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STRONG WAVEFRONT LEMMA AND COUNTING LATTICE POINTS IN SECTORS

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ABSTRACT

We compute the asymptotics of the number of integral quadratic forms with prescribed orthogonal decompositions and more generally, the asymptotics of the number of lattice points lying in sectors of affine symmetric spaces. A new key ingredient in this article is the strong wavefront lemma, which shows that the generalized Cartan decomposition associated to a symmetric space is uniformly Lipschitz.

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1. Introduction

One of the motivations of this paper is a certain counting problem in the space of quadratic forms. Let $\mathcal{S}_W$ be the vector space of all quadratic forms on a Euclidean space W of dimension d. We fix an integral structure on W, and hence on $\mathcal{S}_W$. Let $\mathcal{Q}_W$ denote the subset of $\mathcal{S}_W$ consisting of quadratic forms of determinant ± 1, and set $\mathcal{Q}_W(\mathbb{Z}) = \mathcal{Q}_W \cap \mathcal{S}_W(\mathbb{Z})$. Let $\|\cdot\|$ be any norm on $\mathcal{S}_W$. It follows from the main result of Duke, Rudnick and Sarnak [DRS], as well as of Eskin and McMullen [EM], that for $d \geq 3$ there exists a constant $c > 0$ such that

$$\#\{\boldsymbol{q} \in \mathcal{Q}_W(\mathbb{Z}) : \|\boldsymbol{q}\| < T\} \sim_{T\to\infty} c \cdot T^{d(d-1)/2}. \tag{1.1}$$

Here we will consider a refinement of this problem that concerns counting quadratic forms with prescribed structure. Fix an orthogonal decomposition

$$W = \bigoplus_{i=1}^{n} W_i, \tag{1.2}$$

and for $\Omega \subset \mathrm{SO}(W)$ and $\Omega' \subset \mathcal{Q}_{W_1} \times \cdots \times \mathcal{Q}_{W_n}$, set

$$N_T(\Omega, \Omega') = \#\left\{ \boldsymbol{q} \in \mathcal{Q}_W(\mathbb{Z}) : \begin{array}{l} \|\boldsymbol{q}\| < T, \\ \boldsymbol{q}(k\cdot x) = a_1\boldsymbol{q}_1(x) + \cdots + a_n\boldsymbol{q}_n(x) \\ \text{for some } k \in \Omega,\ (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in \Omega', \\ \text{and } a_1 > \cdots > a_n > 0 \end{array} \right\}. \tag{1.3}$$

For example, if we choose W_i's to be one dimensional, then we are counting the number of quadratic forms in a ball of radius T which can be diagonalized via conjugation by an element from a prescribed set Ω of orthogonal transformations to obtain a form with distinct eigenvalues in decreasing order of absolute values, and with prescribed sign ($\pm$) in each diagonal entry.

Assuming that Ω and Ω' are bounded measurable sets such that the subset $\Omega\Omega'$ has positive measure and boundary of measure zero,[1] we prove the following:

THEOREM 1.4: *For $d \geq 3$,*

$$N_T(\Omega, \Omega') \sim_{T\to\infty} c \cdot T^{d(d-\dim W_n)/2}$$

for some $c = c(\Omega\Omega') > 0$.

[1] The measure of $\Omega\Omega'$ is understood in terms of the identification (3.2) and (3.3).

Theorem 1.4 is an example of our general result (Theorem 1.13) on counting lattice points in sectors of affine symmetric spaces. In ([DRS], [EM]) it is shown that the number of integral points in an affine symmetric $\mathbb{Q}$-variety in a sequence of growing subsets S_T is asymptotic to the volume of S_T, provided the sets S_T are *well-rounded*. A family of subsets S_T being well-rounded means roughly that the volumes of neighborhoods of the boundaries of S_T are uniformly negligible compared to the total volumes of S_T (see (1.11) for the precise condition). In ([DRS], [EMS]), it is shown that the norm balls are well-rounded. However, in most situations, given a sequence of subsets S_T which arises naturally in the geometric or number-theoretic contexts in the category of affine symmetric spaces, it is highly non-trivial to determine whether the family S_T is well-rounded.

The main result of this paper is to show that *sectors* in affine symmetric spaces define a well-rounded family of growing subsets, and consequently, we obtain the asymptotic counting of lattice points in sectors. The main technical lemma needed is what we call the 'strong wave front lemma', a terminology reflecting it being a stronger version of the wavefront lemma introduced by Eskin and McMullen [EM].

Now we introduce notation that we use throughout the paper. Let G be a connected noncompact semisimple Lie group with finite center. A closed subgroup H of G is called symmetric if its identity component coincides with the identity component of the set of fixed points of an involution, say σ, of G. In this case, the homogeneous space G/H is called an **affine symmetric space**. Recall that a maximal compact subgroup of G is a symmetric subgroup associated to a Cartan involution on G. Affine symmetric spaces have many features similar to Riemannian symmetric spaces. In particular, a generalized Cartan decomposition holds:

$$G = KAH$$

where K is a maximal compact subgroup of G compatible with H, and A is a Cartan subgroup corresponding to the pair (K, H).

More precisely, there exists a Cartan involution θ of G which commutes with σ, and let $K = \{g \in G : \theta(g) = g\}$, which is a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{k}$ denote the Lie algebras associated to G, H and K, respectively. Let θ and σ also denote their differentials on $\mathfrak{g}$. Since H and K are θ stable, we have the following orthogonal decomposition with respect to the killing form on

$\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are the (-1)-eigenspaces of θ and σ, respectively. Let $\mathfrak{a}$ denote the maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$ which can be extended to a maximal abelian subalgebra, say $\mathfrak{b}$, of $\mathfrak{p}$. Let A denote the analytic subgroup of G associated to $\mathfrak{a}$. This A is called the Cartan subgroup corresponding to the symmetric pair (K, H).

Wavefront Lemma (Eskin and McMullen [EM]). Given any neighborhood $\mathcal{O}$ of e in G, there exists a neighborhood $\tilde{\mathcal{O}}$ of e in G such that

$$\tilde{\mathcal{O}} g \subset g\mathcal{O}H, \quad \forall g \in KA.$$

Next we will strengthen this result for uniformly regular elements of $g \in G$. For this we will need additional notation (cf. [Sc, Ch. 7], [HS, Part II] or [GOS]). Let $\mathfrak{g}^\alpha$ denote a simultaneous eigenspace for $\operatorname{ad}\mathfrak{a}$ action on $\mathfrak{g}$ associated to the linear character $\alpha \in \mathfrak{a}^*$. Let $\Sigma_\sigma = \{\alpha \in \mathfrak{a}^* : \mathfrak{g}^* \neq 0\}$. Then $\mathfrak{g} = \sum_{\alpha \in \Sigma_\sigma \cup \{0\}} \mathfrak{g}^\alpha$, and Σ_σ forms a root system. Choose a closed positive Weyl chamber $A^+ \subset A$. Let Σ_σ^+ denote the set of positive roots and Δ_σ the corresponding system of positive simple roots. The associated Weyl group is given by $\mathcal{W}_\sigma = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. One can choose a set $\mathcal{W} \subset N_K(\mathfrak{a}) \cap N_K(\mathfrak{b})$ of coset representatives of $N_K(\mathfrak{a})/N_{K\cap H}(\mathfrak{a})Z_K(\mathfrak{a})$. Then

$$G = \bigcup_{w \in \mathcal{W}} KA^+wH. \tag{1.5}$$

For any $c > 0$, an element $g = kawh \in KA^+\mathcal{W}H$ will be called **c-regular** if $\alpha(\log a) \geq c$ for all $\alpha \in \Delta_\sigma$ (here and later, our notation indicates that $k \in K$, $a \in A^+$, $w \in \mathcal{W}$, and $h \in H$). Otherwise, we call such an element **c-singular**.

We fix a Riemannian metric on G and denote by $\mathcal{O}_\varepsilon$ the ε-ball at identity.

Theorem 1.6 (Strong wavefront lemma-I): *Given $c > 0$, there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for every c-regular $g = kawh \in KA^+wH$ and $0 < \varepsilon < \varepsilon_0$,*

$$\mathcal{O}_\varepsilon \cdot g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (A \cap \mathcal{O}_{\ell\varepsilon})a \cdot w(H \cap \mathcal{O}_{\ell\varepsilon})h.$$

The continuity of the Cartan decomposition for Riemannian symmetric spaces (that is, when $H = K$) was independently shown in Nevo [N, Proposition 7.3] and by Gorodnik and Oh [GO, Theorem 2.1]. While the proof of [N] uses embeddings of G in linear groups, the proof of [GO] is based on geometric properties of the Riemannian symmetric spaces. The strong wavefront lemma was used in [N] to prove maximal inequalities for cube averages on semisimple groups and in [GO] to compute the asymptotics of the number of lattice points lying in sectors.

Theorem 1.6 fails on the set of singular elements; for example, in $\mathrm{SL}_2(\mathbb{R})$, if Ω is a small neighborhood of the e, then $(\Omega \cap K)(\Omega \cap A)(\Omega \cap K)$ does not contain a neighborhood of the e in $\mathrm{SL}_2(\mathbb{R})$. To state a version of the strong wavefront lemma that holds for singular elements, we introduce additional notation. Given $J \subset \Delta_\sigma$, an element $kawh \in KA^+\mathcal{W}H$ is called (J,c)**-regular** if $\alpha(\log a) \geq c$ for all $\alpha \in J$. Let $I = \Delta_\sigma \setminus J$. We set $A_I = \exp(\ker I) \subset A$. Let M_I be the analytic semisimple subgroup whose Lie algebra is generated by $\mathfrak{g}^{\pm\beta}$, $\beta \in \Sigma_\sigma^+ \cap \langle I \rangle$. Then M_I centralizes A_I. Now

$$G = \bigcup_{w \in \mathcal{W}} KM_I A_I^+ wH \quad \text{and} \quad M_I \cap A_I = \{e\},$$

where $A_I^+ = A_I \cap A^+$.

Theorem 1.7 (Strong wavefront lemma-II): *Given $c > 0$, there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for any $I \subset \Delta_\sigma$ and $J = \Delta_\sigma \setminus I$, and every $g = kawh \in KA^+\mathcal{W}H$ and $0 < \varepsilon < \varepsilon_0$, if g is (J,c)-regular, then*

$$\mathcal{O}_\varepsilon \cdot g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (M_I \cap \mathcal{O}_{\ell\varepsilon}) \cdot (A_I \cap \mathcal{O}_{\ell\varepsilon})a \cdot w(H \cap \mathcal{O}_{\ell\varepsilon})h.$$

Remark 1.8: Observe that by [GOS, Corollary 4.7], since wv_0 is fixed by the symmetric subgroup $M_I \cap wHw^{-1}$ of M_I, the orbit $M_I(wv_0)$ is closed. Since $M_I \subset Z_G(A_I)$, we have $M_I awv_0 = aM_I wv_0$ is closed. Thus, the set $KM_I awv_0$ is closed for any $a \in A_I$. Moreover, the natural map $KM_I/(M_I \cap wHw^{-1}) \to KM_I awv_0$ given by $km(M_I \cap wHw^{-1}) \mapsto kmawv_0$ is a homeomorphism.

A natural generalization of the Cartan decomposition for Riemannian symmetric spaces is the decomposition

$$G = K\tilde{A}^+H \tag{1.9}$$

where $\tilde{A^+}$ is a Weyl chamber in A with respect to the Weyl group

$$(N_G(A) \cap K \cap H)/(Z_G(A) \cap K \cap H).$$

In Section 4, we will obtain the strong wavefront lemmas with respect to the decomposition (1.9), which generalize Theorem 1.6 and Theorem 1.7.

Well-roundedness of sectors. Let $\iota : G \to \mathrm{GL}(W)$ be an irreducible representation of G and $v_0 \in W$ such that if H denotes the stabilizer of v_0 then H is a symmetric subgroup of G. Therefore by [GOS, Corollary 4.7] the orbit $V = Gv_0$ is closed. Hence it can be realized as an affine symmetric space G/H. Let Γ be a lattice in G. We suppose that $H \cap \Gamma$ is also a lattice in H. In particular, $H\Gamma$ is closed in G, and hence Γv_0 is a discrete subset of W. For a norm $\|\cdot\|$ on W, we set

$$B_T = \{w \in W : \|w\| < T\}.$$

It was shown in [DRS, EM] that the orbit Γv_0 is "equidistributed" with respect to the sets $V \cap B_T$ in the following sense:

$$\#(\Gamma v_0 \cap B_T) \sim_{T\to\infty} \mathrm{Vol}(V \cap B_T) \tag{1.10}$$

where Vol is the G-invariant measure on $V \cong G/H$ determined by the Haar measures on G and H chosen such that $\mathrm{Vol}(G/G\cap\Gamma) = \mathrm{Vol}(H/H\cap\Gamma) = 1$. In fact, it was shown in [EM] that (1.10) holds for any *well-rounded* family of sets $S_T \subset V$ in place of $V \cap B_T$. Recall that a family $\{S_T\}$ is called well-rounded if for any $\varepsilon > 0$ there exists a neighborhood $\mathcal{O}$ of e in G such that

$$\frac{\mathrm{Vol}(\mathcal{O}\cdot\partial S_T)}{\mathrm{Vol}(S_T)} < \varepsilon \tag{1.11}$$

for all sufficiently large $T > 0$.

For any $I \subset \Delta_\sigma$, $w \in \mathcal{W}$ and $\Omega \subset KM_I/(M_I \cap wHw^{-1})$, we consider a family of sets

$$S_T(\Omega, w) = \tilde{\Omega} A_I^+ w v_0 \cap B_T, \tag{1.12}$$

where $\tilde{\Omega} \subset KM_I$ is such that $\Omega = \tilde{\Omega}(M_I \cap wHw^{-1})$; the set $S_T(\Omega, w)$ is well defined because $mawv_0 = awv_0$ for all $a \in A_I$ and $m \in (M_I \cap wHw^{-1})$.

Using the strong wavefront lemma and the volume computation in [GOS] (cf. Proposition 3.8) we obtain the following:

Theorem 1.13: *For every $I \subset \Delta_\sigma$, $w \in \mathcal{W}$, and a bounded measurable set $\Omega \subset KM_I/(M_I \cap wHw^{-1})$ with positive measure and boundary of measure zero,*[2] *the family $\{S_T(\Omega, w)\}_{T\to\infty}$ is well-rounded. In particular,*

$$\#(\Gamma v_0 \cap S_T(\Omega, w)) \sim_{T\to\infty} \mathrm{Vol}(S_T(\Omega, w)) \sim_{T\to\infty} C_I(\Omega, w)\cdot T^{a_I}(\log T)^{b_I - 1},$$

[2] The measure on $KM_I/(M_I \cap wHw^{-1})$ is understood in terms of the identification (3.2) and (3.3).

where $a_I \in \mathbb{Q}^+$, $b_I \in \mathbb{N}$, *and* $C_I(\Omega, w) > 0$.

We will give explicit formulas for a_I, b_I, and $C_I(\Omega, w)$ in section 3.2. In particular, $C_I(\Omega, w)$ can be computed using a G-invariant measure supported on one of the components of the Satake boundary of V.

Remark 1.14: (1) Although a similar counting question was considered in [GOS], the sets $S_T(\Omega, w)$ do not fit into the framework of [GOS]. For the space of quadratic forms $\mathcal{Q}_W$, the counting results in [GOS] are always of order $T^{(\dim W)(\dim W-1)/2}$ (see [GOS, Section 2.3]). On the other hand, Theorem 1.4 exhibits different asymptotic behaviors depending on the choice of the decomposition (1.2).

(2) In order to deduce Theorem 1.13 from Theorem 1.7, which applies only to (J, c)-regular elements, we show that the set of non-(J, c)-regular elements in $S_T(\Omega, w)$ has negligible volume compared to the volume of $S_T(\Omega, w)$ for sufficiently small values of c.

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2. Strong wavefront lemma

This section is devoted to the proofs of Theorems 1.6 and 1.7. We use the same notation as in the introduction. Since any two Riemannian metrics are bi-Lipschitz in a neighborhood of identity, it suffices to prove the theorems for one such metric. It will be convenient to work with the right-invariant Riemannian metric d induced by the positive definite form

$$B(X, Y) = -\operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad}(\theta(Y))), \quad X, Y \in \mathfrak{g}.$$

We will use the following properties of B:

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{for all } \alpha \neq \beta \in \Sigma_\sigma \cup \{0\},$$
$$B^\theta = B^\sigma = B.$$

Remark 2.1: In many of the results stated in the introduction, we fix $w \in \mathcal{W}$ representing a Weyl group element. The explanation given below shows that for proofs, we can assume that $w = e$ and have simpler notation.

Let i_w denote the inner conjugation on G by w; that is, $i_w(g) = wgw^{-1}$ for all $g \in G$. Then $\sigma_w := i_w \circ \sigma \circ i_w^{-1}$ is also an involution of G and wHw^{-1} is the associated symmetric subgroup. Note that $\sigma_w(a) = a^{-1}$ for any $a \in A$. Also $\theta \circ \sigma_w = \sigma_w \circ \theta$. Therefore, in order to prove some of the results stated in the introduction for a fixed $w \in \mathcal{W}$, we can replace σ by σ_w, H by wHw^{-1}, and v_0 by wv_0, and assume that $w = e$.

For $\varepsilon > 0$ and $S \subset G$, we set

$$S_\varepsilon = \{s \in S : d(s, e) < \varepsilon\}.$$

For $I \subset \Delta_\sigma$ and $c > 0$, we define

$$A_I^+(c) = \{a \in A^+ : \beta(\log a) \geq c \text{ if } \beta \in \Delta_\sigma - I \text{ and } \beta(\log a) < c \text{ if } \beta \in I\}.$$

We also set $\mathfrak{a}_I = \ker(I) \subset \mathfrak{a}$ and denote by Z_I the centralizer of $\mathfrak{a}_I$ in G.

THEOREM 2.2: *For $I \subset \Delta_\sigma$ and $c > 0$, there exist $\varepsilon_0 > 0$ and $\ell_1 > 1$ such that for every $0 < \varepsilon < \varepsilon_0$ and $a \in A_I^+(c)$,*

$$G_\varepsilon \cdot a \subset K_{\ell_1 \varepsilon} \cdot Z_{I, \ell_1 \varepsilon} \cdot a \cdot H_{\ell_1 \varepsilon}.$$

We consider the Lie subalgebra

$$\mathfrak{n}_I^+ = \bigoplus_{\beta \in \Sigma_\sigma^+ : \beta|_{\mathfrak{a}_I} \neq 0} \mathfrak{g}_\beta \quad \text{and} \quad \mathfrak{n}_I^- = \bigoplus_{\beta \in \Sigma_\sigma^+ : \beta|_{\mathfrak{a}_I} \neq 0} \mathfrak{g}_{-\beta},$$

and the corresponding analytic subgroups N_I^+ and N_I^-. Note that the Lie algebra of Z_I is given by

$$\mathfrak{z}_I = \bigoplus_{\beta \in \Sigma_\sigma \cup \{0\} :, \beta|_{\mathfrak{a}_I} = 0} \mathfrak{g}_\beta,$$

and we have the decomposition

$$\mathfrak{g} = \mathfrak{n}_I^- \oplus \mathfrak{z}_I \oplus \mathfrak{n}_I^+. \tag{2.3}$$

LEMMA 2.4: *There exist $\ell_2 > 1$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,*

$$G_\varepsilon \subset N_{I, \ell_2 \varepsilon}^- Z_{I, \ell_2 \varepsilon} H_{\ell_2 \varepsilon} \quad \text{and} \quad G_\varepsilon \subset K_{\ell_2 \varepsilon} Z_{I, \ell_2 \varepsilon} N_{I, \ell_2 \varepsilon}^+.$$

Proof. Since $\sigma|_{\mathfrak{a}} = -id$, we have $\sigma(\mathfrak{n}_I^-) \subset \mathfrak{n}_I^+$, and for every $x \in \mathfrak{n}_I^+$,

$$x = (x + \sigma(x)) - \sigma(x) \in \mathfrak{h} + \mathfrak{n}_I^-.$$

Hence, it follows from (2.3) that

$$\mathfrak{g} = \mathfrak{n}_I^- + \mathfrak{z}_I + \mathfrak{h}.$$

Since $\mathfrak{n}_I^- \cap \mathfrak{h} = 0$, there exists a subspace $\mathfrak{z}_0$ of $\mathfrak{z}_I$ such that

$$\mathfrak{g} = \mathfrak{n}_I^- \oplus \mathfrak{z}_0 \oplus \mathfrak{h}.$$

Then the product map $N_I^- \times \exp(\mathfrak{z}_0) \times H \to G$ is a diffeomorphism at a neighborhood of the identity. In particular, it is bi-Lipschitz, and the first claim follows. The proof of the second claim is similar. ∎

LEMMA 2.5: *For $I \subset \Delta_\sigma$ and $c > 0$, there exist $\varepsilon_0 > 0$ and $\ell_3 \in (0,1)$ such that for every $0 < \varepsilon < \varepsilon_0$ and $a \in A_I^+(c)$,*

$$a^{-1} N_{I,\varepsilon}^+ a \subset N_{I,\ell_3\varepsilon}^+ \quad \text{and} \quad a N_{I,\varepsilon}^- a^{-1} \subset N_{I,\ell_3\varepsilon}^-.$$

Proof. For

$$X = \sum_{\beta \in \Sigma_\sigma^+, \beta|_{\mathfrak{a}_I} \neq 0} X_\beta \in \mathfrak{n}_I^+, \quad X_\beta \in \mathfrak{g}_\beta,$$

we have

$$\mathrm{Ad}(a^{-1})X = \sum_\beta \mathrm{Ad}(a^{-1})X_\beta = \sum_\beta e^{-\beta(\log a)} X_\beta.$$

Note that if $\beta = \sum_{\alpha \in \Delta_\sigma} n_\alpha \alpha \in \Sigma_\sigma^+$ with $n_\alpha \geq 0$ satisfies $\beta|_{\mathfrak{a}_I} \neq 0$, then $n_\alpha \geq 1$ for some $\alpha \in \Delta_\sigma - I$. Hence, for $a \in A_I^+(c)$, we have $\beta(\log a) \geq c$ and

$$\|\mathrm{Ad}(a^{-1})X_\beta\| \leq e^{-c}\|X_\beta\|.$$

Since the root spaces $\mathfrak{g}_\beta$ are orthogonal to each other,

$$\|\mathrm{Ad}(a^{-1})X\| \leq e^{-c}\|X\|. \tag{2.6}$$

Since the differential of the exponential map $\exp : \mathfrak{n}_I^+ \to N_I^+$ is identity at 0, we can find a small ball U at 0 in $\mathfrak{n}_I^+$ such that for every $Y \in U$,

$$e^{-c/3}\|Y\| \leq d(\exp(Y), e) \leq e^{c/3}\|Y\|. \tag{2.7}$$

Note that for $a \in A^+$, we have $\mathrm{Ad}(a^{-1})U \subset U$. Combining (2.6) and (2.7), we deduce that for $a \in A_I^+(c)$ and $n = \exp(X) \in \exp(U)$,

$$\begin{aligned} d(a^{-1}na, e) &= d(\exp(\mathrm{Ad}(a^{-1})X), e) \leq e^{c/3}\|\mathrm{Ad}(a^{-1})X\| \\ &\leq e^{-2c/3}\|X\| \leq e^{-c/3} d(n,e). \end{aligned}$$

This proves the claim for N_I^+. The claim for N_I^- is proved similarly. ∎

LEMMA 2.8: *For $I \subset \Delta_\sigma$ and $\ell_4 > 1$, there exists $\varepsilon_0 > 0$ such that for every $z \in Z_{I,\varepsilon_0}$ and $0 < \varepsilon < \varepsilon_0$,*

$$zN^+_{I,\varepsilon}z^{-1} \subset N^+_{I,\ell_4\varepsilon} \quad \text{and} \quad zN^-_{I,\varepsilon}z^{-1} \subset N^-_{I,\ell_4\varepsilon}.$$

Proof. It is easy to check that Z_I normalizes $N_I^{\pm}$.

We can choose $\varepsilon_0 > 0$ so that

$$\begin{aligned} \|\mathrm{Ad}(z)X\| &\le \ell_4^{1/3}\|X\|, & z \in Z_{I,\varepsilon_0},\ X \in \mathfrak{n}_I^+, \\ \ell_4^{-1/3}\|X\| \le d(\exp(X), e) &\le \ell_4^{1/3}\|X\|, & X \in \mathrm{Ad}(Z_{I,\varepsilon_0})\exp^{-1}(N^+_{I,\varepsilon_0}). \end{aligned}$$

Then for every $n = \exp(X) \in N^+_{I,\varepsilon_0}$,

$$\begin{aligned} d(znz^{-1}, e) = d(\exp(\mathrm{Ad}(z)X), e) &\le \ell_4^{1/3}\|\mathrm{Ad}(z)X\| \\ &\le \ell_4^{2/3}\|X\| \le \ell_4 d(n, e). \end{aligned}$$

This proves the first part of the lemma. The proof of the second part is similar. ∎

LEMMA 2.9: *For $I \subset \Delta_\sigma$ and $\ell_5 > 1$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$,*

$$N^+_{I,\varepsilon} \subset N^-_{I,\ell_5\varepsilon} Z_{I,\varepsilon} H_{2\ell_5\varepsilon} \quad \text{and} \quad N^-_{I,\varepsilon} \subset K_{2\ell_5\varepsilon} Z_{I,\varepsilon} N^+_{I,\ell_5\varepsilon}.$$

Proof. As in the proof of Lemma 2.4, we choose a subspace $\mathfrak{z}_0$ of $\mathfrak{z}_I$ such that the product map $N_I^- \times \exp(\mathfrak{z}_0) \times H \to G$ is a diffeomorphism in a neighborhood of the identity. Denote by f the local inverse of the product map:

$$f = (f_1, f_2, f_3) : U \to N_I^- \times \exp(\mathfrak{z}_0) \times H$$

where U is a neighborhood of identity in G. For $X \in \mathfrak{n}_I^+$, the derivative $(df)_e$ is given by

$$(df)_e(X) = (-\sigma(X), 0, X + \sigma(X)) \in \mathfrak{n}_I^- \oplus \mathfrak{z}_0 \oplus \mathfrak{h}.$$

Since the Riemannian metric at identity is invariant under σ, we have for $X \in \mathfrak{n}_I^+$,

$$\|(df_1)_e(X)\| = \|X\|, \quad (df_2)_e = 0, \quad \|(df_3)_e(X)\| \le 2\|X\|.$$

This implies that for sufficiently small $\varepsilon > 0$,

$$f(N^+_{I,\varepsilon}) \subset N^-_{I,\ell_5\varepsilon} \times Z_{I,\varepsilon} \times H_{2\ell_5\varepsilon}.$$

This proves the first claim. The proof of the second claim is similar. ∎

Lemma 2.10: *For $I \subset \Delta_\sigma$ and $c > 0$, there exist $0 < \ell_6 < 1$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon, \delta < \varepsilon_0$ and $a \in A_I^+(c)$,*

$$K_\varepsilon Z_{I,\varepsilon} a Z_{I,\varepsilon} N_{I,\delta}^+ H_\varepsilon \subset K_{\varepsilon+4\delta} Z_{I,\varepsilon+4\delta} a Z_{I,\varepsilon+4\delta} N_{I,\ell_6\delta}^+ H_{\varepsilon+4\delta}.$$

Proof. For simplicity, we write $N_I^\pm = N^\pm$ and $Z_I = Z$.

Choose $\ell_3 = \ell_3(c) \in (0,1)$ as in Lemma 2.5, $\ell_5 \in (1,2)$ so that $\ell_3\ell_5^2 < 1$, and $\ell_4 > 1$ so that $\ell_4^5\ell_3\ell_5^2 < 1$. Let $\varepsilon_0 > 0$ be such that Lemma 2.5, Lemma 2.8, and Lemma 2.9 hold. Fixing $0 < \varepsilon < \varepsilon_0$, let $k_0 \in K_\varepsilon$, $x_0, y_0 \in Z_\varepsilon$, $n_0^+ \in N_\delta^+$, and $h_0 \in H_\varepsilon$. Then

$$\begin{aligned}
& k_0x_0ay_0n_0^+h_0 \\
&= k_0x_0ay_0(n_1^-y_1h_1)h_0 && \text{by Lemma 2.9} \\
& && \text{with } n_1^- \in N_{\ell_5\delta}^-, y_1 \in Z_\delta, h_1 \in H_{2\ell_5\delta} \\
&= k_0n_2^-x_0ay_0y_1h_1h_0 && \text{by Lemma 2.8 and Lemma 2.5} \\
& && \text{with } n_2^- \in N_{\ell_4^2\ell_3\ell_5\delta}^- \\
&= k_0(k_2x_2n_2^+)x_0ay_0y_1h_1h_0 && \text{by Lemma 2.9,} \\
& && \text{with } k_2 \in K_{2\ell_4^2\ell_3\ell_5^2\delta}, x_2 \in Z_{\ell_4^2\ell_3\ell_5\delta}, n_2^+ \in N_{\ell_4^2\ell_3\ell_5^2\delta}^+ \\
&= k_0k_2(x_2x_0ay_0y_1)n_3^+h_1h_0 && \text{by Lemma 2.8 and Lemma 2.5} \\
& && \text{with } n_3^+ \in N_{\ell_4^5\ell_3^2\ell_5^2\delta}^+.
\end{aligned}$$

Since $\ell_4^5\ell_3\ell_5^2 < 1$, we have

$$k_0k_2 \in K_{\varepsilon+4\delta}, \;\; x_2x_0, y_0y_1 \in Z_{\varepsilon+4\delta}, \;\; n_3^+ \in N_{\ell_6\delta}^+, \;\; h_1h_0 \in H_{\varepsilon+4\delta},$$

where $\ell_6 = \ell_4^5\ell_3^2\ell_5^2 < 1$. ∎

Proof of Theorem 2.2. Set $N_I^\pm = N^\pm$ and $Z_I = Z$ for simplicity. In view of Remark 2.1, without loss of generality we may assume that $w = e$.

We choose $\varepsilon_0 > 0$ so that Lemma 2.4 (for some $\ell_2 > 1$), Lemma 2.5, and Lemma 2.10 hold. Because of Lemma 2.4, it suffices to show that

$$K_\varepsilon Z_\varepsilon N_\varepsilon^+ \cdot a \subset K_{\ell_1\varepsilon}(Z_{\ell_1\varepsilon}a)H_{\ell_1\varepsilon}$$

for some $\ell_1 > 1$. Also by Lemma 2.5,

$$K_\varepsilon Z_\varepsilon N_\varepsilon^+ \cdot a \subset K_\varepsilon(Z_\varepsilon a Z_\varepsilon)N_\varepsilon^+ H_\varepsilon.$$

Now we can apply Lemma 2.10 inductively. We consider $\varepsilon > 0$ such that

$$\varepsilon + \frac{4\varepsilon}{1-\ell_6} < \varepsilon_0. \tag{2.11}$$

Setting $\varepsilon_1 = \delta_1 = \varepsilon$, we apply Lemma 2.10 to find

$$\varepsilon_{i+1} < \varepsilon_i + 4\delta_i, \quad \delta_{i+1} < \ell_6\delta_i$$

such that for every $a \in A_I^+(c)$,

$$K_{\varepsilon_i} Z_{\varepsilon_i} a Z_{\varepsilon_i} N^+_{\delta_i} H_{\varepsilon_i} \subset K_{\varepsilon_{i+1}} Z_{\varepsilon_{i+1}} a Z_{\varepsilon_{i+1}} N^+_{\delta_{i+1}} H_{\varepsilon_{i+1}}.$$

Note that

$$\delta_i < \varepsilon\ell_6^i \quad \text{and} \quad \varepsilon_i < \varepsilon + 4\varepsilon\frac{1-\ell_6^i}{1-\ell_6}.$$

Hence by (2.11), $\varepsilon_i, \delta_i < \varepsilon_0$, and we can continue this process indefinitely.

It follows that for every $g \in K_\varepsilon(Z_\varepsilon a Z_\varepsilon)N_\varepsilon^+ H_\varepsilon$, there exist sequences $k_i \in K_{\varepsilon_i}, x_i, y_i \in Z_{\varepsilon_i}, n_i \in N^+_{\delta_i}, h_i \in H_{\varepsilon_i}$ such that $g = k_i x_i a y_i n_i h_i$ for all $i \geq 1$. Since $\delta_i \to 0$, $n_i \to e$. Also, passing to a subsequence, we may assume that $k_i \to k, x_i \to x, y_i \to y, h_i \in h$. Then

$$g = kxayh \subset K_{\ell_7\varepsilon} Z_{\ell_7\varepsilon} a Z_{\ell_7\varepsilon} H_{\ell_7\varepsilon}$$

with $\ell_7 = 1 + 4(1-\ell_6)^{-1}$. We have decomposition $a = a_1 a_2$ where $a_1 \in A_I^+$ and a_2 is in the fixed compact set determined by c. This implies that for some $\ell' > 1$,

$$a Z_{\ell_7\varepsilon} a^{-1} \subset Z_{\ell'\ell_7\varepsilon},$$

and the theorem follows. ∎

Proof of Theorem 1.7. There exists $\ell' > 1$ such that $k^{-1}\mathcal{O}_\varepsilon k \subset \mathcal{O}_{\ell'\varepsilon}$ for every $k \in K$. Then for $g = kawh \in KA^+\mathcal{W}H$, we have

$$\mathcal{O}_\varepsilon \cdot g \subset k(\mathcal{O}_{\ell'\varepsilon} a)wh.$$

Due to Remark 2.1, without loss of generality, we may assume that $w = e$.

Since $M_{I_1} \subset M_{I_2}$ for $I_1 \subset I_2$, we may assume that J is maximal such that a is (J, c)-regular. Then $a \in A_I^+(c)$. We have the decomposition

$$\mathfrak{z}_I = (\mathfrak{z}_I \cap \mathfrak{k}) \oplus (\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q}) \oplus \mathfrak{a}_I \oplus (\mathfrak{z}_I \cap \mathfrak{h}) \tag{2.12}$$

(see [GOS, equation (4.24)]). Hence, the product map

$$(Z_I \cap K) \times \exp(\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q}) \times A_I \times (Z_I \cap H) \to Z_I$$

is a diffeomorphism in a neighborhood of identity, and there exists $\ell'' > 1$ such that for sufficiently small $\varepsilon > 0$,

$$Z_{I,\varepsilon} \subset (Z_I \cap K)_{\ell''\varepsilon} \exp(\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q})_{\ell''\varepsilon} A_{I,\ell''\varepsilon} (Z_I \cap H)_{\ell''\varepsilon}.$$

Therefore, it follows from Theorem 2.2 that

$$\mathcal{O}_\varepsilon \cdot a \subset K_{\ell_1\varepsilon} Z_{\ell_1\varepsilon} a H_{\ell_1\varepsilon} \subset K_{(\ell_1+\ell_1\ell'')\varepsilon} M_{I,\ell_1\ell''\varepsilon} (A_{I,\ell_1\ell''\varepsilon} a) H_{(\ell_1+\ell_1\ell'')\varepsilon}.$$

This proves the theorem. ∎

Proof of Theorem 1.6. Suppose that in Theorem 1.7 we have $J = \Delta_\sigma$. Then $Z = C_G(A)$ is σ- and θ-invariant, and

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}) \oplus (\mathfrak{z} \cap \mathfrak{p} \cap \mathfrak{q}) \oplus (\mathfrak{z} \cap \mathfrak{h}).$$

Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, $\mathfrak{z}\cap\mathfrak{p}\cap\mathfrak{q} = \mathfrak{a}$. Hence, decomposition (2.12) becomes

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}) \oplus \mathfrak{a} \oplus (\mathfrak{z} \cap \mathfrak{h}),$$

and we complete the proof as in Theorem 1.7. ∎

3. Well-roundedness of sectors $S_T(\Omega, w)$

First we need a precise description of the measure on the set

$$KM_I(wv_0) \cong KM_I/(M_I \cap wHw^{-1}).$$

3.1. Description of a measure on $KM_I/(M_I \cap wHw^{-1})$. Fix $w \in \mathcal{W}$. Let $\sigma_w = i_w \circ \sigma \circ i_w^{-1}$ be the involution as in Remark 2.1. Then $\sigma_w \circ \theta = \theta \circ \sigma_w$. Also, the semisimple group M_I is stable under σ_w and θ, and hence M_I admits the generalized Cartan decomposition (see [GOS, Proposition 4.22]):

$$M_I = (M_I \cap K) A^I (M_I \cap wHw^{-1}) = (M_I \cap K) A^{I,+} \mathcal{W}_I (M_I \cap wHw^{-1}), \tag{3.1}$$

where A^I is the orthogonal complement of A_I in A and it is the Cartan subalgebra of M_I associated to the symmetric pair $(M_I \cap K, M_I \cap wHw^{-1})$, and $A^{I,+} = \{a \in A^I : \alpha(\log a) \geq 0, \forall \alpha \in I\}$ is a positive Weyl chamber; and $\mathcal{W}_I \subset M_I$ is a set of representatives of the associated Weyl group, which is generated by the reflections $\{s_\alpha\}_{\alpha \in I}$. An invariant measure, say λ on $M_I/(M_I \cap wHw^{-1})$, is given as follows: for any $f \in C_c(M_I/M_I \cap wHw^{-1})$,

$$\int f d\lambda = \sum_{w_1 \in \mathcal{W}_I} \int_{K \cap M_I} dk \int_{A^{I,+}} f(kaw_1(M_I \cap wHw^{-1}))\delta_I(a)\, da$$

where

$$\delta_I(a) = \prod_{\alpha\in\Sigma_\sigma^+\cap\langle I\rangle} (\sinh\alpha(\log a))^{l_\alpha^+}(\cosh\alpha(\log a))^{l_\alpha^-},$$

and $l_\alpha^\pm$ denote the dimensions of the (± 1)-eigenspaces of $\sigma\theta$ on $\mathfrak{g}^\alpha$.

Therefore we can identify

$$KM_I/(M_I\cap wHw^{-1}) \cong K\times A^{I,+}\times \mathcal{W}_I, \tag{3.2}$$

and treat $KM_I/(M_I\cap wHw^{-1})$ as a product measure space.

On the other hand, once we fix a measurable section $s_1 : K/(K\cap M_I)\to K$ for the natural quotient map, we can identify $K\times A^{I,+}\times\mathcal{W}_I$ with

$$K/(K\cap M_I)\times M_I/(M_I\cap wHw^{-1}).$$

We consider the measure on $K\times A^{I,+}\times\mathcal{W}_I$ such that it corresponds to the product of the invariant measures on the product space

$$K/(K\cap M_I)\times M_I/(M_I\cap wHw^{-1}),$$

where the Haar measures on K and $K\cap M_I$ are normalized. This measure, in view of (3.2), will give rise to the integral $d\bar{m}$ on

$$KM_I/(M_I\cap wHw^{-1})$$

given as follows: for any $f\in C_c(KM_I/M_I\cap wHw^{-1})$,

$$\int f(\bar{m})d\bar{m} := \sum_{w_1\in\mathcal{W}_I}\int_K dk\int_{A^{I,+}} f(kaw_1(M_I\cap wHw^{-1}))\delta_I(a)\,da. \tag{3.3}$$

3.2. Volume estimate for the sectors $S_T(\Omega,w)$. Let λ_ι denote the highest weight for the irreducible representation ι. We express

$$\lambda_\iota = \sum_{\alpha\in\Delta_\sigma} m_\alpha\alpha \tag{3.4}$$

and the sum of positive roots (with multiplicities)

$$2\rho = \sum_{\alpha\in\Delta_\sigma} u_\alpha\alpha. \tag{3.5}$$

Let $I\subset\Delta_\sigma$. Set

$$a_I = \max\left\{\frac{u_\alpha}{m_\alpha} : \alpha\in\Delta_\sigma - I\right\}, \tag{3.6}$$

$$b_I = \#\left\{\alpha\in\Delta_\sigma - I : \frac{u_\alpha}{m_\alpha} = a_I\right\}. \tag{3.7}$$

PROPOSITION 3.8: *For any $w \in \mathcal{W}$ and a bounded measurable set*

$$\Omega \subset KM_I/(M_I \cap wHw^{-1})$$

with positive measure and zero boundary measure, there exists $C_I(\Omega, w) > 0$ such that

$$\mathrm{Vol}(S_T(\Omega, w)) \sim_{T\to\infty} C_I(\Omega, w) \cdot T^{a_I} (\log T)^{b_I - 1}.$$

Proof. From [HS, Theorem 2.5] (see also [GOS]) one deduces that a G-invariant measure on G/H is given by

$$(3.9)\quad \int_{G/H} f\, d\mu = \sum_{w\in\mathcal{W}} \int_{\bar{m}\in KM_I/(M_I\cap wHw^{-1})} \int_{a\in A_I^+} f(\bar{m}awH)\xi_I(a)\, da d\bar{m}, f \in C_c(G/H),$$

where da denotes a Haar measure on A_I, and $d\bar{m}$ is described in the paragraph following (3.2), and

$$(3.10)\qquad \xi_I(a) = \prod_{\alpha\in\Sigma_\sigma^+ - \langle I\rangle} \sinh(\alpha(\log a))^{l_\alpha^+} \cosh(\alpha(\log a))^{l_\alpha^-}.$$

Here $l_\alpha^\pm$ denote the dimensions of the (± 1)-eigenspaces of $\sigma\theta$ in $\mathfrak{g}_\alpha$. We decompose ξ_I as a linear combination of functions $\exp(\chi(a))$ where χ's are characters of A_I. Note that 2ρ is the maximal character in this decomposition. In view of equations (3.4), (3.5), and (3.6), we define

$$I_0 = I \cup \{\alpha \in \Delta_\sigma - I : \frac{u_\alpha}{m_\alpha} < a_I\}.$$

By the computation using [GOS, Theorem 6.1], as done in the proof of [GOS, Theorem 6.4], applied to $\mathfrak{a}_I$ in place of $\mathfrak{a}$, there exists a locally finite measure $\eta_{I,w}$ on W such that for every $f \in C_c(W)$,

$$(3.11)\qquad \lim_{T\to\infty} \frac{1}{T^{a_I}(\log T)^{b_I-1}} \int_{a\in A_I^+} f(awv_0/T)\xi_I(a)\, da = \int_W f\, d\eta_{I,w},$$

where the measure $\eta_{I,w}$ can be described as follows:

$$(3.12)\qquad \int_W f\, d\eta_{I,w} = \int_{\bar{b}\in D^+} f(b(wv_0)^{I_0})\, \tilde{\xi}_I(b)\, d\bar{b},$$

where $D^+ = \exp \mathfrak{d}^+$,

$$\mathfrak{d}^+ = \{\bar{b} \in \mathfrak{a}_I/(\mathfrak{a}_{I_0} \cap \ker\rho) : \alpha(b) \geq 0,\ \forall\alpha \in I_0\},$$

$d\bar{b}$ denotes the Haar measure on $A_I/(A_I \cap \exp(\ker \rho))$, $v_0^{I_0}$ is the projection of v_0 to the sum of the weight spaces with weights of the form $\lambda_\iota - \sum_{\alpha \in I_0} m_\alpha \alpha$, $m_\alpha \geq 0$, and

$$\tilde{\xi}_I(b) = \Bigg(\prod_{\alpha \in (\Sigma_\sigma^+ \cap \langle I_0 \rangle) - \langle I \rangle} \sinh(\alpha(\log b))^{l_\alpha^+} \cosh(\alpha(\log b))^{l_\alpha^-} \Bigg) \times \exp\Bigg(\sum_{\alpha \in \Sigma_\sigma^+ - \langle I_0 \rangle} u_\alpha \alpha(\log b) \Bigg). \tag{3.13}$$

Moreover, it follows from (3.11) that $\eta_{I,w}$ is a homogeneous measure of degree a_I.

Fix any $m \in KM$. Let $c > 1$ and take a continuous function $\psi : [0, \infty] \to [0, 1]$ such that $\mathrm{supp}(\psi) \subset [0, c]$ and $\psi = 1$ on $[0, 1]$. Setting $f(y) = \psi(\|my\|)$, we have

$$\int_{A_I^+} \chi_{B_T}(mav_0)\xi_I(a)da \leq \int_{A_I^+} f(awv_0/T)\xi_I(a)da. \tag{3.14}$$

Now by (3.11) and (3.14),

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T^{a_I}(\log T)^{b_I-1}} \int_{A_I^+} \chi_{B_T}(mawv_0)\xi_I(a)da \leq& \int_W f\, d\eta_{I,w} \\ \leq& c^{a_I}\eta_{I,w}(m^{-1}B_1).\end{aligned}$$

The lower estimate for lim inf is proved similarly.

Hence, taking $c \to 1^+$, we obtain

$$\lim_{T\to\infty} \frac{1}{T^{a_I}(\log T)^{b_I-1}} \int_{A_I^+} \chi_{B_T}(mawv_0)\xi_I(a)da = \eta_{I,w}(m^{-1}B_1). \tag{3.15}$$

In view of (3.2) let $s : KM_I/(M_I \cap wHw^{-1}) \to KM_I$ denote the measurable section of the obvious quotient map. Since

$$S_T(\Omega, w) = \Omega A_I^+ wv_0 \cap B_T,$$

$$\mathrm{Vol}(S_T(\Omega, w)) = \int_{\bar{m}\in\Omega} \int_{a \in A_I^+} \chi_{B_T}(s(\bar{m})awv_0)\xi(a)\, da d\bar{m}. \tag{3.16}$$

Therefore from (3.15), using the dominated convergence theorem, we deduce that

$$C_I(\Omega, w) := \lim_{T\to\infty} \frac{\mathrm{Vol}(S_T(\Omega, w))}{T^{a_I}(\log T)^{b_I-1}} = \int_{\bar{m}\in\Omega} \eta_{I,w}(s(\bar{m})^{-1}B_1)d\bar{m}. \tag{3.17}$$

Note that there exists $\delta > 0$ such that $s(\bar{m})^{-1}B_1 \supset B_\delta$ for all $\bar{m} \in \Omega$, and because η_{I,w_0} is homogeneous, $\eta_{I,w}(B_\delta) > 0$. Hence $C_I(\Omega, w) > 0$. ∎

Remark 3.18: The value of the parameter $C_I(\Omega, w)$ in the statement of Proposition 3.8 is given by

$$C_I(\Omega, w) = \nu_{I_0,w}(B_1 \cap \Omega D^+(wv_0)^{I_0}), \tag{3.19}$$

where $\nu_{I_0,w}$ is a G-invariant measure on the G-orbit $G(wv_0)^{I_0}$.

This formula can be justified as follows: combining (3.11), (3.12), (3.15), (3.17) and (3.3) we get

$$\begin{aligned} C_I(\Omega, w) &= \int_{\bar{m}\in\Omega} d\bar{m} \int_{\bar{b}\in D^+} \chi_{B_1}(\bar{m}\bar{b}(wv_0)^{I_0})\xi_I(b)\, d\bar{b} \\ &= \int_{k\in K} dk \int_{a\in A^{I,+}} \int_{\bar{b}\in D^+} \chi_\Omega(ka)\chi_{B_1}(kab(wv_0)^{I_0})\delta_I(a)\tilde{\xi}_I(b)\, da d\bar{b}, \end{aligned} \tag{3.20}$$

where

$$\delta_I(a) = \prod_{\alpha\in(\Sigma_\sigma^+\cap\langle I\rangle)} \sinh(\alpha(\log a))^{l_\alpha^+} \cosh(\alpha(\log a))^{l_\alpha^-}; \tag{3.21}$$

$l_\alpha^\pm$ are the dimensions of the (± 1)-eigenspaces of $\sigma\theta$ acting on $\mathfrak{g}^\alpha$.

Since

$$\mathfrak{a}_{I_0} \cap \ker \rho = \mathfrak{a}_{I_0} \cap \ker \lambda_\iota,$$

it follows from [GOS, Theorem 5.1] that the orbit $G(wv_0)^{I_0}$ supports a G-invariant measure ν_{I_0}. Now comparing the formula (3.20) with the formula (5.3) in [GOS, Theorem 5.1], we obtain (3.19).

3.2.1. *Upper estimate of volume for (J, c)-singular elements in $S_T(\Omega, w)$.* For $c > 0$, $I \subset \Delta_\sigma$, and a bounded measurable $\Omega \subset KM_I$, we set

$$V_{I,w}(c) = \left\{ mawv_0 : m \in \Omega,\, a \in A_I^+ \text{ with } \alpha(\log a) \le c \text{ for some } \alpha \in \Delta_\sigma - I \right\}.$$

Note that this set is the set of (J, c)-singular elements for $J = \Delta_\sigma \setminus I$.

Proposition 3.22: *For small $c > 0$ and sufficiently large $T > 0$,*

$$\mathrm{Vol}(V_{I,w}(c) \cap B_T) \ll c \cdot T^{a_I}(\log T)^{b_I-1}.$$

Proof. For $\alpha \in \Delta_\sigma$, set

$$U_c(\alpha) = \{a \in A_I^+ : \alpha(\log a) \le c\}.$$

There exists $\delta > 1$ such that $m^{-1}B_T \subset B_{\delta T}$ for all $T > 0$. By (3.9), this gives the estimate

$$\operatorname{Vol}(V_{I,w}(c) \cap B_T) \ll \sum_{\alpha \in \Delta_\sigma - I} \int_{a \in A_I^+ \cap U_c(\alpha) : \|av_0\| < \delta T} \xi_I(a)da. \tag{3.23}$$

Now we use the volume computation from [GOS] (see the proof of Theorem 6.4 in [GOS]) to show for every nonnegative $f \in C_c(W)$,

$$\int_{A_I^+ \cap U_c(\alpha)} f(av_0/T)\xi_I(a)da \ll \left(\int_{A_I^+ \cap U_c(\alpha)} f(av^{I_0})\tilde{\xi}_I(a)da \right) \cdot T^{a_I}(\log T)^{b_I - 1},$$

where $I \subset I_0 \subset \Delta_\sigma$, $v^{I_0} \in W$ and $\tilde{\xi}_I \in C(A^+)$ are as defined in section 3.2. By [GOS, Corollary 4.7] the projection of $v_0^{I_0}$ on the λ_ι-eigenspace is nonzero, and the map $A^+ \to \mathbb{R} : a \mapsto \lambda_\iota(a)$ is proper. Therefore the map $A_I^+ \to W : a \mapsto av_0^{I_0}$ is proper. This implies that there exists a compact $L \subset A_I^+$ such that

$$L \supset \{a \in A_I^+ : av_0^{I_0} \in \operatorname{supp} f\}.$$

Then

$$\begin{aligned}\int_{A_I^+ \cap U_c(\alpha)} f(av_0/T)\xi_I(a)da &\ll \max(f) \cdot \operatorname{Vol}(L \cap U_c(\alpha)) \cdot T^{a_I}(\log T)^{b_I - 1} \\ &\ll_f c \cdot T^{a_I}(\log T)^{b_I - 1}.\end{aligned}$$

Taking a function f satisfying $\chi_{B_1} \le f$, we obtain

$$\begin{aligned}\int_{a \in A_I^+ \cap U_c(\alpha) : \|av_0\| < T} \xi_I(a)da &\ll \left(\int_{A_I^+ \cap U_c(\alpha)} f(av)\tilde{\xi}_I(a)da \right) \cdot T^{a_I}(\log T)^{b_I - 1} \\ &\ll_f c \cdot T^{a_I}(\log T)^{b_I - 1}.\end{aligned}$$

Therefore, by (3.23),

$$\operatorname{Vol}(V_{I,w}(c) \cap B_T) \ll c \cdot (\delta T)^{a_I}(\log(\delta T))^{b_I - 1}.$$

This completes the proof. ∎

The following corollary of Theorem 1.7 will be used in the proof of Theorem 1.13:

COROLLARY 3.24: *Let $\Delta_\sigma = I \sqcup J$ and B be a bounded subset of KM_I. Then given $c > 0$, there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for every (J, c)-regular $g = bah \in BA_I H$ and $0 < \varepsilon < \varepsilon_0$,*

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) b (M_I \cap \mathcal{O}_{\ell\varepsilon})(A_I \cap \mathcal{O}_{\ell\varepsilon}) aH.$$

Proof. Let $b = km$ for $k \in K$ and $m \in M_I$. Note that $m \in KB \cap M_I$, which is bounded. By (3.1) there exist $k_0 \in M_I \cap K$, $a_0 \in A^I$ and $h_0 \in M_I \cap H$ such that $m = k_0 a_0 h_0$. By Theorem 1.7,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) k k_0 (M_I \cap \mathcal{O}_{\ell\varepsilon})(A_I \cap \mathcal{O}_{\ell\varepsilon}) a_0 aH.$$

There exists $\ell_1 > 1$ such that for every $k \in K$ and small $\varepsilon > 0$, $k\mathcal{O}_\varepsilon k^{-1} \subset \mathcal{O}_{\ell_1\varepsilon}$. Hence,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) k (M_I \cap \mathcal{O}_{\ell_1\ell\varepsilon}) k_0 a_0 h_0 (A_I \cap \mathcal{O}_{\ell\varepsilon}) aH.$$

There exists $\ell_2 > 1$ such that for every $m \in KB$ and small $\varepsilon > 0$, $m^{-1}\mathcal{O}_\varepsilon m \subset \mathcal{O}_{\ell_2\varepsilon}$. Hence,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) km (M_I \cap \mathcal{O}_{\ell_2\ell_1\ell\varepsilon})(A_I \cap \mathcal{O}_{\ell\varepsilon}) aH$$

as required. ∎

Proof of Theorem 1.13. Due to Remark 2.1, without loss of generality, we may assume that $w = e$. We will denote $S_T(\Omega, e)$ by $S_T(\Omega)$.

Let $c, \varepsilon \in (0, 1)$.

Let $s : KM_I/(M_I \cap H) \to KM_I$ be a measurable section such that $s(\Omega)$ is bounded and measurable. For neighborhoods U_1 of e in K and U_2 of e in M_I, we set

$$\Omega^+ = U_1 s(\Omega) U_2 (M_I \cap H),$$
$$\Omega^- = \bigcap_{u_1 \in U_1, u_2 \in U_2} u_1 s(\Omega) u_2 (M_I \cap H).$$

One can check that as U_1 and U_2 shrink to $\{e\}$, we have

$$\Omega^+ \downarrow \bar{\Omega} \qquad \text{and} \qquad \Omega^- \uparrow \operatorname{int}(\Omega).$$

Since $\operatorname{Vol}(\partial\Omega) = 0$, we have $\operatorname{Vol}(\Omega^+ - \Omega^-) \to 0$. Hence, it follows from (3.17) that we can choose U_1 and U_2 so that

$$C_I(\Omega^+) - C_I(\Omega^-) < \varepsilon. \tag{3.25}$$

Fix a set $\tilde{\Omega} \supset \Omega$ such that $\bar{\Omega} \subset \mathrm{int}(\tilde{\Omega})$, set

$$V_I = \Omega A_I^+ v_0 \quad \text{and} \quad \tilde{V}_I = \tilde{\Omega} A_I^+ v_0,$$

and define $V_I(c) = V_{I,e}(c)$ and $\tilde{V}_I(c) = \tilde{V}_{I,e}(c)$ as in Proposition 3.22. We can choose U_1 and U_2 so that $\Omega^+ \subset \tilde{\Omega}$.

We claim that there exists a neighborhood $\mathcal{O}'$ of e in G such that

$$\mathcal{O}' \cdot S_T(\Omega) \subset S_{(1+\varepsilon)T}(\Omega^+) \cup (\tilde{V}_I(c) \cap B_{(1+\varepsilon)T}). \tag{3.26}$$

By Corollary 3.24, there exists a neighborhood $\mathcal{O}_1$ such that

$$\mathcal{O}_1^{-1} \cdot (V_I - \tilde{V}_I(c)) \subset \tilde{V}_I - V_I(c/2).$$

This implies that

$$\mathcal{O}_1 \cdot V_I(c/2) \subset \tilde{V}_I(c).$$

Also, by Corollary 3.24 and continuity of operator norm, there exists a neighborhood $\mathcal{O}_2$ of e in G such that for every $v = mav_0 \in V_I - V_I(c/2)$,

$$\mathcal{O}_2 v \subset (U_1 m U_2) A_I^+ v_0$$

and

$$\mathcal{O}_2 \cdot B_T \subset B_{(1+\varepsilon)T}.$$

Hence,

$$\mathcal{O}_2 \cdot (S_T(\Omega) - V_I(c/2)) \subset S_{(1+\varepsilon)T}(\Omega^+).$$

Setting $\mathcal{O}' = \mathcal{O}_1 \cap \mathcal{O}_2$, we deduce the claim (3.26).

A similar argument shows there exists a neighborhood $\mathcal{O}''$ of e in G such that

$$S_{(1-\varepsilon)T}(\Omega^-) \subset \Big(\bigcap_{g \in \mathcal{O}''} g S_T(\Omega) \Big) \cup \tilde{V}_I(c). \tag{3.27}$$

Combining (3.26) and (3.27), we deduce that for $\mathcal{O} = \mathcal{O}' \cap \mathcal{O}''$,

$$\begin{aligned} \mathrm{Vol}(\mathcal{O} \cdot \partial S_T(\Omega)) &\le \mathrm{Vol}\left(\mathcal{O} S_T(\Omega) - \cap_{g \in \mathcal{O}} g S_T(\Omega)\right) \\ &\le \mathrm{Vol}(S_{(1+\varepsilon)T}(\Omega^+)) - \mathrm{Vol}(S_{(1-\varepsilon)T}(\Omega^-)) + \mathrm{Vol}(\tilde{V}_I(c) \cap B_{(1+\varepsilon)T}). \end{aligned} \tag{3.28}$$

By Proposition 3.22,

$$\limsup_{T \to \infty} \frac{\mathrm{Vol}(\tilde{V}_I(c) \cap B_{(1+\varepsilon)T})}{T^{a_I} (\log T)^{b_I - 1}} \ll c.$$

By Proposition 3.8,

$$\lim_{T\to\infty} \frac{\mathrm{Vol}(S_{(1+\varepsilon)T}(\Omega^+))}{T^{a_I}(\log T)^{b_I-1}} = (1+\varepsilon)^{a_I} C_I(\Omega^+),$$
$$\lim_{T\to\infty} \frac{\mathrm{Vol}(S_{(1-\varepsilon)T}(\Omega^-))}{T^{a_I}(\log T)^{b_I-1}} = (1-\varepsilon)^{a_I} C_I(\Omega^-).$$

Hence, it follows from (3.28) and (3.25) that

$$\limsup_{T\to\infty} \frac{\mathrm{Vol}(\mathcal{O}\cdot\partial S_T(\Omega))}{T^{a_I}(\log T)^{b_I-1}} \ll (1+\varepsilon)^{a_I} C_I(\Omega^+) - (1-\varepsilon)^{a_I} C_I(\Omega^-) + c$$
$$\ll \varepsilon + c.$$

Since ε and c can be taken arbitrary small, this proves that the family of sets $S_T(\Omega)$ is well-rounded. Hence, it follows from [DRS, EM] that

$$\#(\Gamma v_0 \cap S_T(\Omega)) \sim_{T\to\infty} \mathrm{Vol}(S_T(\Omega)).$$

This proves the theorem. ∎

Proof of Theorem 1.4. To deduce Theorem 1.4 from Theorem 1.13, we observe that (see [GOS, §2.3])

$$\mathcal{Q}_W \simeq \bigcup_{p+q=d} \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(p,q), \quad d = \dim W,$$

and $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(p,q)$ is an affine symmetric space. We set

$$G = \mathrm{SL}_d(\mathbb{R}),$$
$$K = \mathrm{SO}(d),$$
$$A = \{\mathrm{diag}(s_1,\ldots,s_d) : s_i \in \mathbb{R}^+,\ s_1\cdots s_d = 1\},$$
$$H = \mathrm{SO}(p,q).$$

Then we have the generalized Cartan decomposition $G = KAH$. The set of simple roots on $\mathrm{Lie}(A) = \{s = (s_1,\ldots,s_d) : s_i \in \mathbb{R},\ s_1+\cdots+s_d = 0\}$ is

$$\Delta_\sigma = \{\alpha_i(s) = s_i - s_{i+1} : i = 1,\ldots,d-1\}.$$

In view of (1.2) and (1.3), set

$$i_k = \sum_{i=1}^{k} \dim W_i, \quad 1 \le k \le n-1.$$

Let $I = \Delta_\sigma \setminus \{\alpha_{i_1},\ldots,\alpha_{i_{n-1}}\}$. Then

$$M_I \simeq \mathrm{SL}_{i_1}(\mathbb{R}) \times \mathrm{SL}_{i_2-i_1}(\mathbb{R}) \times \cdots \times \mathrm{SL}_{d-i_{n-1}}(\mathbb{R}),$$

and A_I is the centralizer of M_I in A.

Since the set of integral quadratic forms in the question is a finite union of $\mathrm{SL}_d(\mathbb{Z})$-orbits, we conclude that the proof of the theorem reduces to the computation of the asymptotics of $\#(\mathrm{SL}_d(\mathbb{Z})\boldsymbol{q}_0 \cap S_T(\Omega\Omega'))$ where $\boldsymbol{q}_0 \in \mathcal{Q}_W(\mathbb{Z})$. This shows that Theorem 1.4 is a particular case of Theorem 1.13; it may be noted that since $d \geq 3$ the subgroups $\mathrm{SO}(p,q)$ are semisimple and $\mathrm{SO}(p,q) \cap \mathrm{SL}_d(\mathbb{Z})$ is a lattice in $\mathrm{SO}(p,q)$.

It remains to compute the parameters a_I and b_I, which are determined by the volume asymptotics in Proposition 3.8.

If we restrict the character 2ρ, which is the sum of all roots in Σ_σ^+, then we get

$$\rho|_{\mathrm{Lie}(A_I)} = \sum_{k=1}^{n-1} u_{i_k}\alpha_{i_k}, \quad \text{where } u_{i_k} = i_k(d-i_k).$$

The highest weight, say λ_ι, of the representation of $\mathrm{SL}_d(\mathbb{R})$ on the space of quadratic forms restricted to $\mathrm{Lie}(A_I)$ is

$$\lambda_\iota|_{\mathrm{Lie}(A_I)} = \sum_{k=1}^{n-1} m_{i_k}\alpha_{i_k} \quad \text{where } m_i = 2(d-i_k)/d.$$

By (3.6) and (3.7),

$$a_I = \max\left\{\frac{u_{i_k}}{m_{i_k}} : 1 \leq k \leq n-1\right\} = d i_{n-1}/2,$$
$$b_I = \#\left\{i_k : 1 \leq k \leq n-1, \frac{u_{i_k}}{m_{i_k}} = a_I\right\} = 1.$$

This proves the theorem. ∎

4. Another version of the strong wavefront Lemma

In this section, we obtain a version of the strong wavefront lemma for a generalized Cartan decomposition with a different Weyl chamber $\tilde{A}^+$ defined below.

Let $G^{\sigma\theta} = \{g \in G : \sigma\theta(g) = g\}$, the symmetric subgroup associated to the involution $\sigma\theta$ of G, and $\mathfrak{g}^{\sigma\theta}$, be the associated Lie subalgebra. Then A is the maximal $\mathbb{R}$-split Cartan subalgebra of $G^{\sigma\theta}$. Set

$$\tilde{\Sigma}_{\sigma,\theta} = \{\alpha \in \Sigma_\sigma : \mathfrak{g}^\alpha \cap \mathfrak{g}^{\sigma\theta} \neq \{0\}\} \quad \text{and} \quad \tilde{\Sigma}^+_{\sigma,\theta} = \Sigma^+_\sigma \cap \tilde{\Sigma}_{\sigma,\theta}.$$

Then $\tilde{\Sigma}_{\sigma,\theta}$ is a root system on A, and we denote by $\tilde{\Delta}_{\sigma,\theta} \subset \tilde{\Sigma}^+_{\sigma,\theta}$ the set of simple roots on A. Let $\tilde{A^+}$ denote the associated closed Weyl chamber of A.

Then $A^+ \subset \tilde{A^+}$. Also, the following generalized Cartan decomposition holds:

$$G = K\tilde{A}^+ H.$$

Note that $\tilde{A}^+ \neq A^+$ in general (see [HS, p. 109]).

Given $c > 0$, an element $g = kah \in K\tilde{A}^+ H$ is called c-regular for $\tilde{\Delta}_{\sigma,\theta}$ if $\alpha(\log a) > c$ for all $\alpha \in \tilde{\Delta}_{\sigma,\theta}$.

Theorem 4.1 (Strong wavefront Lemma-III): *Given $c > 0$, there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for every $g = kah \in K\tilde{A}^+ H$ which is c-regular for $\tilde{\Delta}_{\sigma,\theta}$ and every $0 < \varepsilon < \varepsilon_0$,*

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (A \cap \mathcal{O}_{\ell\varepsilon})a \cdot (H \cap \mathcal{O}_{\ell\varepsilon})h.$$

This result is stronger than Theorem 1.6 because any c-regular element is also c-regular for $\tilde{\Delta}_{\sigma,\theta}$, but the converse implication does not hold in general.

Now we consider the situation involving singular elements. Let $\tilde{I} \subset \tilde{\Delta}_{\sigma,\theta}$ and $\tilde{J} = \tilde{\Delta}_{\sigma,\theta} \setminus \tilde{I}$. For $c > 0$, we say that an element $g = kah \in KA^+H$ is $(\tilde{J}, c)$-regular if $\alpha(\log a) > c$ for all $\alpha \in \tilde{J}$. Let $A_{\tilde{I}} = \exp(\ker \tilde{I})$. Let $M^{\sigma\theta}_{\tilde{I}}$ denote the analytic semisimple subgroup of $G^{\sigma\theta}$ whose Lie algebra is generated by $\mathfrak{g}^{\pm\beta} \cap \mathfrak{g}^{\sigma\theta}$ for all $\beta \in \Sigma^+_{\sigma,\theta} \cap \langle \tilde{I} \rangle$. Then $M^{\sigma\theta}_{\tilde{I}}$ is contained in the centralizer of $A_{\tilde{I}}$, and

$$G = KM^{\sigma\theta}_{\tilde{I}} A^+_{\tilde{I}} H,$$

where $A^+_{\tilde{I}} = \tilde{A^+} \cap A_{\tilde{I}}$.

Theorem 4.2 (Strong wave front Lemma-IV): *Given $c > 0$, there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for every $\tilde{I} \subset \tilde{\Delta}_{\sigma\theta}$, $\tilde{J} = \tilde{\Delta}_{\sigma,\theta} \setminus \tilde{I}$, $g = kah \in K\tilde{A}^+ H$ which is $(\tilde{J}, c)$-regular, and $0 < \varepsilon < \varepsilon_0$,*

$$\mathcal{O}_\varepsilon \cdot g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (M^{\sigma\theta}_{\tilde{I}} \cap \mathcal{O}_{\ell\varepsilon}) \cdot (A_{\tilde{I}} \cap \mathcal{O}_{\ell\varepsilon})a \cdot (H \cap \mathcal{O}_{\ell\varepsilon})h.$$

This result strengthens Theorem 1.7.

Lemma 4.3: *For any $a \in A$,*

$$\mathfrak{g} = \mathfrak{q} \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus \operatorname{Ad} a(\mathfrak{p} \cap \mathfrak{h}).$$

Proof. Since $\mathfrak{g} = \mathfrak{q} \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{h})$, it is enough to show that

$$\operatorname{Ad} a(\mathfrak{p} \cap \mathfrak{h}) \cap (\mathfrak{q} + (\mathfrak{k} \cap \mathfrak{h})) = \{0\}.$$

To prove this, let $X \in \mathfrak{p} \cap \mathfrak{h}$ such that $\operatorname{Ad} a(X) \in \mathfrak{q} \oplus (\mathfrak{k} \cap \mathfrak{h})$. Therefore,

$$\sigma(\operatorname{Ad} a(X)) = \operatorname{Ad} \sigma(a)(\sigma(X)) = (\operatorname{Ad} a)^{-1}(X),$$
$$\theta(\operatorname{Ad} a(X)) = \operatorname{Ad} \theta(a)(\theta(X)) = (\operatorname{Ad} a)^{-1}(-X),$$

and

$$\sigma(\operatorname{Ad} a(X)) = -\theta(\operatorname{Ad} a(X)) = (\operatorname{Ad} a)^{-1}(X). \tag{4.4}$$

Now we write $\operatorname{Ad} a(X) = Y_1 + Y_2 + Y_3$, where $Y_1 \in \mathfrak{q} \cap \mathfrak{k}$, $Y_2 \in \mathfrak{q} \cap \mathfrak{p}$, and $Y_3 \in \mathfrak{k} \cap \mathfrak{h}$. Then

$$\sigma(\operatorname{Ad} a(X)) = -Y_1 - Y_2 + Y_3,$$
$$\theta(\operatorname{Ad} a(X)) = Y_1 - Y_2 + Y_3,$$

and it follows from (4.4) that $Y_2 = 0$ and $Y_3 = 0$. Hence, $\operatorname{Ad} a(X) \in \mathfrak{k} \cap \mathfrak{q}$ and $\sigma(\operatorname{Ad} a(X)) = -\operatorname{Ad} a(X)$. Then by (4.4),

$$(\operatorname{Ad} a)^2(X) = -X.$$

If $X \neq 0$, this gives a contradiction because $\operatorname{Ad} a$ is self-adjoint. ∎

As a consequence of the above lemma, we obtain the following:

Corollary 4.5: *Given $c > 0$ there exist $\ell > 1$ and $\varepsilon_0 > 0$ such that for any $a \in A$ such that $|\alpha(\log a)| \leq c$ for all $\alpha \in \Delta_\sigma$, and any $0 < \varepsilon < \varepsilon_0$, we have*

$$\mathcal{O}_\varepsilon a \subset (\mathcal{O}_{\ell\varepsilon} \cap K)(\mathcal{O}_{\ell\varepsilon} \cap \exp(\mathfrak{p} \cap \mathfrak{q}))a(\mathcal{O}_{\ell\varepsilon} \cap \exp(\mathfrak{p} \cap \mathfrak{h})).$$

Proof of Theorem 4.2. Let $w \in \mathcal{W}$ be such that $waw^{-1} = b \in A^+$. We set

$$I = \{\alpha \in \Delta_\sigma : \alpha(\log b) < c/n_0\} \quad \text{and} \quad J = \Delta_\sigma \setminus I,$$

where $n_0 \in \mathbb{N}$ is such that any positive root is a sum of at most n_0 simple roots counted with multiplicity. We apply Theorem 1.7 to the involution $\sigma_w := i_w \circ \sigma \circ i_w^{-1}$ in place of σ. Since the element $(kw^{-1})b(whw^{-1})$ is $(J, c/n_0)$-regular,

$$\begin{aligned}\mathcal{O}_\varepsilon(kah) =& \mathcal{O}_\varepsilon(kw^{-1})b(whw^{-1})w \\ \subset& (\mathcal{O}_{\ell\varepsilon} \cap K)(kw^{-1})(\mathcal{O}_{\ell\varepsilon} \cap M_I)(\mathcal{O}_{\ell\varepsilon} \cap A_I)b(\mathcal{O}_{\ell\varepsilon} \cap wHw^{-1})(whw^{-1})w \\ =& (\mathcal{O}_{\ell\varepsilon} \cap K)k(w^{-1}\mathcal{O}_{\ell\varepsilon}w \cap w^{-1}M_Iw)(w^{-1}\mathcal{O}_{\ell\varepsilon}w \cap w^{-1}A_Iw)a \\ &\times (w^{-1}\mathcal{O}_{\ell\varepsilon}w \cap H)h.\end{aligned} \tag{4.6}$$

There exists $\ell_1 > 1$ such that

$$w^{-1}\mathcal{O}_{\ell\varepsilon}w \subset \mathcal{O}_{\ell_1\varepsilon}$$

for all $0 < \varepsilon < \varepsilon_0$. Since M_I is σ_w- and θ-stable, $M_I^w := w^{-1}M_I w$ is σ- and θ-stable and $A = (A \cap M_I^w)(w^{-1}A_I w)$. Let $a_1 \in A \cap M_I^w$ be such that $a \in a_1(w^{-1}A_I w^{-1})$. We now apply Corollary 4.5 to M_I^w in place of G, and conclude that for some $\ell_2 \geq \ell_1$,

$$(4.7) \qquad \begin{aligned}(\mathcal{O}_{\ell_1\varepsilon} \cap M_I^w)a_1 \subset &(\mathcal{O}_{\ell_2\varepsilon} \cap K \cap M_I^w)(\mathcal{O}_{\ell_2\varepsilon} \cap \exp(\mathfrak{p} \cap \mathfrak{q}) \cap M_I^w)a_1 \\ &\times (\mathcal{O}_{\ell_2\varepsilon} \cap \exp(\mathfrak{p} \cap \mathfrak{h}) \cap M_I^w).\end{aligned}$$

Since M_I^w commutes with $w^{-1}A_I w$, combining (4.6) and (4.7), we obtain that for some $\ell_3 \geq \ell_2$

$$(4.8) \qquad \begin{aligned}\mathcal{O}_\varepsilon(kah) \subset &(\mathcal{O}_{\ell_3\varepsilon} \cap K)k(\mathcal{O}_{\ell_3\varepsilon} \cap \exp(\mathfrak{p} \cap \mathfrak{q}) \cap M_I^w) \\ &\times (\mathcal{O}_{\ell_3\varepsilon} \cap w^{-1}A_I w)a(\mathcal{O}_{\ell_3\varepsilon} \cap H)h.\end{aligned}$$

By the definition of I, each eigenvalue of $\mathrm{ad}(\log b)$ on the Lie algebra of M_I is at most c. Hence every eigenvalue of $\mathrm{ad}(\log a)$ on the Lie algebra of M_I^w is at most c. Since a is given to be $(\tilde{J}, c)$-regular, we conclude that

$$M_I^w \cap \exp(\mathfrak{p} \cap \mathfrak{q}) \subset M_I^w \cap G^{\sigma\theta} \subset M_{\tilde{I}}.$$

Therefore, the conclusion of the theorem follows from (4.8). ∎

Note that Theorem 4.1 follows from Theorem 4.2.

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