dagger}| > 0$ *if* $[\Gamma : H\cap\Gamma] = \infty$*.*

Letting dm the probability invariant measure on M and writing $h = sm \in C_0^\dagger \times M$, $dh = d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s)dm$ is a Haar measure on H, and the following defines a Haar measure on G: for $g = ha_tk \in HA^+K$,

$$dg = 4\sinh t \cdot \cosh t\; dhdtdm_{p_0}(k)$$

where $dm_{p_0}(k) := dm_{p_0}((k.X_0)^+)$.

We denote by $d\lambda$ the unique measure on $H\backslash G$ which is compatible with the choice of dg and dh: for $\psi \in C_c(G)$,

$$\int_G \psi\; dg = \int_{[g]\in H\backslash G} \int_{h\in H} \psi(h[g])\; dhd\lambda[g].$$

Hence for $[e] = H$, $d\lambda([e]a_tk) = 4\sinh t \cdot \cosh t dt dm_{p_0}(k)$.

3. Density of the Burger-Roblin measure on $\mathrm{T}^1_{p_0}(\mathbb{B})$

Fixing p_o and o in $\mathbb{B}$, let $g_0 \in G$ be such that $g_0(p_0) = o$. Let $\Gamma < G$ be a non-elementary discrete subgroup of G. We use the same notation for $K = \mathrm{Stab}_G(p_0)$, A, A^+, X_0, M, etc as in section 2.

Let N denote the expanding horospherical subgroup of G for A^+:

$$N = \{g \in G : a_t g a_t^{-1} \to e \quad \text{as } t \to \infty\}.$$

The product map $A \times N \times K \to G$ is a diffeomorphism.

We fix a Borel subset $E \subset \mathbb{S}^2$ for the rest of this section.

Definition 3.1. Define a function $\mathfrak{R}_E$ on G as follows: for $g = a_t nk \in ANK$,

$$\mathfrak{R}_E(g) := e^{-\delta_\Gamma t} \cdot \chi_{(g_0^{-1}E)_{p_0}}(k^{-1})$$

where $(g_0^{-1}E)_{p_0} := \{u \in K : uX_0^- \in g_0^{-1}(E)\}$ and $\chi_{(g_0^{-1}E)_{p_0}}$ is its characteristic function.

Lemma 3.2. *For any Borel subset $E \subset \mathbb{S}^2$,*

$$\int_{k \in K/M} \mathfrak{R}_E(k^{-1}g_0) d\nu_{p_0}(kX_0^-) = \nu_o(E).$$

Proof. Write $k^{-1}g_0 = a_t nk_0 \in ANK$. Since $X_0^- = \lim_{s\to\infty} a_{-s}(p_0)$ and $\lim_{s\to\infty} a_{s+t} n a_{-s-t} = e$, we obtain

$$\begin{aligned}
&\beta_{kX_0^-}(o, p_0) = \beta_{X_0^-}(k^{-1}o, p_0) \\
&= \lim_{s\to\infty} d(k^{-1}g_0 p_0, a_{-s}p_0) - d(p_0, a_{-s}p_0) \\
&= \lim_{s\to\infty} d(a_t n p_0, a_{-s}p_0) - d(p_0, a_{-s}p_0) \\
&= \lim_{s\to\infty} d((a_{s+t} n a_{-s-t}) a_{s+t} p_0, p_0) - d(p_0, a_{-s}p_0) \\
&= \lim_{s\to\infty} ((s+t) - s) = t.
\end{aligned}$$

On the other hand, since NA fixes X_0^-, $k_0^{-1}(X_0^-) = g_0^{-1}k(X_0^-)$, and hence

$$\chi_{(g_0^{-1}E)_{p_0}}(k_0^{-1}) = \chi_{g_0^{-1}E}(k_0^{-1}X_0^-) = \chi_E(k(X_0^-)).$$

So

$$\mathfrak{R}_E(k^{-1}g_0) = e^{-\delta_\Gamma \beta_{kX_0^-}(o,p_0)} \chi_E(k(X_0^-)).$$

Therefore by the conformal property of $\{\nu_x\}$,

$$\int_{k \in K/M} \mathfrak{R}_E(k^{-1}g_0) d\nu_{p_0}(kX_0^-) = \int_{\xi \in E} e^{-\delta_\Gamma \beta_\xi(o,p_0)} d\nu_{p_0}(\xi) = \nu_o(E).$$

$\square$

Fixing a left-invariant metric on G, we denote by U_ϵ an ϵ-ball around e, and for $S \subset G$, we set $S_\epsilon = S \cap U_\epsilon$.

Lemma 3.3. *(cf.* [16, Lem. 6.1]*) There exists $\ell \geq 1$ such that for any $a_t nk \in ANK$ and small $\epsilon > 0$,*

$$a_t nk(g_0^{-1} U_\epsilon g_0) \subset A_{\ell\epsilon} a_t N K_{\ell\epsilon} k.$$

For each small $\epsilon > 0$, we choose a non-negative function $\psi^\epsilon \in C_c(G)$ supported inside U_ϵ and of integral $\int_G \psi^\epsilon dg$ one, and define $\Psi^\epsilon \in C_c(\Gamma\backslash G)$ by

$$\Psi^\epsilon(g) = \sum_{\gamma\in\Gamma} \psi^\epsilon(\gamma g). \tag{3.4}$$

Definition 3.5. Define a function Ψ_E^ϵ on $\Gamma\backslash G$ by

$$\Psi_E^\epsilon(g) = \int_{k^{-1}\in(g_0^{-1}E)_{p_0}} \Psi^\epsilon(gkg_0^{-1}) dm_{p_0}(k).$$

For each $\epsilon > 0$, define

$$E_\epsilon^+ := g_0 U_\epsilon g_0^{-1}(E) \quad \text{and} \quad E_\epsilon^- := \cap_{u\in U_\epsilon} g_0 u g_0^{-1}(E). \tag{3.6}$$

By Lemma 3.3, it follows that there exists $c > 0$ such that for all $g \in U_\epsilon$ and $g_1 \in G$,

$$(1 - c\epsilon)\mathfrak{R}_{E_\epsilon^-}(g_1 g_0) \leq \mathfrak{R}_E(g_1 g g_0) \leq (1 + c\epsilon)\mathfrak{R}_{E_\epsilon^+}(g_1 g_0). \tag{3.7}$$

Proposition 3.8. *There exists $c > 0$ such that for all small $\epsilon > 0$,*

$$(1 - c\epsilon)\nu_o(E_\epsilon^-) \leq m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) \leq (1 + c\epsilon)\nu_o(E_\epsilon^+).$$

Proof. Using the decomposition $G = ANK$, we have for $g = a_t nk$,

$$dg = dt dn dm_{p_0}(k)$$

where dn is the Lebesgue measure on N.

We use the following formula for $\tilde{m}^{\mathrm{BR}}$ (cf. [16]): for any $\psi \in C_c(G)^M$,

$$\tilde{m}^{\mathrm{BR}}(\psi) = \int_K \int_A \int_N \psi(ka_t n) e^{-\delta_\Gamma t}\, dn dt d\nu_{p_0}(k(X_0^-)).$$

For $\psi_E^\epsilon(g) := \int_{k^{-1}\in(g_0^{-1}E)_{p_0}} \psi^\epsilon(gkg_0^{-1}) dm_{p_0}(k)$, we have

$$\begin{aligned}
&m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) = \tilde{m}^{\mathrm{BR}}(\psi_E^\epsilon) \\
&= \int_{g\in G} \int_{k^{-1}\in(g_0^{-1}E)_{p_0}} \psi^\epsilon(gkg_0^{-1}) dm_{p_0}(k) d\tilde{m}^{\mathrm{BR}}(g) \\
&= \int_{KAN} \int_{k\in K} \psi^\epsilon(k_0 a_t nkg_0^{-1}) \chi_{(g_0^{-1}E)_{p_0}}(k^{-1}) dm_{p_0}(k) e^{-\delta_\Gamma t} dn dt d\nu_{p_0}(k_0 X_0^-) \\
&= \int_{k_0\in K} \int_{ANK} \psi^\epsilon(k_0(a_t nk)g_0^{-1}) \mathfrak{R}_E(a_t nk) dm_{p_0}(k) dn dt d\nu_{p_0}(k_0 X_0^-) \\
&= \int_{k_0\in K} \int_G \psi^\epsilon(kgg_0^{-1}) \mathfrak{R}_E(g)\, dg\, d\nu_{p_0}(k_0 X_0^-) \\
&= \int_{k_0\in K} \int_G \psi^\epsilon(g) \mathfrak{R}_E(k^{-1} g g_0) dg\, d\nu_{p_0}(k_0 X_0^-).
\end{aligned}$$

Hence applying (3.7), the identity $\int \psi^\epsilon dg = 1$ and Lemma 3.2, we deduce that

$$\begin{aligned} m_\Gamma^{\rm BR}(\Psi_E^\epsilon) &\le (1+c\epsilon)\int_{k\in K}\left(\int_G \psi^\epsilon(g)dg\right)\mathfrak{R}_{E_\epsilon^+}(k^{-1}g_0)d\nu_{p_0}(kX_0^-) \\ &= (1+c\epsilon)\int_{k\in K}\mathfrak{R}_{E_\epsilon^+}(k^{-1}g_0)d\nu_{p_0}(kX_0^-) \\ &= (1+c\epsilon)\nu_o(E_\epsilon^+). \end{aligned}$$

The other inequality follows similarly. □

4. Simpler proof of Theorem 1.7 for the special case of $E = \mathbb{S}^2$.

The result in this section is covered by the proof of Theorem 1.7 (for general E) given in section 6. However we present a separate proof for this special case as it is considerably simpler and uses a different interpretation of the counting function.

We may assume without loss of generality that $\mathcal{P} = \Gamma(C_0)$. We use the notations from section 2.

Set

$$\mathcal{N}_T(\mathcal{P}) = \#\{C \in \mathcal{P} : d(\hat{C}, o) < t\}.$$

Lemma 4.1. *For $T > 1$,*

$$\mathcal{N}_T(\mathcal{P}) = \#[e]\Gamma \cap [e]A_T^+Kg_0^{-1}$$

where $[e] = H \in H\backslash G$ and $A_T^+ = \{a_t : 0 \le t \le T\}$.

Proof. Note that $N_T(\mathcal{P})$ is equal to the number of hyperbolic planes $\gamma(\hat{C}_0)$ such that $d(o, \gamma(\hat{C}_0)) < T$, or equivalently, $d(\gamma^{-1}(o), \hat{C}_0) < T$. Since $\{x \in \mathbb{B} : d(x, \hat{C}_0) < T\} = HA_T^+(p_0)$, $N_T(\mathcal{P})$ is equal to the number of $[\gamma] \in \Gamma/\operatorname{Stab}_\Gamma(\hat{C}_0)$ such that $\gamma^{-1}g_0p_0 \in HA_T^+p_0$, or alternatively, the number of $[\gamma] \in H\cap\Gamma\backslash\Gamma$ such that $\gamma g_0 \in HA_T^+K$, which is equal to $\#[e]\Gamma g_0 \cap [e]A_T^+K$. □

Define the following counting function F_T on $\Gamma\backslash G$ by

$$F_T(g) := \sum_{\gamma\in\Gamma\cap H\backslash\Gamma} \chi_{B_T}([e]\gamma g)$$

where $B_T = [e]A_T^+Kg_0^{-1} \subset H\backslash G$. Note that $F_T(e) = \mathcal{N}_T(\mathcal{P})$.

By the strong wave front lemma (see [7]), for all small $\epsilon > 0$, there exists $\ell > 1$ and $t_0 > 0$ such that for all $t > t_0$,

$$Ka_tkg_0^{-1}U_\epsilon \subset Ka_tA_{\ell\epsilon}kK_{\ell\epsilon}g_0^{-1}. \tag{4.2}$$

It follows that for all $T \gg 1$,

$$(B_T - B_{t_0})U_\epsilon \subset B_{T+\ell\epsilon} \text{ and } (B_{T-\ell\epsilon} - B_{t_0}) \subset \cap_{u\in U_\epsilon} B_Tu.$$

Hence there exists $m_0 \ge 1$ such that for all $g \in U_\epsilon$ and $T \gg 1$,

$$F_{T-\ell\epsilon}(g) - m_0 \le F_T(e) \le F_{T+\ell\epsilon}(g) + m_0.$$

Integrating against Ψ^ϵ (see (3.4)), we obtain

$$\langle F_{T-\ell\epsilon}, \Psi^\epsilon\rangle - m_0 \le F_T(e) \le \langle F_{T+\ell\epsilon}, \Psi^\epsilon\rangle + m_0,$$

where the inner product is taken with respect dg.

Setting $\Xi_t = 4\sinh t \cdot \cosh t$, we have

$$\begin{aligned}\langle F_{T+\ell\epsilon}, \Psi^\epsilon\rangle &= \int_{g\in\Gamma\cap H\backslash G} \chi_{B_T}([e]g)\Psi^\epsilon(gg_0^{-1})\, dg \\ &= \int_{k\in K}\int_0^{T+\ell\epsilon}\int_{s\in\Gamma\cap H\backslash C_0^\dagger}\left(\int_{m\in M}\Psi^\epsilon(sa_tmkg_0^{-1})\, dm\right)\Xi_t\, d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s)dtdm_{p_0}(k) \\ &= \int_{k\in K}\int_0^{T+\ell\epsilon}\int_{s\in\Gamma\cap H\backslash C_0^\dagger}\Psi^\epsilon_{kg_0^{-1}}(sa_t)\Xi_t\, d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s)dtdm_{p_0}(k).\end{aligned}$$

where $\Psi^\epsilon_{g_1}\in C_c(\Gamma\backslash G)^M$ is given by $\Psi^\epsilon_{g_1}(g)=\int_{m\in M}\Psi^\epsilon(gmg_1)\, dm$.

Hence by Theorem 2.5, and using $\Xi_t \sim e^{2t}$, we deduce that as $T\to\infty$,

$$\langle F_{T+\ell\epsilon}, \Psi^\epsilon\rangle \sim \frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}_\Gamma|} m^{\mathrm{BR}}_\Gamma(\Psi^\epsilon_{\mathbb{S}^2})e^{\delta_\Gamma(T+\ell\epsilon)}$$

where

$$\Psi^\epsilon_{\mathbb{S}^2}(g) = \int_{k\in K}\Psi^\epsilon(gkg_0^{-1})dm_{p_0}(k).$$

By Prop. 3.8,

$$m^{\mathrm{BR}}_\Gamma(\Psi^\epsilon_{\mathbb{S}^2}) = (1+O(\epsilon))|\nu_o|.$$

Therefore it follows, as $\epsilon>0$ is arbitrary,

$$\limsup_T \frac{F_T(e)}{e^{\delta_\Gamma T}} \le \frac{|\nu_o|\cdot|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}_\Gamma|}.$$

Similarly

$$\liminf_T \frac{F_T(e)}{e^{\delta_\Gamma T}} \ge \frac{|\nu_o|\cdot|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}_\Gamma|}.$$

This finishes the proof, as $|\mu^{\mathrm{PS}}_{C_0^\dagger}| = \mathrm{sk}_\Gamma(\mathcal{P})$.

5. Uniform distribution along $\mathfrak{b}_T(\mathcal{W})$

In this section, fix a Borel subset $\mathcal{W}\subset K$ with $M\mathcal{W}=\mathcal{W}$.

Definition 5.1. For $T>1$, we set

$$\mathfrak{b}_T(\mathcal{W}) = H\backslash HKA_T^+\mathcal{W}\subset H\backslash G$$

where $A_T^+ = \{a_t\in A : 0\le t\le T\}$.

Theorem 5.2. *We assume that the natural projection map* $\Gamma\cap H\backslash\hat{C}_0 \to \Gamma\backslash\mathbb{B}$ *is proper. Suppose that* $|m_\Gamma^{\mathrm{BMS}}| < \infty$ *and that* $\mathrm{sk}_\Gamma(C_0) < \infty$. *For any* $\psi \in C_c(\Gamma\backslash G)$, *we have*

$$\int_{g\in\mathfrak{b}_T(\mathcal{W})}\int_{h\in\Gamma\cap H\backslash H}\psi(hg)dhd\lambda(g) \sim \frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}|}\cdot\int_{k\in\mathcal{W}} m_\Gamma^{\mathrm{BR}}(\psi_k)\,dm_{p_0}(k)\cdot e^{\delta_\Gamma T}$$

as $T\to\infty$, *where* $\psi_k \in C_c(\Gamma\backslash G)^M$ *is given by* $\psi_k(g) = \int_{m\in M}\psi(gmk)dm$.

Proof. (cf. [15, Thm 4.3])
Set $K'_\epsilon = \cup_{k\in K} kK_\epsilon k^{-1}$ and define $\psi_\epsilon^\pm \in C_c(\Gamma\backslash G)$ by

$$\psi_\epsilon^+(g) := \sup_{u\in K'_\epsilon}\psi(gu) \quad\text{and}\quad \psi_\epsilon^-(g) := \inf_{u\in K'_\epsilon}\psi(gu).$$

Note that for a given $\eta > 0$, there exists $\epsilon = \epsilon(\eta) > 0$ such that for all $g \in \Gamma\backslash G$, $|\psi_\epsilon^+(g) - \psi_\epsilon^-(g)| \le \eta$ by the uniform continuity of ψ.

We can deduce from Theorem 2.5 that for all $t > T_1(\eta) \gg 1$,

$$\int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_tk)dh = (1+O(\eta))\frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{|m_\Gamma^{\mathrm{BMS}}|}m_\Gamma^{\mathrm{BR}}(\psi_{\epsilon,k}^+)e^{(\delta-2)t}$$

where $\psi_{\epsilon,k}^+$ is defined similarly as ψ_k and the implied constant can be taken uniformly over all $k \in K$. Defining

$$K_T(t) := \{k\in K : a_tk \in HKA_T^+\},$$

by Prop. 4.8 and Corollary 4.11 in [15], we have $HKA_T^+ = \cup_{0\le t\le T} Ha_tK_T(t)$ and there exists a sufficiently large $T_0(\epsilon) > T_1(\eta)$ such that $e \in K_T(t) \subset K_\epsilon M$ for all $T_0(\epsilon) < t < T$.

For $[e] = H \in H\backslash G$ and $s > 0$, set

$$V_T(s) := (\cup_{s\le t\le T}[e]a_tK_T(t))\mathcal{W}$$

so that

$$\mathfrak{b}_T(\mathcal{W}) = V_T(s)\cup(\mathfrak{b}_T(\mathcal{W}) - V_T(s)).$$

Let $[g] = [e]a_tkk_1 \in V_T(T_0(\epsilon))$ where $k_1\in K$ and $k\in\mathcal{W}$. For $t > T_0(\epsilon)$, there exist $h_0 \in H$ and $u \in K'_\epsilon$ such that $a_tk_1k = h_0a_tku$ and hence

$$\psi^H(g) := \int_{h\in\Gamma\cap H\backslash H}\psi(hg)dh$$
$$= \int_{h\in\Gamma\cap H\backslash H}\psi(hh_0a_tku)dh \le \int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_tk)dh.$$

Therefore

$$\int_{V_T(T_0(\epsilon))}\psi^H(g)d\lambda(g) \le \int_{k\in\mathcal{W}}\int_{T_0(\epsilon)<t\le T}\int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_tk)\Xi_t\,dhdtdm_{p_0}(k)$$

where $\Xi_t = 4\sinh t\cosh t$.

Using $\Xi_t \sim e^{2t}$, we then deduce

$$\int_{k\in\mathcal{W}}\int_{T_0(\epsilon)<t<T}\int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_tk)\Xi_t\, dhdtdm_{p_0}(k)$$
$$= (1+O(\eta))\frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}|}\cdot\int_{k\in\mathcal{W}} m_\Gamma^{\mathrm{BR}}(\psi_k)dm_{p_0}(k)\cdot(e^{\delta_\Gamma T}-e^{\delta_\Gamma T_0(\epsilon)})$$

since $m_\Gamma^{\mathrm{BR}}(\psi^+_{\epsilon,k}) = (1+O(\eta))m_\Gamma^{\mathrm{BR}}(\psi_k)$.

Hence

$$\limsup_T \frac{\int_{V_T(T_0(\epsilon))}\psi^H(g)d\lambda(g)}{e^{\delta_\Gamma T}} = (1+O(\eta))\frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\int_{\mathcal{W}} m_\Gamma^{\mathrm{BR}}(\psi_k)dm_{p_0}(k).$$

On the other hand, it follows from the assumption that $\Gamma\backslash\Gamma H$ is a proper subset of $\Gamma\backslash G$ and that

$$\int_{[g]\in\mathfrak{b}_T(\mathcal{W})-V_T(T_0(\epsilon))}\int_{h\in\Gamma\cap H\backslash H}\psi(hg)dhd\lambda(g) = O(1).$$

As $\eta>0$ is arbitrary and $\epsilon(\eta)\to 0$ as $\eta\to 0$, it follows that

$$\limsup_T \frac{\int_{[g]\in\mathfrak{b}_T(\mathcal{W})}\psi^H(g)d\lambda(g)}{e^{\delta_\Gamma T}} \le \frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\int_{k\in\mathcal{W}} m_\Gamma^{\mathrm{BR}}(\psi_k)dm_{p_0}(k).$$

By a similar argument, one can prove

$$\liminf_T \frac{\int_{[g]\in\mathfrak{b}_T(\mathcal{W})}\psi^H(g)d\lambda(g)}{e^{\delta_\Gamma T}} \ge \frac{|\mu^{\mathrm{PS}}_{C_0^\dagger}|}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\int_{k\in\mathcal{W}} m_\Gamma^{\mathrm{BR}}(\psi_k)dm_{p_0}(k).$$

□

6. Proof of Theorem 1.7

Without loss of generality, we may assume that $\mathcal{P}=\Gamma(C_0)$. We keep the notations from section 2.

Definition 6.1. A subset $E\subset\mathbb{S}^2$ is said to be $\mathcal{P}$-admissible if, for any $C\in\mathcal{P}$, $C^\circ\cap E\neq\emptyset$ implies $C^\circ\subset E$, possibly except for finitely many circles.

For a subset $E\subset\mathbb{S}^2$, we define $E_{p_0}\subset K$ by

$$E_{p_0} := \{k\in K : k(X_0^-)\in E\}.$$

We also set

$$\mathcal{N}_T(\mathcal{P},E) := \{C\in\mathcal{P} : C\cap E\neq\emptyset,\ \ d(\hat{C},o)<T\}.$$

Lemma 6.2. *Fix a $\mathcal{P}$-admissible subset $E\subset\mathbb{S}^2$. There exists $c_0>0$ such that for all $T>1$,*

$$\#[e]\Gamma\cap[e]KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1}-c_0 \le \mathcal{N}_T(\mathcal{P},E) \le \#[e]\Gamma\cap[e]KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1}+c_0$$

where $[e]=H\in H\backslash G$.

Proof. Since g_0K/M represents the set of all unit vectors based at o, and the set $\{u \in \mathrm{T}^1_o(\mathbb{B}) : u^- \in E\}$ is identified with $g_0(g_0^{-1}E)_{p_0} = \{g_0k[M] : kX_0^- \in g_0^{-1}E\}$, the set $g_0(g_0^{-1}E)_{p_0}A^-(p_0)$ represents the set of all points in $\mathbb{B}$ lying in the cone consisting of geodesic rays connecting o with a point in E. Therefore the condition $C \subset E$ is equivalent to that $\hat{C} \subset g_0(g_0^{-1}E)_{p_0}A^-(p_0)$. Hence by the $\mathcal{P}$-admissibility condition, we may assume without loss of generality that $\mathcal{N}_T(\mathcal{P}, E)$ is equal to the number of hyperbolic planes $\gamma(\hat{C}_0)$ such that $d(o, \gamma(\hat{C}_0)) < T$ and $\gamma(\hat{C}_0) \subset g_0(g_0^{-1}E)_{p_0}A^-(p_0)$. Since $\{x \in \mathbb{B} : d(o,x) < T\} = g_0KA_T^-(p_0)$ where $A_T^- = \{a_{-t} : 0 \le t \le T\}$, the former condition is again same as $\gamma(\hat{C}_0) \cap g_0KA_T^-(p_0) \neq \emptyset$. Hence

$$\begin{aligned}
&\mathcal{N}_T(\mathcal{P}, E)\\
&= \#\{\gamma(C_0) : \gamma(\hat{C}_0) \cap g_0KA_T^-(p_0) \neq \emptyset,\ \gamma(\hat{C}_0) \subset g_0(g_0^{-1}E)_{p_0}A^-(p_0)\}\\
&= \#\{[\gamma] \in \Gamma/\Gamma \cap H : \gamma \in g_0KA_T^-KH \cap g_0(g_0^{-1}E)_{p_0}A^-KH\}\\
&= \#\{[\gamma] \in \Gamma/\Gamma \cap H : \gamma \in g_0(g_0^{-1}E)_{p_0}A_T^-KH\}.
\end{aligned}$$

In the last equality, we have used the fact that if $a_{-t} \in KA_T^-KH$ for some $t > 0$, then $t < T$ (see [15, Lem 4.10]).

By taking the inverse, we obtain that

$$\mathcal{N}_T(\mathcal{P}, E) = [e]\Gamma \cap [e]KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1}.$$

□

Fixing a Borel subset $E \subset \mathbb{S}^2$, recall the definition of $E_\epsilon^\pm$ from (3.6):

$$E_\epsilon^+ := g_0U_\epsilon g_0^{-1}(E) \quad \text{and} \quad E_\epsilon^- := \cap_{u \in U_\epsilon} g_0ug_0^{-1}(E).$$

We can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}_\epsilon^+$ such that $E \subset \tilde{E}_\epsilon^+ \subset E_\epsilon^+$ by adding all the open disks inside E_ϵ^+ intersecting the boundary of E. Similarly we can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}_\epsilon^-$ such that $E_\epsilon^- \subset \tilde{E}_\epsilon^- \subset E$ by adding all the open disks inside E intersecting the boundary of E_ϵ^-. By the local finiteness of $\mathcal{P}$, there are only finitely many circles intersecting $\partial(E)$ (resp. $\tilde{E}_\epsilon^-$) which are not contained in $\tilde{E}_\epsilon^+$ (resp. E). Therefore there exists $q_\epsilon \ge 1$ (independent of T) such that

$$\mathcal{N}_T(\mathcal{P}, \tilde{E}_\epsilon^-) - q_\epsilon \le \mathcal{N}_T(\mathcal{P}, E) \le \mathcal{N}_T(\mathcal{P}, \tilde{E}_\epsilon^+) + q_\epsilon. \tag{6.3}$$

Setting

$$B_T(E) := [e]KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1} \subset H\backslash G,$$

we define functions $F_T^{\epsilon,\pm}$ on $\Gamma\backslash G$:

$$F_T^{\epsilon,\pm}(g) := \sum_{\gamma \in \Gamma \cap H\backslash\Gamma} \chi_{B_{T\pm\ell\epsilon}(E^\pm_{(\ell+1)\epsilon})}([e]\gamma g).$$

Lemma 6.4. *There exists $m_\epsilon \ge 1$ such that for all $g \in U_\epsilon$ and $T \gg 1$,*

$$F_T^{\epsilon,+}(g) - m_\epsilon \le \mathcal{N}_T(\mathcal{P}, E) \le F_T^{\epsilon,+}(g) + m_\epsilon.$$

Proof. It follows from (4.2) that

$$B_T(E_\epsilon^+)U_\epsilon \subset B_{T+\ell\epsilon}(E^+_{(\ell+1)\epsilon}) \text{ and } B_{T-\ell\epsilon}(E^-_{(\ell+1)\epsilon}) \subset \cap_{u\in U_\epsilon} B_T(E_\epsilon^-)u.$$

Hence for any $g \in U_\epsilon$, as U_ϵ is symmetric,

$$\#[e]\Gamma \cap B_T(\tilde{E}_\epsilon^+) \le \#[e]\Gamma \cap B_T(\tilde{E}_\epsilon^+)U_\epsilon g^{-1} \le \#[e]\Gamma g \cap B_{T+\ell\epsilon}(E^+_{(\ell+1)\epsilon}).$$

By Lemma 6.2 and (6.3), it follows that for some fixed $m_\epsilon \ge 1$,

$$\mathcal{N}_T(\mathcal{P}, E) \le F_T^{\epsilon,+}(g) + m_\epsilon.$$

The other inequality can be proved similarly. □

Hence by integrating against Ψ^ϵ (see (3.4)), we obtain

$$\langle F_T^{\epsilon,-}, \Psi^\epsilon\rangle - m_\epsilon \le \mathcal{N}_T(\mathcal{P}, E) \le \langle F_T^{\epsilon,+}, \Psi^\epsilon\rangle + m_\epsilon. \tag{6.5}$$

We note that

$$B_T(E) = \mathfrak{b}_T((g_0^{-1}E)^{-1}_{p_0})\ g_0^{-1}$$

where $\mathfrak{b}_T(\mathcal{W})$ is defined as in Def. 5.1.

Since

$$\begin{aligned}\langle F_T^{\epsilon,+}, \Psi^\epsilon\rangle &= \int_{\Gamma\cap H\backslash G} \chi_{B_{T+\ell\epsilon}(E^+_{(\ell+1)\epsilon})}([e]g)\Psi^\epsilon(g)\, dg \\ &= \int_{[g]\in B_{T+\ell\epsilon}(E^+_{(\ell+1)\epsilon})} \int_{h\in\Gamma\cap H\backslash H} \Psi^\epsilon(hg)dhd\lambda(g) \\ &= \int_{[g]\in \mathfrak{b}_{T+\ell\epsilon}((g_0^{-1}E^+_{(\ell+1)\epsilon})^{-1}_{p_0})} \int_{h\in\Gamma\cap H\backslash H} \Psi^\epsilon(hgg_0^{-1})dhd\lambda(g)\end{aligned}$$

we deduce from Theorem 5.2 that

$$\langle F_T^{\epsilon,+}, \Psi^\epsilon\rangle \sim \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot m_\Gamma^{\mathrm{BR}}(\Psi^\epsilon_{E^+_{(\ell+1)\epsilon}})\cdot e^{\delta_\Gamma(T+\ell\epsilon)} \tag{6.6}$$

where $\Psi^\epsilon_E(g) = \int_{k^{-1}\in(g_0^{-1}E)_{p_0}} \Psi^\epsilon(gkg_0^{-1})dm_{p_0}(k)$ (see Def. 3.5).

Therefore by (6.5) and Prop. 3.8 we have

$$\limsup_T \frac{\mathcal{N}_T(\mathcal{P}, E)}{e^{\delta_\Gamma T}} \le (1+O(\epsilon))\frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\nu_o(E^+_{(\ell+1)\epsilon}).$$

Since $\nu_o(\partial(E)) = 0$ by the assumption, $\nu_o(E^+_{(\ell+1)\epsilon} - E) \to 0$ as $\epsilon \to 0$. As ϵ can be taken arbitrarily small, it follows that

$$\limsup_T \frac{\mathcal{N}_T(\mathcal{P}, E)}{e^{\delta_\Gamma T}} \le \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\nu_o(E).$$

Similarly, we can prove

$$\liminf_T \frac{\mathcal{N}_T(\mathcal{P}, E)}{e^{\delta_\Gamma T}} \ge \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot|m_\Gamma^{\mathrm{BMS}}|}\cdot\nu_o(E).$$

This completes the proof.

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Mathematics department, Brown university, Providence, RI, USA and Korea Institute for Advanced Study, Seoul, Korea
E-mail address: heeoh@math.brown.edu

The Ohio State University, Columbus, OH and Tata Institute of Fundamental Research, Mumbai, India
E-mail address: shah@math.osu.edu