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Locally compact groups with dense orbits under $\mathbb{Z}^d$-actions by automorphisms

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Abstract. We consider locally compact groups G admitting a topologically transitive $\mathbb{Z}^d$-action by automorphisms. It is shown that such a group G has a compact normal subgroup K of G, invariant under the action, such that G/K is a product of (finitely many) locally compact fields of characteristic zero; moreover, the totally disconnected fields in the decomposition can be chosen to be invariant under the $\mathbb{Z}^d$-action and such that the $\mathbb{Z}^d$-action is via scalar multiplication by non-zero elements of the field. Under the additional conditions that G be finite dimensional and 'locally finitely generated' we conclude that K as above is connected and contained in the center of G. We describe some examples to point out the significance of the conditions involved.

1. *Introduction*

It has been shown, in response to an old question of Halmos, that a locally compact group admitting an ergodic automorphism is compact (see [**1, 3, 6, 8, 11, 17**]). It can be seen, on the other hand, that it is possible to have abelian groups of automorphisms acting ergodically, without the group being compact. This holds, in particular, for (finite-dimensional) vector spaces over locally compact fields, where the group of automorphisms can even be chosen to be finitely generated. It was shown in [**4, 5**] that if G is a connected, or more generally almost connected, locally compact group admitting an abelian group of automorphisms whose action has a dense orbit (in particular, if it is ergodic), then G contains a compact (normal) subgroup K invariant under all automorphisms of G, such that G/K is a vector group, namely, it is topologically isomorphic to $\mathbb{R}^n$; the results in [**5**] also contain some further information on K. Generalizing the result, here we prove the following.

THEOREM 1.1. *Let G be a locally compact group, and suppose that there exists a finitely generated abelian group of (bicontinuous) automorphisms, say $\mathfrak{H}$, of G whose action on G*

has a dense orbit. Then there exist a unique compact normal $\mathfrak{H}$-invariant subgroup K of G, a finite-dimensional vector space V over $\mathbb{R}$, and finitely many locally compact non-discrete totally disconnected fields of characteristic zero, say $F_1, \ldots, F_q$, such that the following hold:

(i) *G/K is isomorphic to $V \times F_1 \times \cdots \times F_q$ as a topological group, with respect to the additive structures on V and the F_i;*

(ii) *each of the V and $F_1, \ldots, F_q$ is invariant under the factor action of $\mathfrak{H}$ on G/K;*

(iii) *there exists a continuous homomorphism $\varphi : \mathfrak{H} \to \mathrm{GL}(V) \times F_1^* \times \cdots \times F_q^*$, where F_i^* denotes the multiplicative group of non-zero elements in F_i, such that for any $\alpha \in \mathfrak{H}$ the (factor) action of α on $V \times F_1 \times \cdots \times F_q$ is given by $(v, f_1, \ldots, f_q) \mapsto (\varphi_0(\alpha)(v), \varphi_1(\alpha)f_1, \ldots, \varphi_q(\alpha)f_q)$, for $v \in V$ and $f_i \in F_i$ for $i = 1, \ldots, q$, where $(\varphi_0(\alpha), \varphi_1(\alpha), \ldots, \varphi_q(\alpha)) = \varphi(\alpha)$;*

(iv) *in the topological group $\mathrm{GL}(V) \times F_1^* \times \cdots \times F_q^*$, $\overline{\varphi(\mathfrak{H})}$ (the closure of $\varphi(\mathfrak{H})$) coincides with its centralizer. In particular, the subgroup $\mathbb{R}^* \cdot \mathbf{1}_V \times F_1^* \times \cdots \times F_q^*$ is contained in $\overline{\varphi(\mathfrak{H})}$.*

Remark 1.1. The subgroup $C := K \cap G^0$ is connected, and it is a maximal compact normal subgroup of G^0. Moreover, $G/C \cong V \times T$, where $V \cong G^0/C$ is a finite-dimensional real vector space, and T is a closed $\mathfrak{H}$-invariant totally disconnected subgroup of G/C (see Corollary 5.1).

We recall that by Ostrowski's theorem every non-discrete locally compact field of characteristic zero is either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p. The multiplicative group of non-zero elements in each of these fields admits a finitely generated (abelian) dense subgroup (see Lemma 4.1), say $\mathfrak{H}$; the action of $\mathfrak{H}$ on the additive group of the field has a dense orbit, and also it is topologically irreducible (we call an action on a topological group *topologically irreducible* if there is no non-trivial proper closed normal subgroup invariant under the action). It may be worth noting that Theorem 1.1(i) readily implies the following converse of this.

COROLLARY 1.1. *Let G be a locally compact non-compact group. Suppose that there exists a finitely generated abelian group of automorphisms, say $\mathfrak{H}$, of G such that the following conditions are satisfied:*

(i) *the $\mathfrak{H}$-action has a dense orbit on G;*

(ii) *the $\mathfrak{H}$-action on G is topologically irreducible.*

Then G is either $\mathbb{R}$, or $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some prime p.

The uniqueness of the subgroup K as in Theorem 1.1, when it exists, is easy to settle and it would be convenient to clarify this at the outset. We first note the following.

PROPOSITION 1.1. *Let G be a locally compact group and K be a normal subgroup such that G/K is topologically isomorphic to $V_0 \times V_1 \times \cdots \times V_q$, where $V_0, V_1, \ldots, V_q$ are finite-dimensional vector spaces over $\mathbb{R}$ and $\mathbb{Q}_{p_1}, \ldots, \mathbb{Q}_{p_q}$, respectively, where $p_1, \ldots, p_q$ are prime numbers. Let α be a bicontinuous automorphism of G such that $\alpha(K) = K$, the factor action of α on G/K leaves invariant all V_i, $i = 0, \ldots, q$, and all eigenvalues of $\alpha|_{V_i}$, $i = 0, \ldots, q$, are of absolute value less than 1. Then every α-invariant compact subset of G is contained in K.*

Proof. Under the condition in the hypothesis the factor action of α on $V_0 \times V_1 \times \cdots \times V_q$ does not leave any compact subset of $V_0 \times V_1 \times \cdots \times V_q$ other than the point set consisting of the identity. Thus, for any compact α-invariant subset of G its image in G/K is trivial, so it is contained in K. □

COROLLARY 1.2. *Let G and $\mathfrak{H}$ be as in Theorem 1.1. If there exists a subgroup K for which the assertions (i)–(iv) as in the conclusion of the theorem are satisfied (for some choices of V and $F_1, \ldots, F_q$ as in the statement), then such a subgroup K is unique.*

Proof. Let K be a subgroup for which the conditions are satisfied. We note that since, by condition (iv), $\overline{\varphi(\mathfrak{H})}$ contains all automorphisms of $V \times F_1 \times \cdots \times F_q$ whose action is by scalar multiplication componentwise, it follows that there exists $\alpha \in \mathfrak{H}$ such that all eigenvalues of the factor action of α on $V \times F_1 \times \cdots \times F_q$ are of absolute value less than 1. Hence by Proposition 1.1 every α-invariant, and hence every $\mathfrak{H}$-invariant, compact subgroup of G is contained in K. By symmetry of the argument, it follows that K is the only subgroup for which the conditions are satisfied. □

Assertion (iv) in Theorem 1.1 also enables the deduction of the following (we recall that a measure is said to be locally finite if the measure assigned to every compact set is finite).

COROLLARY 1.3. *Let the notation be as in Theorem 1.1. Then:*
(i) *any compact $\mathfrak{H}$-invariant subset of G is contained in K;*
(ii) *any locally finite $\mathfrak{H}$-invariant measure on G is supported on K.*

Proof. As noted in the proof of Corollary 1.2, condition (iv) in Theorem 1.1 implies that there exists $\alpha \in \mathfrak{H}$ such that all eigenvalues of the factor action of α on V and F_i, $i = 1, \ldots, q$, are of absolute value less than one. The conclusions as above are immediate from this. □

In the case of connected Lie groups G it was shown in [**5**] that the subgroup K as in the conclusion of Theorem 1.1 is contained in the center of G; the general assertion for almost connected groups, stated above, is in fact deduced from this. We obtain a generalization of this stronger result, for a class of locally compact groups. We shall say that a locally compact group G is *locally finitely generated* if there exists a finite subset S of G such that the closure of the subgroup of G generated by S is open in G. We note that any connected Lie group, and any closed subgroup of $\mathrm{GL}(n, F)$, where F is any locally compact field of characteristic zero, is locally finitely generated. Using certain properties of locally compact fields together with Theorem 1.1 we deduce the following.

THEOREM 1.2. *Let the notation be as in Theorem 1.1. Suppose, in addition, that G is finite dimensional and locally finitely generated. Then K is connected and contained in the center of G.*

THEOREM 1.3. *Let the notation be as in Theorem 1.2. Suppose, furthermore, that G^0 is a Lie group. Then for each $1 \leq i \leq q$ there exists a closed normal $\mathfrak{H}$-invariant subgroup Φ_i of G isomorphic to F_i as a topological group, and $G = G^0 \times \Phi_1 \times \cdots \times \Phi_q$.*

Theorem 1.1 will be proved first, through Sections 2–4, in the special case of totally disconnected locally compact groups. The arguments involved depend on properties of tidy subgroups of automorphisms of these groups, proved in [**13–16**], which we shall briefly recall. The general case of the theorem will be proved in §5. Theorems 1.2 and 1.3 involving special additional hypotheses will be proved in §6. In the last section (§7) we describe a class of examples to show that there are non-abelian groups admitting finitely generated abelian groups of automorphisms acting with a dense orbit.

2. *Automorphisms of totally disconnected groups and a partial paving of G*

Let G be a locally compact totally disconnected group. Let $\mathrm{Aut}(G)$ denote the group of all (bi-continuous) automorphisms of G.

2.1. *Basic results on tidy subgroups.*

In [**13**] the structure of G with respect to any automorphism was studied by showing the existence of 'tidy subgroups'.

For a compact subgroup U of G and $\alpha \in \mathrm{Aut}(G)$ we define

$$U_{\alpha+} := \bigcap_{n\geq 0} \alpha^n(U), \quad U_{\alpha-} := \bigcap_{n\geq 0} \alpha^{-n}(U),$$
$$U_{\alpha++} := \bigcup_{n\geq 0} \alpha^n(U_{\alpha+}), \quad U_{\alpha--} := \bigcup_{n\geq 0} \alpha^{-n}(U_{\alpha-}),$$

and

$$U_{\alpha 0} := U_{\alpha+} \cap U_{\alpha-} = \bigcap_{n\in\mathbb{Z}} \alpha^n(U).$$

Definition. A compact open subgroup U of G is said to be *tidy* for an $\alpha \in \mathrm{Aut}(G)$ if it satisfies the following conditions:

$$\mathbf{T1}(\alpha)\colon U = U_{\alpha+}U_{\alpha-} \quad \text{and} \quad \mathbf{T2}(\alpha)\colon U_{\alpha++} \text{ is closed.}$$

Moreover, U is said to be a *tidy* for a subgroup $\mathfrak{H} \subset \mathrm{Aut}(G)$ if it is tidy for every $\alpha \in \mathfrak{H}$.

We recall the following results from [**16**, Theorems 3.4, 5.5, 6.8, 6.14]; see also [**15**, Theorems 5.1, 5.2].

THEOREM 2.1. (Willis) *Let $\mathfrak{H}$ be a finitely generated abelian subgroup of* $\mathrm{Aut}(G)$. *Then there exists an open compact subgroup U of G which is tidy for $\mathfrak{H}$. Furthermore:*

(i) *there are distinct closed subgroups $U_0, U_1, \ldots, U_q$ of U such that*

$$U = U_0U_1 \cdots U_q$$

and for each $\alpha \in \mathfrak{H}$ we have $\alpha(U_0) = U_0$ and either

$$\alpha(U_j) \supset U_j \quad \textit{or} \quad \alpha(U_j) \subset U_j \quad \textit{for all } j \in \{1, \ldots, q\};$$

(ii) *the subgroups $\tilde{U}_j := \bigcup_{\alpha\in\mathfrak{H}} \alpha(U_j)$ are closed;*

(iii) $U_0 = \bigcap_{\alpha\in\mathfrak{H}} \alpha(U_j)$;

(iv) *for each $j \in \{1, \ldots, q\}$ there exist $\alpha_j \in \mathfrak{H}$ and a homomorphism $\rho_j : \mathfrak{H} \to \mathbb{Z}$ such that $\alpha_j(U_j) \supset U_j$, $t_j := [\alpha_j(U_j) : U_j] > 1$,*

$$\alpha(U_j) = \alpha_j^{\rho_j(\alpha)}(U_j), \quad \textit{and} \quad |\alpha(U_j)|/|U_j| = t_j^{\rho_j(\alpha)} \quad \textit{for all } \alpha \in \mathfrak{H};$$

(v) *if* $\mathfrak{H}_1 := \{\alpha \in \mathfrak{H} : \alpha(U) = U\}$ *then* $\mathfrak{H}_1 = \bigcap_{j=1}^{q} \ker \rho_j$ *and* $\mathfrak{H}/\mathfrak{H}_1 \cong \mathbb{Z}^r$ *for some* $0 \leq r \leq q$.

LEMMA 2.1. *If* U *is tidy for* $\alpha \in \mathrm{Aut}(G)$ *then* $U_{\alpha++} \cap U_{\alpha--} = U_{\alpha 0}$.

Proof. If $u \in U_{\alpha--}$, then $\alpha^n(u) \in U$ for all sufficiently large $n > 0$. On the other hand, if $u \in U_{\alpha++}$ as well, then there exists $n_0 \geq 0$ such that $u_0 := \alpha^{-n_0}(u) \in U_{\alpha+}$. Hence $\alpha^n(u_0) \in U$ whenever $|n|$ is sufficiently large. By [**14**, Lemma 3.2] this implies that $\alpha^n(u_0) \in U_{\alpha 0}$ for all n. Hence $u \in U_{\alpha 0}$. □

We also note that the ordering of the subgroups $U_0, U_1, \ldots, U_q$, in the product $U_0 U_1 \cdots U_q$ as in Theorem 2.1 arrived at in the course of the proof, is such that the following holds (see [**16**, Proof of Theorem 6.8]).

LEMMA 2.2. *Suppose that* $q \geq 2$. *Then there exist* $\alpha \in \mathfrak{H}$ *and* $0 < t < q$ *such that the action of* α *on* U_j *is strictly expanding for every* $j \in \{1, \ldots, t\}$, *and strictly contracting for every* $j \in \{t+1, \ldots, q\}$. *For this* α *we have that* $U_{\alpha 0} = U_0$, $U_{\alpha+} = U_1 \cdots U_t$ *and* $U_{\alpha-} = U_{t+1} \cdots U_q$. *Also,* $U_{\alpha++} = \tilde{U}_1 \cdots \tilde{U}_t$ *and* $U_{\alpha--} = \tilde{U}_{t+1} \cdots \tilde{U}_q$.

2.2. *A paved set.* From now on, unless specified otherwise, we will assume that $\mathfrak{H} \subset \mathrm{Aut}(G)$ is a finitely generated abelian group and U is an open compact subgroup of G for which the conclusion of Theorem 2.1 holds, and $U_0, U_1, \ldots, U_q$ and $\tilde{U}_1, \ldots, \tilde{U}_q$ are the subgroups as in the statement of the theorem.

For each $j \in \{1, \ldots, q\}$ define

$$X_j^{(n)} = \alpha_j^n(U_j) \setminus \alpha_j^{n-1}(U_j).$$

Then $\tilde{U}_j$ is partitioned as

$$\tilde{U}_j = U_0 \cup \bigcup_{n \in \mathbb{Z}} X_j^{(n)} = \bigcup_{n \in \{-\infty\} \cup \mathbb{Z}} X_j^{(n)}, \tag{1}$$

where $X_j^{(-\infty)} := U_0$. Note that

$$U_0 X_j^{(n)} = X_j^{(n)} = X_j^{(n)} U_0 \quad \text{and} \quad \mathfrak{H}_1 X_j^{(n)} = X_j^{(n)}. \tag{2}$$

For each $\boldsymbol{n} = (n_1, \ldots, n_q) \in (\{-\infty\} \cup \mathbb{Z})^q$ let

$$X^{(\boldsymbol{n})} = \prod_{j=1}^{q} X_j^{(n_j)} = X_1^{(n_1)} \cdots X_q^{(n_q)}.$$

We note that the sets $X_j^{(n_j)}$ may not commute with each other and the product is understood to be in the order of occurrence of the indices j. Note also that $U_j = \bigcup_{-\infty \leq n \leq 0} X_j^{(n)}$ and so, since $U = U_0 U_1 \cdots U_q$, we have that $U = \bigcup_{\boldsymbol{n} \leq \boldsymbol{0}} X^{(\boldsymbol{n})}$, where $\boldsymbol{n} \leq \boldsymbol{0}$ means that $-\infty \leq n_j \leq 0$ for each $j \in \{1, 2, \ldots, q\}$. By (2),

$$U_0 X^{(\boldsymbol{n})} = X^{(\boldsymbol{n})} = X^{(\boldsymbol{n})} U_0 \quad \text{and} \quad \mathfrak{H}_1 X^{(\boldsymbol{n})} = X^{(\boldsymbol{n})}. \tag{3}$$

PROPOSITION 2.1. *The sets* $X^{(\boldsymbol{n})}$, *where* $\boldsymbol{n} \in (\{-\infty\} \cup \mathbb{Z})^q$, *are pairwise disjoint.*

Proof. The proof is by induction on q, the number of factors. When $q = 1$ the claim is nothing but the partitioning of $\tilde{U}_1$ in (1).

Assume that the claim is established for all $1 \le q \le k$ and consider $q = k + 1 \ge 2$. Choose $\alpha \in \mathfrak{H}$ and $0 < t < q$ as in Lemma 2.2. For each $x = \prod_{j=1}^{q} x_j^{(n_j)} \in X^{(\boldsymbol{n})}$ define

$$x_+^{(\boldsymbol{n})} = \prod_{j=1}^{t} x_j^{(n_j)} \in U_{\alpha++} \quad \text{and} \quad x_-^{(\boldsymbol{n})} = \prod_{j=t+1}^{q} x_j^{(n_j)} \in U_{\alpha--}.$$

Then $x = x_+^{(\boldsymbol{n})} x_-^{(\boldsymbol{n})}$.

Suppose now that $x \in X^{(\boldsymbol{m})} \cap X^{(\boldsymbol{n})} \neq \emptyset$ for some $\boldsymbol{m}$ and $\boldsymbol{n}$ in $(\{-\infty\} \cup \mathbb{Z})^q$. Then x factors in two ways as

$$x = x_+^{(\boldsymbol{m})} x_-^{(\boldsymbol{m})} = x_+^{(\boldsymbol{n})} x_-^{(\boldsymbol{n})}.$$

Therefore,

$$y := (x_+^{(\boldsymbol{m})})^{-1} x_+^{(\boldsymbol{n})} = x_-^{(\boldsymbol{m})} (x_-^{(\boldsymbol{n})})^{-1}$$

and hence, by Lemmas 2.1 and 2.2,

$$y \in U_{\alpha++} \cap U_{\alpha--} = U_{\alpha 0} = U_0.$$

It follows that, by (2),

$$x_+^{(\boldsymbol{m})} = x_+^{(\boldsymbol{n})} y^{-1} \in \prod_{j=1}^{t} X_j^{(m_j)} \cap \prod_{j=1}^{t} X_j^{(n_j)} \tag{4}$$

and that

$$x_-^{(\boldsymbol{m})} = y x_-^{(\boldsymbol{n})} \in \prod_{j=t+1}^{q} X_j^{(m_j)} \cap \prod_{j=t+1}^{q} X_j^{(n_j)}. \tag{5}$$

The group $U_{\alpha++}$ is a closed subgroup of G and is invariant under $\mathfrak{H}$. Let the restriction of $\mathfrak{H}$ to $U_{\alpha++}$ be denoted by $\mathfrak{H}|_{++} \subset \mathrm{Aut}(U_{\alpha++})$. Then $V := U_{\alpha+}$ is an open subgroup of $U_{\alpha++}$, it is tidy for $\mathfrak{H}|_{++}$ and its factoring as $V = U_1 \cdots U_t$ satisfies the conclusion of Theorem 2.1. Now for each $j \in \{1, \ldots, t\}$ the group $\tilde{U}_j$, its partition in (1), and, in turn, the sets $X_j^{(n)}$, $n \in \{-\infty\} \cup \mathbb{Z}$, are the same whether defined with respect to the $\mathfrak{H}$ action on G or the $\mathfrak{H}|_{++}$ action on $U_{\alpha++}$. Therefore, in view of (4), the induction hypothesis applied to $\mathfrak{H}|_{++}$ acting on $U_{\alpha++}$ implies that $m_j = n_j$ for each $j \in \{1, \ldots, t\}$. That $m_j = n_j$ for each $j \in \{t+1, \ldots, q\}$ follows from (5) by application of the same argument to $U_{\alpha--}$. Hence $X^{(\boldsymbol{m})} = X^{(\boldsymbol{n})}$ and we have established the claim for $q = k + 1$. This completes the inductive step, and hence the proof of the proposition. $\square$

COROLLARY 2.1. *For $i, j \in \{1, \ldots, q\}$ and $i \neq j$, $\tilde{U}_i \cap \tilde{U}_j = U_0$.*

Proof. By (2) for any $n \in \{-\infty\} \cup \mathbb{Z}$, we have $X_j^{(n)} = X^{(\boldsymbol{n})}$ for $\boldsymbol{n} = (n_1, \ldots, n_q)$, where $n_k = -\infty$ if $k \neq j$ and $n_j = n$. Therefore, by (1) and Proposition 2.1, $\tilde{U}_i \cap \tilde{U}_j = U_0$ if $i \neq j$. $\square$

COROLLARY 2.2. *Let $x = x_1 \cdots x_q$, where $x_j \in \tilde{U}_j$ for each $j \in \{1, \ldots, q\}$. If $x \in U$, then $x_j \in U_j$ for each $j \in \{1, \ldots, q\}$.*

Proof. For each $j \in \{1, \ldots, q\}$, $x_j \in X^{(n_j)}$ for some $n_j \in \mathbb{Z} \cup \{-\infty\}$. Since $x \in U$, there exists $\boldsymbol{m} \leq \boldsymbol{0}$ such that $x \in X^{(\boldsymbol{m})}$. Therefore, $x \in X^{(\boldsymbol{n})} \cap X^{(\boldsymbol{m})}$, where $\boldsymbol{n} = (n_1, \ldots, n_q)$. So, by Proposition 2.1, $\boldsymbol{n} = \boldsymbol{m}$, and hence $n_j \leq 0$ for each j. Therefore, $x_j \in X_j^{(n_j)} \subset U_j$ for each j. □

LEMMA 2.3. *Let U be tidy for α. Then $\alpha^N(U_+)\alpha^{-N}(U_-)$ is open for every $N \geq 0$.*

Proof. Since $\alpha^N(U_{\alpha+}) \geq U_{\alpha+}$ and $\alpha^{-N}(U_{\alpha-}) \geq U_{\alpha-}$, we have

$$\begin{aligned}\alpha^N(U_{\alpha+})\alpha^{-N}(U_{\alpha-}) &= \alpha^N(U_{\alpha+})U_{\alpha+}U_{\alpha-}\alpha^{-N}(U_{\alpha-})\\ &= \alpha^N(U_{\alpha+})U\alpha^{-N}(U_{\alpha-}),\end{aligned}$$

which is open because U is. □

PROPOSITION 2.2. *$X^{(\boldsymbol{n})}$ is open for every $\boldsymbol{n} \in \mathbb{Z}^q$.*

Proof. The case of $q = 0$ is trivial. If $q = 1$ then the conclusion is clear from the definition of $X^{(\boldsymbol{n})}$. Now assume that $q \geq 2$. Let $\boldsymbol{n} = (n_1, \ldots, n_q) \in \mathbb{Z}^q$ and set $N = \max\{0, n_1, \ldots, n_q\}$. Choose $\alpha \in \mathfrak{H}$ and $0 < t < q$ as in Lemma 2.2. Then $\prod_{j=1}^t X_j^{(n_j)} \subset \alpha^N(U_{\alpha+})$ and $\prod_{j=t+1}^q X_j^{(n_j)} \subset \alpha^{-N}(U_{\alpha-})$. Hence $X^{(\boldsymbol{n})}$ is contained in the open set $\alpha^N(U_{\alpha+})\alpha^{-N}(U_{\alpha-})$.

For each $j \in \{1, \ldots, q\}$, let α_j be such that statement (iv) of Theorem 2.1 holds. Now define

$$\begin{aligned}C_j &= \alpha_j^{n_j-1}(U_j), \quad \text{and}\\ F_j &= \begin{cases}\alpha^N(U_j) & \text{if } j \in \{1, \ldots, t\},\\ \alpha^{-N}(U_j) & \text{if } j \in \{t+1, \ldots, q\}.\end{cases}\end{aligned}$$

As statement (iv) of Theorem 2.1 holds for α_j, there exist integers $N_j \geq 0$ such that $\alpha_j^{N_j}(U_j) = F_j$. Now for all $j = 1, \ldots, q$, define

$$Y_j = \left(\prod_{i=1}^{j-1} F_i\right) C_j \left(\prod_{i=j+1}^{q} F_i\right)$$

and let

$$B = \alpha^N(U_{\alpha+})\alpha^{-N}(U_{\alpha-}) \Big\backslash \bigcup_{j=1}^q Y_j.$$

Then each Y_j is compact, because C_j and F_i are compact, and hence B is open. Clearly, B is the union of the sets $X^{(\boldsymbol{m})}$ such that $\boldsymbol{m} = (m_1, \ldots, m_q)$ satisfies $n_j \leq m_j \leq N_j$ for $1 \leq j \leq q$. By Proposition 2.1, the sets $X^{(\boldsymbol{m})}$ are disjoint, and hence we have

$$X^{(\boldsymbol{n})} = B \Big\backslash \Big(\bigcup \{X^{(\boldsymbol{m})} : \boldsymbol{m} \neq \boldsymbol{n} \text{ and } n_j \leq m_j \leq N_j \text{ for } 1 \leq j \leq q\}\Big).$$

Since B is open and each $X^{(\boldsymbol{m})}$ is compact, this shows that $X^{(\boldsymbol{n})}$ is open. □

3. *Groups of automorphisms acting with a dense orbit*

For the results of this section, we will assume the notation of §2. We further assume that the action of $\mathfrak{H}$ has a dense orbit on G.

PROPOSITION 3.1. *There exists a surjective homomorphism $\rho : \mathfrak{H} \to \mathbb{Z}^q$ such that*

$$\alpha(X^{(\boldsymbol{n})}) = X^{(\boldsymbol{n}+\rho(\alpha))} \quad \textit{for all } \alpha \in \mathfrak{H} \textit{ and } \boldsymbol{n} \in \mathbb{Z}^q \tag{6}$$

and $\ker \rho = \mathfrak{H}_1$.

Proof. Let $\rho_j : \mathfrak{H} \to \mathbb{Z}$, $j = 1, \ldots, q$, be the homomorphisms as in (iv) of Theorem 2.1. We define $\rho : \mathfrak{H} \to \mathbb{Z}^q$ by setting $\rho(\alpha) = (\rho_1(\alpha), \ldots, \rho_q(\alpha))$ for all $\alpha \in \mathfrak{H}$. Then $\mathfrak{H}_1 = \ker \rho$. Now $\alpha(X_j^{(n)}) = X_j^{(n+\rho_j(\alpha))}$ for all $\alpha \in \mathfrak{H}$, $n \in \mathbb{Z}$, and $1 \leq j \leq q$. Therefore, (6) holds. Since $\mathfrak{H}$ has a dense orbit in G and $X^{(\boldsymbol{n})}$ is open for each $\boldsymbol{n} \in \mathbb{Z}^q$, $\mathfrak{H}$ acts transitively on $\{X^{(\boldsymbol{n})} : \boldsymbol{n} \in \mathbb{Z}^q\}$ and it follows that ρ is surjective. □

COROLLARY 3.1. *There exists $\alpha \in \mathfrak{H}$ such that*

$$\alpha(U) \supset U, \quad \bigcap_{n \geq 0} \alpha^{-n}(U) = U_0, \quad \textit{and}$$

$$U_{\alpha++} = \bigcup_{n \geq 0} \alpha^n(U) = \bigcup_{\boldsymbol{n} \in (\{-\infty\} \cup \mathbb{Z})^q} X^{(\boldsymbol{n})} = G.$$

Proof. Since ρ as in Proposition 3.1 is surjective, there exists $\alpha \in \mathfrak{H}$ such that $\rho(\alpha) = (1, \ldots, 1)$. By (iv) of Theorem 2.1, for all $1 \leq j \leq q$,

$$\alpha(U_j) \supset U_j, \quad \bigcup_{n \geq 0} \alpha^n(U_j) = \tilde{U}_j, \quad \text{and} \quad \bigcap_{n \geq 0} \alpha^{-n}(U_j) = U_0.$$

Therefore,

$$\begin{aligned} U_{\alpha++} &= \bigcup_{n \geq 0} \alpha^n(U) = U_0 \tilde{U}_1 \cdots \tilde{U}_q \\ &= \bigcup_{\beta \in \mathfrak{H}} \beta(U) = \bigcup \{X^{(\boldsymbol{n})} : \boldsymbol{n} \in (\{-\infty\} \cup \mathbb{Z})^q\}. \end{aligned}$$

Also $\mathfrak{H}$ has a dense orbit on G and U is open in G. Therefore, $U_{\alpha++}$ is dense in G. But $U_{\alpha++}$ is closed in G, since U is tidy for α. Therefore, $U_{\alpha++} = G$. □

COROLLARY 3.2. *The set $S = \bigcup \{X^{(\boldsymbol{n})} : \boldsymbol{n} \in (\{-\infty\} \cup \mathbb{Z})^q \setminus \mathbb{Z}^q\}$ is a closed nowhere dense subset of G.*

Proof. By Proposition 2.1 and Corollary 3.1, $S = G \setminus \bigcup_{\boldsymbol{n} \in \mathbb{Z}^q} X^{(\boldsymbol{n})}$. Therefore, by Proposition 2.2, S is closed. Also, S is $\mathfrak{H}$-invariant. Hence if it had an interior point we would have $S = G$, which is not the case. This shows that S is nowhere dense. □

COROLLARY 3.3. *$\mathfrak{H}_1$ has a dense orbit on $X^{(\boldsymbol{n})}$ for every $\boldsymbol{n} \in \mathbb{Z}^q$.*

Proof. Let $\boldsymbol{n} \in \mathbb{Z}^q$. Since $X^{(\boldsymbol{n})}$ is open, there exists $x \in X^{(\boldsymbol{n})}$ such that $\mathfrak{H}(x) = \{\alpha(x) \mid \alpha \in \mathfrak{H}\}$ is dense in G. By Proposition 3.1 and Proposition 2.1,

$$X^{(\boldsymbol{n})} \cap \beta(X^{(\boldsymbol{n})}) = X^{(\boldsymbol{n})} \cap X^{(\rho(\beta)+\boldsymbol{n})} = \emptyset \quad \text{for all } \beta \in \mathfrak{H} \setminus \mathfrak{H}_1. \tag{7}$$

As $X^{(\boldsymbol{n})}$ is open and compact,

$$X^{(\boldsymbol{n})} = \overline{X^{(\boldsymbol{n})} \cap \mathfrak{H}(x)} = \overline{X^{(\boldsymbol{n})} \cap \mathfrak{H}_1(x)} = \overline{\mathfrak{H}_1(x)};$$

the second equality holds due to (7) and the last equality holds due to (3). □

PROPOSITION 3.2. *Any $\mathfrak{H}_1$-invariant open subgroup of U contains U_0.*

Proof. Let V be a $\mathfrak{H}_1$-invariant open subgroup of U. Then V is closed, and hence compact. By Corollaries 3.2 and 3.1, there exists $\boldsymbol{n} \in \mathbb{Z}^q$ such that $V \cap X^{(\boldsymbol{n})}$ is non-empty. Then, by Corollary 3.3, $X^{(\boldsymbol{n})}$ is contained in V. Let $x \in X^{(\boldsymbol{n})}$ and consider $u \in U_0$. Then, by (3), $ux \in X^{(\boldsymbol{n})} \subset V$, and hence $u = (ux)x^{-1} \in V$. Thus, $U_0 \subset V$. □

PROPOSITION 3.3. *U_0 is a normal subgroup of G.*

Proof. Since U_0 is $\mathfrak{H}$-invariant and $\mathfrak{H}$ has a dense orbit in G, it is enough to show that U_0 is normal in U. Choose $\alpha \in \mathfrak{H}$ as in Corollary 3.1. Let $k \in \mathbb{N}$. Then $\alpha^{-k}(U)$ is an open subgroup of U. Since U is compact, $\alpha^{-k}(U)$ is a subgroup of finite index in U. Let $V_k = \bigcap_{u \in U} u\alpha^{-k}(U)u^{-1}$. Then V_k coincides with an intersection of finitely many conjugates of $\alpha^{-k}(U)$, and hence is an open normal subgroup of U. Since U and $\alpha^{-k}(U)$ are $\mathfrak{H}_1$-invariant, V_k is $\mathfrak{H}_1$-invariant. Therefore, by Proposition 3.2, $U_0 \subset V_k$. Now $\bigcap_{k=1}^{\infty} V_k \subset \bigcap_{k=1}^{\infty} \alpha^{-k}(U) = U_0$, and hence U_0 is normal in U. □

COROLLARY 3.4. *For distinct $j, k \in \{1, 2, \ldots, q\}$, $[\tilde{U}_j, \tilde{U}_k] \subset U_0$.*

Proof. Let $x \in \tilde{U}_j$ and $y \in \tilde{U}_k$. Then by Corollary 3.1 there exist $n_i \in \{-\infty\} \cup \mathbb{Z}$ and $u_i \in X_i^{(n_i)}$, for all $1 \leq i \leq q$, such that

$$xyx^{-1}y^{-1} = u_1 \cdots u_q \in X^{(\boldsymbol{n})}, \tag{8}$$

where $\boldsymbol{n} = (n_1, \ldots, n_q)$.

By Proposition 3.1 there exists $\gamma \in \mathfrak{H}$ such that

$$\gamma(U_j) \subset U_j, \quad \bigcap_{n=1}^{\infty} \gamma^n(U_j) = U_0, \quad \text{and} \quad \gamma(U_i) = U_i \quad \text{for all } i \neq j.$$

Therefore, there exists a sequence $a_m \to \infty$ of positive integers such that, as $m \to \infty$, the sequences $\{\gamma^{a_m} x\}$, $\{\gamma^{a_m} y\}$, and $\{\gamma^{a_m} u_i\}$ converge, say $\gamma^{a_m} x \to x_0 \in U_0$, $\gamma^{a_m} y \to y_0 \in \tilde{U}_k$, and $\gamma^{a_m} u_i \to z_i$, for $i = 1, \ldots, q$, with $z_i \in X_i^{(n_i)}$ if $i \neq j$, and $z_j \in U_0$. Then

$$\gamma^{a_m}(xyx^{-1}y^{-1}) \to x_0 y_0 x_0^{-1} y_0^{-1} \quad \text{as } m \to \infty. \tag{9}$$

Since U_0 is a normal subgroup of $\tilde{U}_k$, $x_0 y_0 x_0^{-1} y_0^{-1} \in U_0$. Also

$$\gamma^{a_m}(u_1 \cdots u_q) \to z_1 \cdots z_q \in X^{(\boldsymbol{n}')} \quad \text{as } m \to \infty, \tag{10}$$

where $\boldsymbol{n}' = (n_1', \ldots, n_q')$, $n_i' = n_i$ if $i \neq j$, and $n_j' = -\infty$.

By (8)–(10), and Proposition 2.1, $n_i' = -\infty$ for all i. This proves that $u_i \in U_0$ for all $i \neq j$. Therefore, $xyx^{-1}y^{-1} \in \tilde{U}_j$.

By a similar argument with k in place of j, we have $xyx^{-1}y^{-1} \in \tilde{U}_k$. Hence $xyx^{-1}y^{-1} \in \tilde{U}_j \cap \tilde{U}_k = U_0$ by Corollary 2.1. □

3.1. *Structure of G/U_0.* Let $\bar{G} = G/U_0$ and $\pi : G \to \bar{G}$ denote the quotient homomorphism. Consider the factor action of $\mathfrak{H}$ on $\bar{G}$.

For $1 \leq j \leq q$, let $G_j = \pi(\tilde{U}_j)$. Then G_j is a closed subgroup of $\bar{G}$. By Corollary 3.4, for $i \neq j$, the elements of G_i commute with the elements of G_j. By Corollary 2.1, $G_i \cap G_j = \{e\}$. Therefore,

$$\bar{G} = G_1 \times \cdots \times G_q. \tag{11}$$

Clearly the projection of $\bar{G}$ onto G_j is $\mathfrak{H}$-equivariant for each j. Hence $\mathfrak{H}$ has a dense orbit on G_j. Moreover, $\bar{U}_j := \pi(U_j)$ is an open compact subgroup of G_j which is tidy for the $\mathfrak{H}$-action on G_j, and the conclusion of Theorem 2.1 holds for $q = 1$; moreover, $\bigcap_{\beta \in \mathfrak{H}} \beta(\bar{U}_j) = \{e\}$.

4. *The case of $q = 1$ and $U_0 = \{e\}$*

In view of (11), in order to analyze the group theoretic structure of $\bar{G}$ and the algebraic properties of the action of $\mathfrak{H}$ on $\bar{G}$, it is enough to study the structure of G_j and the action of $\mathfrak{H}$ restricted to G_j for each j separately. Therefore, unless specified otherwise, in this section we assume that G is a locally compact totally disconnected group, $\mathfrak{H} \subset \mathrm{Aut}(G)$ is a finitely generated abelian group whose action on G has a dense orbit, and the following holds: (i) there exists an open compact subgroup U of G which is tidy for $\mathfrak{H}$; (ii) there exists an element $\alpha \in \mathfrak{H}$ such that $\alpha(U) \supset U$, $\bigcap_{n \geq 0} \alpha^{-n}(U) = \{e\}$; and (iii) there exists a surjective homomorphism $\rho : \mathfrak{H} \to \mathbb{Z}$ such that $\beta(U) = \alpha^{\rho(\beta)}(U)$ for all $\beta \in \mathfrak{H}$.

Let $\mathfrak{H}_1 = \ker \rho = \{\beta \in \mathfrak{H} : \beta(U) = U\}$. For $n \in \mathbb{Z}$ we put $X^{(n)} = \alpha^n(U) \setminus \alpha^{n-1}(U)$. By Corollary 3.1,

$$U_{\alpha++} = \bigcup_{n \geq 0} \alpha^n(U) = \bigcup_{n \in \{-\infty\} \cup \mathbb{Z}} X^{(n)} = G, \tag{12}$$

where $X^{(-\infty)} = U_0 = \{e\}$.

PROPOSITION 4.1. *U is a normal subgroup of G. In particular, $\alpha^k(U)$ is normal in G for all $k \in \mathbb{Z}$.*

Proof. (Cf. Proposition 3.3) For every $k \in \mathbb{N}$, let V_k denote the unique maximal compact open normal subgroup of U contained in $\alpha^{-k}(U)$. Then V_k is an open $\mathfrak{H}_1$-invariant subgroup. By (12) there exists an integer $n_k \geq k$ such that $V_k \subset \alpha^{-n_k}(U)$ and $V_k \cap X^{(-n_k)} \neq \emptyset$. Therefore, by Corollary 3.3, $X^{(-n_k)} \subset V_k$. Since

$$\alpha^{-n_k-1}(U)X^{(-n_k)} = X^{(-n_k)} \subset V_k$$

and V_k is a group, we have that $\alpha^{-n_k-1}(U) \subset V_k$. Therefore, $V_k = \alpha^{-n_k}(U)$. Hence $\alpha^{-n_k}(U)$ is normal in U. Therefore, U is normal in $\alpha^{n_k}(U)$, where $n_k \geq k$. Since $k \in \mathbb{N}$ can be chosen arbitrarily large, by (12) we have that U is normal in G. □

PROPOSITION 4.2. *Let $F = U/\alpha^{-1}(U)$. Then every element of F, other than the identity, has order p, where p is a prime number.*

Proof. Since U and $\alpha^{-1}(U)$ are $\mathfrak{H}_1$-invariant, $\mathfrak{H}_1$ acts on F via group automorphisms, and the natural quotient map from U to F is $\mathfrak{H}_1$-equivariant. Therefore, by Corollary 3.3,

$\mathfrak{H}_1$ has a dense orbit on $F \setminus \{e\}$, and as F is discrete, it follows that the $\mathfrak{H}_1$-action on $F \setminus \{e\}$ is transitive. Therefore, all elements of $F \setminus \{e\}$ have the same order, say p. If $p = rs$, with $r, s \in \mathbb{N}$ and $r \geq 2$, then r is the order of an element in $F \setminus \{e\}$ and so $r = p$ and $s = 1$, which shows that p is a prime. □

COROLLARY 4.1. *For any $k \in \mathbb{N}$, the order of the group $U/\alpha^{-k}(U)$ is a prime power. In particular, $U/\alpha^{-k}(U)$ is a nilpotent group.*

Proof. For all $k \in \mathbb{N}$,

$$|U/\alpha^{-k}(U)| = |U/\alpha^{-(k-1)}(U)| \cdot |\alpha^{-(k-1)}(U)/\alpha^{-k}(U)|,$$

and $\alpha^{-(k-1)}(U)/\alpha^{-k}(U) \cong U/\alpha^{-1}(U)$. Therefore, we get $|U/\alpha^{-k}(U)| = |U/\alpha^{-1}(U)|^k$, which is a prime power because by Proposition 4.2 $|U/\alpha^{-1}(U)|$ is a power of a prime. □

PROPOSITION 4.3. *$\mathfrak{H}_1$ acts minimally on $X^{(n)}$ for every $n \in \mathbb{Z}$.*

Proof. Let Y be a compact $\mathfrak{H}_1$-invariant subset of $X^{(n)}$. Let k be any integer less than n, and let $V = \alpha^k(U)$. Then V is $\mathfrak{H}_1$-invariant, and $VX^{(n)} = X^{(n)}$. Therefore, VY is an open compact $\mathfrak{H}_1$-invariant subset of $X^{(n)}$. Since $\mathfrak{H}_1$ has a dense orbit in $X^{(n)}$, we have that $VY = X^{(n)}$. This shows that, for any $x \in X^{(n)}$, $Y \cap Vx \neq \emptyset$.

Since $U_0 = \{e\}$, any neighbourhood of e contains $\alpha^k(U)$ for some integer $k < n$. The above observation therefore shows that Y is dense in $X^{(n)}$, and since Y is compact it follows that $Y = X^{(n)}$. □

For open compact subgroups V and W of G, let

$$[V, W] = \{vwv^{-1}w^{-1} : v \in V,\ w \in W\}.$$

PROPOSITION 4.4. *G is abelian.*

Proof. We first show that, for any $m, n \in \mathbb{Z}$, $[\alpha^m(U), \alpha^n(U)] = \alpha^k(U)$ for some $k \in \mathbb{Z} \cup \{-\infty\}$. Let $C = [\alpha^m(U), \alpha^n(U)]$. Then C is compact and $\mathfrak{H}_1$-invariant. If $C \neq \{e\}$, then there exists a maximal integer k satisfying $C \cap X^{(k)} \neq \emptyset$. Then, by Proposition 4.3, $X^{(k)} \subset C$. Since $\alpha^{-1}(C) \subset C$, by (12), $C = \alpha^k(U)$.

Now, for all integers $m \geq 0$ let $V_m = \alpha^{-m}(U)$, and $V_\infty = \{e\}$. By Proposition 4.1 V_m is a normal subgroup of U. Also, by the above observation, for all m there exists $k_m \in \mathbb{Z} \cup \{\infty\}$ such that $[U, V_m] = V_{m+k_m}$. By Corollary 4.1, U/V_{m+1} is a finite nilpotent group, so $[U/V_{m+1}, V_m/V_{m+1}]$ is a proper subgroup of V_m/V_{m+1}, and hence k_m is in fact contained in $\mathbb{N} \cup \{\infty\}$. Now,

$$\begin{aligned} V_{1+m+k_m} &= \alpha^{-1}(V_{m+k_m}) = [\alpha^{-1}(U), V_{m+1}] \\ &\subset [U, V_{m+1}] = V_{(m+1)+k_{(m+1)}}. \end{aligned}$$

Therefore, $1 \leq k_{m+1} \leq k_m$ for all $m \geq 0$. If $k_0 = \infty$, then U is abelian, and hence, by (12), G is abelian. Now suppose that, if possible, $k_0 \in \mathbb{N}$; we shall show that this leads to a contradiction. Then $k_m \in \mathbb{N}$ for all m and hence there exists $M \geq 0$ such that $k_m = k_M \geq 1$ for all $m \geq M$.

Let $W = V_{M+k_M+k_0}$. Then we have

$$[U, V_{M+k_0}] = \alpha^{-k_M}(V_{M+k_0}) = W \tag{13}$$

and

$$\begin{aligned}[[U, U], V_{M+k_0}] = [V_{k_0}, V_{M+k_0}] &= \alpha^{-k_0}([U, V_M]) \\ &= \alpha^{-k_0}(V_{M+k_M}) = W. \end{aligned} \tag{14}$$

By Proposition 4.1, W is a normal subgroup of U. Let $\psi : U \to U/[U, W]$ denote the quotient homomorphism. For any $u \in U$ and $v \in V_{M+k_0}$, by (13), $[\psi(u), \psi(v)] = \psi([u, v]) \in \psi(W)$, which is central in $U/[U, W]$. Therefore, the map $u \mapsto [\psi(u), \psi(v)]$ is a homomorphism from U into $\psi(W)$. Then, as $\psi(W)$ is abelian, we get $[\psi([U, U]), \psi(v)] = \{e\}$. Hence

$$[[U, U], V_{M+k_0}] \subset [U, W] = \alpha^{-k_M}(W).$$

Comparing with (14) we get that $\alpha^{-k_M}(W) = W$, and hence that $\alpha^{-k_M}(U) = U$, which is a contradiction. This completes the proof. □

Now, since G is abelian, we will denote the group operation on G by $+$ and its identity element by 0.

PROPOSITION 4.5. *There exists a binary operation* $*$ *on G such that* $(G, +, *)$ *is a (locally compact) field, with* $(G \setminus \{0\}, *)$ *as the multiplicative (topological) group of the field. Also* $\{\gamma(1) : \gamma \in \mathfrak{H}\}$*, where* 1 *denotes the identity element of* $(G \setminus \{0\}, *)$*, is a dense subgroup of* $(G \setminus \{0\}, *)$*, and the action of any* $\gamma \in \mathfrak{H}$ *is given by* $\gamma(g) = \gamma(1) * g$ *for all* $g \in G$.

Proof. Let $x_0 \in X^{(0)}$, such that $\mathfrak{H}(x_0)$ is dense in G. We shall first define a group structure on $X^{(0)}$ as follows.

Let $n \in \mathbb{N}$. Now, $X^{(0)}$ is invariant under the action of $\alpha^{-n}(U)$ and there are only finitely many orbits under the action. Let X_n denote the finite orbit space $X^{(0)}/\alpha^{-n}(U)$, and $\phi_n : X^{(0)} \to X_n$ be the natural quotient map. Since $\alpha^{-n}(U)$ is $\mathfrak{H}_1$-invariant, the $\mathfrak{H}_1$-action on U induces a $\mathfrak{H}_1$-action on X_n such that ϕ_n is $\mathfrak{H}_1$-equivariant. By Corollary 3.3, $\mathfrak{H}_1$ has a dense orbit on $X^{(0)}$. Therefore, the $\mathfrak{H}_1$-action on X_n is transitive. For every $n \in \mathbb{N}$, let $x_n = \phi_n(x_0) \in X_n$. Then every element of X_n can be expressed as $\alpha(x_n)$ for some $\alpha \in \mathfrak{H}_1$. We now define a binary operation $*_n$ on X_n by

$$(\alpha(x_n)) *_n (\beta(x_n)) = (\alpha\beta)(x_n) \quad \text{for all } \alpha, \beta \in \mathfrak{H}_1; \tag{15}$$

it can be seen that, as $\mathfrak{H}_1$ is abelian, $*_n$ is well defined and $(X_n, *_n)$ is isomorphic to a quotient group of $\mathfrak{H}_1$.

Now, for $n > m \geq 1$, let $\phi_{n,m} : X_n \to X_m$ be the natural quotient map. Then $\phi_{n,m}$ is a homomorphism from the group $(X_n, *_n)$ to $(X_m, *_m)$. Therefore, the inverse limit of the finite groups $(X_n, *_n)$ as $n \to \infty$ exists, and it is the topological space $X^{(0)}$, with a group structure, which we shall denote by $*$, such that ϕ_n is a homomorphism for all n. Then $(X^{(0)}, *)$ is a topological group with x_0 as the unit element.

Next let $\beta \in \mathfrak{H}_1$ and $y \in X^{(0)}$ be arbitrary. For any $n \in \mathbb{N}$, there exists $\gamma \in \mathfrak{H}_1$ such that $\phi_n(y) = \gamma(x_n)$. By (15) for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\phi_n(\beta(y)) = \beta(\phi_n(y)) &= \beta(\gamma(x_n)) \\ &= \beta(x_n) *_n \gamma(x_n) = \phi_n(\beta(x_0)) *_n \phi_n(y).\end{aligned}$$

Therefore,

$$\beta(y) = \beta(x_0) * y \quad \text{for all } \beta \in \mathfrak{H}_1 \text{ and } y \in X^{(0)}. \tag{16}$$

We now define a binary operation $*$ on G as follows. For $i \in \{1, 2\}$ and $g_i \in G$, there exist a unique $k_i \in \{-\infty\} \cup \mathbb{Z}$ and a $y_i \in X^{(0)}$ such that $g_i = \alpha^{k_i}(y_i)$; here $\alpha^{-\infty}(z) := 0$ for any $z \in X^{(0)}$. We define

$$g_1 * g_2 = \alpha^{k_1+k_2}(y_1 * y_2), \tag{17}$$

with the convention that $k_1 + k_2 = -\infty$ if $k_1 = -\infty$ or $k_2 = -\infty$.

Since $\alpha^k(X^{(0)})$ is open in G for all $k \in \mathbb{Z}$, and $\alpha^{-k}(z) \to 0$ as $k \to \infty$, for every $z \in Z$, the binary operation $*$ is continuous on $G \times G$. Now it is straightforward to verify that $(G \setminus \{0\}, *)$ is a topological group. From (15) and (17), and the commutativity of $\mathfrak{H}$, we get that it is an abelian group. We also note that

$$\alpha(g) = \alpha(x_0) * g \quad \text{for all } g \in G. \tag{18}$$

By (16) and (18),

$$\gamma(x_0) * g = \gamma(g) \quad \text{for all } \gamma \in \mathfrak{H}. \tag{19}$$

Therefore, since γ is an automorphism of $(G, +)$,

$$\gamma(x_0) * (z_1 + z_2) = \gamma(z_1 + z_2) = \gamma(z_1) + \gamma(z_2) = (\gamma(x_0) * z_1) + (\gamma(x_0) * z_2).$$

Also $\mathfrak{H}(x_0)$ is dense in G, and the binary operations $+$ and $*$ are continuous on $G \times G$. Therefore, $*$ is distributive over $+$. This proves that $(G, +, *)$ is a locally compact field.

The last statement in the proposition follows from (19). □

LEMMA 4.1. *Let K be a locally compact field, then K^* contains a finitely generated dense subgroup if and only if* $\mathrm{Char}(K) = 0$.

Proof. First suppose that $\mathrm{Char}(K) = 0$. Then K is a finite extension of $\mathbb{Q}_p$ for some prime p (see [**12**]). Let Z denote the compact ring of elements of norm at most 1 in K, and let U be the group of units in Z. The exponential map from $pZ \to U$ is a homomorphism, and a local homeomorphism. Therefore, for some $k > 0$ the exponential map is an isomorphism of the additive group $p^k Z$ and an open compact subgroup U_k of U for some $k > 0$. Now $p^k Z \cong Z$ and $Z/\mathbb{Z}_p^d$ is finite, where $d = [K : \mathbb{Q}_p]$. Since $\mathbb{Z}_p$ contains $\mathbb{Z}$ as a dense subgroup, we conclude that U_k, and hence U, contains a finitely generated dense subgroup. Since K^* is generated by p and U, this proves the 'if' part.

Now suppose that $\mathrm{Char}(K) = p > 0$. Then $K = F((t))$, the topological field of Laurent power series over a finite field F of characteristic p (see [**12**]). Put $U = F[[t]]^* = F^* + tF[[t]]$. Note that $U_n := F^* + t^n F[[t]]$ is an open compact subgroup of U and $|U/U_n| = q^{n-1}$, where $q = |F|$. For any $f \in U_n$, $f^p \in U_{pn}$. Let $n > 1$ and $r \in \mathbb{N}$ be such that $p^{r-1} < n \leq p^r$. Since $U = U_1$, every element of U/U_n is of order at most p^r. Therefore, any set of generators of U/U_n has at least q^{n-1}/p^r generators.

Now, if possible, let $\mathfrak{H}$ be a finitely generated dense subgroup of K^*. Then $\mathfrak{H} \cap U$ is generated by finitely many, say N, elements and is dense in U. This implies that the image of $\mathfrak{H} \cap U$ on U/U_n is surjective, and hence U/U_n is generated by at most N elements. Therefore, $q^{n-1}/p^r \leq N$ for all $r \in \mathbb{N}$ and $p^{r-1} < n \leq p^r$, which is not possible. This proves the 'only if' part. □

Combining Section 3.1, Proposition 4.5, and Lemma 4.1 we obtain the following.

THEOREM 4.1. *Let G be a locally compact totally disconnected group. Let $\mathfrak{H}$ be a finitely generated abelian group of automorphisms on G whose action has a dense orbit on G. Then there exist a compact normal $\mathfrak{H}$-invariant subgroup U_0 of G, locally compact totally disconnected fields $F_1, \ldots, F_q$ of characteristic zero, and an isomorphism $\psi : G/U_0 \to F_1 \times \cdots \times F_q$ (the Cartesian product of the additive topological groups of the F_i). Moreover, there exists a homomorphism $\phi : \mathfrak{H} \to F_1^* \times \cdots \times F_q^*$ such that, for any $\alpha \in \mathfrak{H}$ and $x \in G/U_0$, we have $\psi(\alpha(x)) = \phi(\alpha)\psi(x)$, where $\prod_i F_i^*$ acts on $\prod_i F_i$ via coordinate-wise multiplications. Also $\phi(\mathfrak{H})$ is dense in $F_1^* \times \cdots \times F_q^*$.*

5. *The general case*

In this section we prove Theorem 1.1 for a general locally compact group, using the results of the previous sections for totally disconnected groups and results from [**5**] for connected locally compact groups.

A locally compact group G is said to be *almost connected* if G/G^0 is compact, where G^0 denotes the connected component of the identity in G. From a well-known theorem of Montgomery and Zippin (see [**10**, Theorem 4.6]) the following was deduced in [**5**, Lemma 5.1], for this class of groups.

LEMMA 5.1. *Let G be an almost connected locally compact group. Then G has a unique maximal compact normal subgroup C; moreover, G/C is a Lie group.*

We next note the following.

LEMMA 5.2. *Let G be a finite-dimensional almost connected locally compact group such that G^0 has no non-trivial totally disconnected compact normal subgroup. Then G admits a unique maximal totally disconnected compact normal subgroup N. Also, G/N is a Lie group, and G^0N is an open subgroup of G.*

Proof. Since G is finite dimensional, by a theorem of Montgomery and Zippin [**10**, Theorem 4.6], there exist totally disconnected compact normal subgroups N such that G/N is a Lie group. Then G/G^0N is a compact totally disconnected Lie group, and hence it is finite. We choose N to be a subgroup from this class of subgroups, with the additional condition that the order of G/G^0N is the minimum possible.

Now let N' be any totally disconnected compact normal subgroup of G. Then NN' is also a totally disconnected compact normal subgroup. By the choice of N the G/G^0NN' has the same order as G/G^0N. Therefore, $G^0NN' = G^0N$, and so $N' \subset G^0N$. Also $NN' \cap G^0$ is a totally disconnected compact normal subgroup of G^0, and hence by the condition in the hypothesis it is trivial. This shows that $NN' = N$, and so $N' \subset N$. This shows that N is the unique maximal totally disconnected compact normal subgroup N of G. The choice of N as above shows also that the second statement in the lemma holds. □

PROPOSITION 5.1. *Let G be a finite-dimensional locally compact group and $\mathfrak{H}$ be a finitely generated abelian group of automorphisms of G. Suppose that the $\mathfrak{H}$-action has a dense orbit on G. Suppose also that G^0 has no non-trivial totally disconnected* *compact normal subgroup. Then there exists an* Aut(G)*-invariant totally disconnected closed subgroup T of G such that $G = G^0T$ and $G^0 \cap T$ is trivial. Thus, G is a direct product of G^0 and T, and the $\mathfrak{H}$-action on G^0 (as well as on T) has dense orbits.*

Proof. Let $\bar{G} = G/G^0$. Then $\bar{G}$ is totally disconnected and $\mathfrak{H}$ has a dense orbit on $\bar{G}$. As in §2 let U be a compact open subgroup of $\bar{G}$ which is tidy for $\mathfrak{H}$. Let G' denote the inverse image of U in G. Since $G'/G^0 = U$ is compact, by Lemma 5.2 there exists a unique maximal totally disconnected compact normal subgroup of G', say N; also G'/N is a Lie group and G^0N is open in G'. Since G' is $\mathfrak{H}_1$-invariant, so is N. Therefore $\bar{N}$, the image of N in $G'/G^0 = U$, is a compact open $\mathfrak{H}_1$-invariant subgroup of U. By Proposition 3.1 there exists $\alpha \in \mathfrak{H}$ such that

$$\alpha(U) \supset U, \quad \bigcup_{n=0}^{\infty} \alpha^n(U) = \bar{G}, \quad \text{and} \quad \bigcap_{n=0}^{\infty} \alpha^{-n}(U) = U_0. \tag{20}$$

By Proposition 3.2, $U_0 \subset \bar{N}$. Therefore, there exists $n \geq 0$ such that $\alpha^{-n}(U) \subset \bar{N}$, or equivalently $\alpha^n(\bar{N}) \supset U$. Hence $\alpha^n(N)G^0 \supset G'$. Therefore, $G' = N'G^0$, where $N' := \alpha^n(N) \cap G'$. Now $\alpha^n(N)$ is normal in $\alpha^n(G')$ and $\alpha(G') \supset G'$. Therefore, N' is a totally disconnected compact normal subgroup of G'. So by the maximality, $N \supset N'$.

Recall that by hypothesis G^0 has no non-trivial compact totally disconnected normal subgroup. Hence $N \cap G^0 = \{e\}$. Thus we have $N \supset N'$, $NG^0 = N'G^0$, and $N \cap G^0 = \{e\} = N' \cap G^0$. This implies that $N = N'$. Hence $\bar{N} = U$ and so $\alpha^{-k}(U) \subset \bar{N}$ for all $k \geq 0$. Therefore, by an argument as above, $N = \alpha^k(N) \cap G'$ for all $k \geq 0$. The argument also shows that $\alpha^k(N)$ is an open normal subgroup of $\alpha^l(N)$ for all integers $k \leq l$.

Now let $T = \bigcup_{n=0}^{\infty} \alpha^n(N)$. Then T is a locally compact totally disconnected subgroup of G, and, by (20), $G = TG^0$. Since N is normalized by G^0, $\alpha^n(N)$ is normalized by G^0, and hence T is normalized by G^0. Therefore, T is normal in G. Also, from the choice of N and T we see that $T \cap G^0 = \{e\}$. Also, for any $\beta \in$ Aut(G), $\beta(T)T/T$ is a union of a sequence of totally disconnected compact normal subgroups of $G/T \cong G^0$, and therefore by the condition on G^0 it is trivial. This shows that T is Aut(G)-invariant. The second statement in the proposition readily follows from the first one. □

In proving Theorem 1.1 we shall use the following special case proved in [**5**].

THEOREM 5.1. [**5**, Theorem 1.1] *Let G be a connected Lie group. Suppose that there exists an abelian subgroup $\mathfrak{H}$ of* Aut(G) *such that the $\mathfrak{H}$-action on G has a dense orbit. Then there exists a compact subgroup C contained in the center of G such that $G/C \cong V$, where V is a vector group, namely the additive (topological) group of a finite-dimensional real vector space.*

Together with Proposition 5.1, Theorem 5.1 implies the following.

COROLLARY 5.1. *Let G be a locally compact group and $\mathfrak{H}$ be a finitely generated abelian group of automorphisms of G, such that the $\mathfrak{H}$-action has a dense orbit on G. Let C be the maximal compact normal subgroup of G^0 (see Lemma 5.1). Then C is connected, G^0/C is a vector group, and $G/C \cong G^0/C \times T$, where T is a totally disconnected closed subgroup of G/C, invariant under the factor action of $\mathfrak{H}$ on G/C.*

Proof. We note that, since G^0/C is a Lie group, G/C is finite dimensional. Applying Proposition 5.1 to the $\mathfrak{H}$-action on G/C, we get that there exists an $\mathrm{Aut}(G/C)$-invariant totally disconnected closed subgroup T of G/C such that $G/C \cong G^0/C \times T$. Then the $\mathfrak{H}$-action on G^0/C has a dense orbit, and, since G^0/C has no non-trivial compact normal subgroup, Theorem 5.1 implies that G^0/C is a vector group. Then C has to be connected, since if C' is an open subgroup of C then the quotient homomorphism of G^0/C' onto G^0/C is a covering map, and it has to be a homeomorphism since G^0/C is simply connected. This proves the corollary. □

Proof of Theorem 1.1. Recall that by Corollary 1.2 if a subgroup K with the desired properties exists then it is unique. We now prove the existence of the subgroup K, satisfying the conditions. Let C be the maximal compact normal subgroup of G^0 (see Lemma 5.1). By Corollary 5.1 there exists a totally disconnected normal subgroup T of G/C, invariant under the $\mathfrak{H}$-action on G/C, such that $G/C \cong G^0/C \times T$. Let U_0 be the subgroup of T such that the conclusion of Theorem 4.1 holds (for T in the place of G there). Let K be the closed subgroup of G containing C, such that K/C is the subgroup of G/C corresponding to U_0 in the decomposition as above. Then K is a $\mathfrak{H}$-invariant compact normal subgroup of G, and $G/K \cong G^0/C \times T/U_0$. Now the assertions (i)–(iii) of the theorem follow from Theorems 5.1 and 4.1.

The assertion (iv) is inspired by [**5**, §2]; however, while [**5**] depends on a result from [**2**], here we present a direct argument. First note that the additive group $W := V \times F_1 \times \cdots \times F_q$ is a finitely generated module over the ring $\mathcal{R} := \mathbb{R} \times F_1 \times \cdots \times F_q$, and $\mathrm{End}_{\mathcal{R}}(W) = \mathrm{End}_{\mathbb{R}}(V) \times F_1 \times \cdots \times F_q$. The centralizer of $\varphi(\mathfrak{H})$ in $\mathrm{End}_{\mathcal{R}}(W)$, say Z, is a topologically closed subring of $\mathrm{End}_{\mathcal{R}}(W)$ containing the group ring $\mathcal{R}[\varphi(\mathfrak{H})]$. By our assumption, there exists $w \in W$ such that $\varphi(\mathfrak{H})w$ is dense in W. In particular, $\mathcal{R}[\varphi(\mathfrak{H})]w = W$. Therefore, the homomorphism $\psi : Z \to W$ of the additive groups, given by $\psi(z) = zw$ for all $z \in Z$, is surjective. Also if $z \in \ker(\psi)$, then $z(hw) = h(zw) = 0$ for all $h \in \varphi(\mathfrak{H})$, and since $\varphi(\mathfrak{H})w$ is dense in W this implies that $z = 0$. Thus, ψ is bijective. Since Z and W are locally compact second countable groups, it follows that ψ is a homeomorphism. Since $\psi(\varphi(\mathfrak{H}))$ is dense in W, we conclude that $\varphi(\mathfrak{H})$ is dense in Z. Since the centralizer of $\varphi(\mathfrak{H})$ in $\mathrm{GL}(V) \times F_1^* \times \cdots \times F_q^*$ is contained in Z, this proves (iv). □

6. *The case of finite-dimensional locally finitely generated G*

In this section we use various properties of locally compact fields together with Theorem 1.1 to obtain proofs of Theorems 1.2 and 1.3.

6.1. *Theorem 1.2 for totally disconnected G.*

We first consider the case when G is totally disconnected and locally finitely generated. Let the notation be as in the statement of Theorem 1.1. Let $U_0 = K$. Then G/U_0 is a product of the additive subgroups of locally compact fields of characteristic zero. Let U be an open compact normal subgroup of G containing U_0.

6.1.1. *To show that* $\mathrm{Aut}(U)$ *is profinite.* Since G is locally finitely generated, there exists an open subgroup, say O, containing a dense finitely generated subgroup. By part (vi) of Theorem 1.1 there exists $\alpha \in \mathfrak{H}$ such that $\alpha(U)$ contains the finitely generated subgroup, and hence $O \subset \alpha(U)$. Since O is open it is of finite index in $\alpha(U)$. It follows that $\alpha(U)$ contains a dense subgroup which is finitely generated, and hence so does U. This implies, in particular, that for any natural number m, U has only finitely many subgroups of index m (cf. [**9**, Corollary 1.1.2]). Since there is a basis of neighborhoods of the identity consisting of open normal subgroups this implies that the group $\mathrm{Aut}(U)$, of bicontinuous automorphisms of U, is a profinite group.

6.1.2. *To show that $G = Z_G(U_0)U_0$.* Let $Z_G(U)$ denote the centralizer of U in G. Then $Z_G(U)U$ is a closed normal subgroup of G because U is normal. We note that $G/Z_G(U)U$ is a quotient of G/U_0, which is a product of locally compact fields of characteristic zero. Hence $G/Z_G(U)U$ has no non-trivial normal subgroup of finite index. On the other hand, the conjugation action of G on U induces an injective homomorphism $\psi : G/Z_G(U)U \to \mathrm{Aut}(U)/\mathrm{Int}(U)$, where $\mathrm{Int}(U)$ is the normal subgroup of $\mathrm{Aut}(U)$ consisting of all inner automorphisms of U. Since $G/Z_G(U)U$ has no non-trivial subgroups of finite index, and $\mathrm{Aut}(U)$ is profinite, it follows that ψ is trivial. Hence $G = Z_G(U)U$.

Since $U_0 \subset U$, it follows also that $G = Z_0U$, where $Z_0 := Z_G(U_0)$. Since U_0 is $\mathfrak{H}$-invariant, so is Z_0, and hence we get that $G = Z_0\alpha(U)$ for all $\alpha \in \mathfrak{H}$. By Corollary 3.1 there exists $\alpha \in \mathfrak{H}$ such that $\alpha(U) \subset U$ and $\bigcap_{i=1}^{\infty} \alpha^i(U) = U_0$, so the preceding conclusion implies that $G = Z_0U_0$.

6.1.3. *To show that $Z_G(U_0)$ is abelian.* Since $G = Z_0U_0$, by Theorem 4.1 it follows that

$$Z_0/C \cong F_1 \times \cdots \times F_q \quad \text{where } C = Z_0 \cap U_0. \tag{21}$$

Then C is contained in the center of Z_0 and Z_0/C is abelian. Therefore, for each $\xi \in Z_0$, the map $\psi_\xi : Z_0 \to C$ given by $\psi_\xi(\eta) = [\xi, \eta]$ for all $\eta \in Z_0$ is a homomorphism. Clearly ψ_ξ factors through Z_0/C. In view of (21), since F_i does not admit a non-trivial continuous homomorphism into a finite group, and hence into a totally disconnected compact group, the map ψ_ξ is a trivial homomorphism. This means that Z_0 is abelian.

6.1.4. *To show that $G = \Phi \times U_0$ for a $\mathfrak{H}$-invariant subgroup Φ.* Let $X = \widehat{Z_0}$, the character group of Z_0, and let X' be the annihilator of C in X. Then, by (21),

$$X' = \widehat{Z_0/C} \cong \widehat{\prod_{i=1}^{q} F_i} = \bigoplus_{i=1}^{q} \hat{F}_i \cong \bigoplus_{i=1}^{q} F_i.$$

Therefore, X' is a torsion-free divisible subgroup of X. Hence there exists a subgroup A of X such that $X = X' \oplus A$ [**7**, Corollary A1.36(i)]. Now $A \cong X/X' = \hat{C}$. Since C is compact, $\hat{C}$ is discrete. Therefore, A is a discrete, and hence a closed, subgroup of X. Therefore,

$$Z_0 = \Phi \times C \quad \text{where } \Phi := \bigcap_{\chi \in A} \ker \chi. \tag{22}$$

Since C is totally disconnected, by [**7**, Corollary 7.70], $A \cong \hat{C}$ is a torsion group. Since X' is torsion free, any element of $X' \oplus A$ that does not belong to A has infinite order. Therefore, A is $\mathrm{Aut}(X)$-invariant. Hence Φ is $\mathrm{Aut}(Z_0)$-invariant.

Now we have $G = U_0 Z_0 = U_0 \Phi$ with U_0 and Φ normal $\mathfrak{H}$-invariant subgroups. By (21), $C = Z_0 \cap U_0$. Hence $U_0 \cap \Phi$ is trivial by (22). Therefore, $G = \Phi \times U_0$.

6.1.5. *To show that* $U_0 = \{e\}$. Now since the $\mathfrak{H}$-action on G has a dense orbit, it follows that the $\mathfrak{H}$-action on $G/\Phi = U_0$ has a dense orbit. Suppose $U_0 \neq \{e\}$. Then U_0 has a proper open normal subgroup, say V. Since $U = (\Phi \cap U) \times U_0$ contains a finitely generated dense subgroup, U_0 has a finitely generated dense subgroup. Therefore, there are only finitely many open subgroups in U_0 with the same index as that of V (see [**9**, Corollary 1.1.2]). Therefore, the finite intersection $\bigcap_{\alpha \in \mathfrak{H}} \alpha(V)$ is a proper open subgroup of U_0, and it is $\mathfrak{H}$-invariant. Since the $\mathfrak{H}$-action has a dense orbit in U_0 this is a contradiction. Therefore, U_0 is trivial. Thus $G = \Phi \cong F_1 \times \cdots \times F_q$. This proves Theorem 1.2 in the case when G is totally disconnected.

6.2. *Theorem 1.2 in the general case.* We begin by noting the following.

LEMMA 6.1. *Let G be a finite-dimensional connected locally compact group. Let Q be the smallest closed subgroup of G containing every totally disconnected compact normal subgroup of G. Then Q is a compact subgroup contained in the center of G. Furthermore, G/Q is a Lie group, and the center of G/Q contains no non-trivial compact subgroup; in particular, G/Q has no non-trivial totally disconnected compact normal subgroup.*

Proof. The first assertion is proved in [**5**, Lemma 5.2], using the well-known theorem of Montgomery and Zippin. In a compact central subgroup of a Lie group, elements of finite order form a dense subset, so to prove the second statement it suffices to show that the center of G/Q does not contain any non-trivial element of finite order. Now let z be an element of finite order contained in the center of G/Q. Let Q' be the normal subgroup of G containing Q, such that Q'/Q is the cyclic subgroup generated by z. Then Q' is abelian, as Q is central in G.

By the Montgomery–Zippin theorem there exists a totally disconnected compact normal subgroup, say N, such that G/N is a Lie group. Then $N \subset Q$, and Q'/N is a normal compact abelian Lie subgroup of G/N. Hence the automorphism group of Q'/N is discrete, and as G/N is connected it follows that Q'/N is contained in the center of G/N. It is therefore the product of its connected component of the identity $(Q'/N)^0$ and a finite subgroup, say F. Let N' be the subgroup of Q' containing N and such that $N'/N = F$. Then N' is a totally disconnected compact normal subgroup of G, and hence by the definition of Q we have $N' \subset Q$. Also, since Q is of finite index in Q', we see that $(Q'/N)^0$ is contained in Q/N. Therefore, $Q' = Q$, which means that z is trivial. As seen above this shows that the center of G/Q has no non-trivial compact subgroup. The last statement in the lemma follows from the fact that with G/Q being connected, every totally disconnected normal subgroup is contained in its center. □

Completion of proof of Theorem 1.2. Let the notation be as in the statement of the theorem. Let Q be the smallest closed subgroup of G^0 containing every compact totally disconnected normal subgroup of G^0. We note that Q is invariant under all automorphisms of G, and in particular $\mathfrak{H}$-invariant. Thus the $\mathfrak{H}$-action on G factors to G/Q.

In view of Lemma 6.1, we can apply Proposition 5.1 to G/Q in place of G and conclude that there exists a totally disconnected $\mathfrak{H}$-invariant closed normal subgroup T of G/Q such that $G/Q \cong G^0/Q \times T$, and $\mathfrak{H}$ admits dense orbits on G^0/Q and T. Therefore, by Theorem 5.1, applied to G^0/Q in the place of G there, and recalling that Q is compact by Lemma 6.1, we see that there exists a compact normal $\mathfrak{H}$-invariant subgroup C of G^0 such that C/Q is contained in the center of G^0/Q, and G^0/C is a vector group. Since, by Lemma 6.1, G^0/Q has no non-trivial compact central subgroup, it follows that $C = Q$. Thus G^0/Q is a vector group. We conclude also that Q is connected, since if Q' is an open subgroup of Q then the quotient map of G^0/Q' onto G^0/Q is a covering map and has to be a homeomorphism.

We next show that Q is contained in the center of G. Since Q is finite dimensional, $\hat{Q} \otimes \mathbb{Q} \cong \mathbb{Q}^d$, where $d = \dim Q \leq \dim G < \infty$ [**7**, Corollary 8.22]. Moreover, since Q is connected, $\hat{Q}$ is torsion free [**7**, Corollary 7.70]. Therefore, the map $i : \hat{Q} \to \hat{Q} \otimes \mathbb{Q}$ given by $i(\chi) = \chi \otimes 1$ is injective. The conjugation action of G on Q, gives an action of G on $\hat{Q}$ via automorphisms. Extending this action to the action on $\hat{Q} \otimes \mathbb{Q}$, we obtain a homomorphism $\phi : G \to \mathrm{GL}(d, \mathbb{Q})$ such that $\ker \phi = Z_G(Q)$, the centralizer of Q in G. Therefore, $Z_G(Q)$ is a closed subgroup of countable index in G. Hence, by the Baire category theorem, $Z_G(Q)$ is an open subgroup of G. Since $Z_G(Q)$ is $\mathfrak{H}$-invariant, and since $\mathfrak{H}$ has a dense orbit on G, we conclude that $Z_G(Q) = G$. Thus Q is contained in the center of G.

Since T is a totally disconnected and has a dense $\mathfrak{H}$ orbit, by §6.1 we conclude that $U_0 = \{e\}$ and $T \cong F_1 \times \cdots \times F_q$, where the F_i are the additive subgroups of locally compact totally disconnected fields of characteristic zero. Therefore, K in the statement of Theorem 1.2 is the same as Q as above; this proves the theorem. □

6.3. *Proof of Theorem 1.3.* We are now given that G^0 is a Lie group. Then K as in Theorem 1.1, which is the same as Q above, is a finite-dimensional torus. Let $\Psi_1, \ldots, \Psi_q$ be the closed normal subgroups of G containing K and such that $F_i = \Psi_i/K$ for all $i = 1, \ldots, r$, where F_i are the additive subgroups of locally compact totally disconnected fields of characteristic zero, as above. Let $\Psi = \Psi_1 \cdot \Psi_2 \cdot \cdots \cdot \Psi_q$. Then Ψ is an $\mathfrak{H}$-invariant closed normal subgroup of G such that $G = G^0\Psi$, and $G^0 \cap \Psi = K$.

We next show that Ψ is contained in the center of G. Since K is central in G and G/K is abelian, it follows that for any $g \in G$ the map θ_g defined by $\theta_g(\xi) = [\xi, g] := \xi g \xi^{-1} g^{-1}$, for all $\xi \in \Psi$, is a continuous homomorphism of Ψ into K; furthermore, for all $g \in G$, θ_g factors to $\Psi/K \cong \prod_{i=1}^q F_i$. Since K is a Lie group it follows that each θ_g is trivial on an open subgroup of Ψ containing K. Since G contains a finitely generated subgroup which is dense in an open subgroup of G, it follows that there is a compact open subgroup, say Ω_1, of Ψ containing K, such that θ_g is trivial on Ω_1 for all g in an open subgroup of G. Thus $Z_G(\Omega_1)$ is an open subgroup of G. Therefore, $\Omega := Z_G(\Omega_1) \cap \Omega_1$ is a compact open abelian subgroup of Ψ containing K, and it centralizes G^0.

Since $G/G_0 \cong F_1 \times \cdots \times F_q$, and $\mathfrak{H}$ has a dense orbit on G/G_0 preserving each F_i, by Theorem 4.1 there exists $\alpha \in \mathfrak{H}$ such that

$$\alpha(\Omega) \supset \Omega \quad \text{and} \quad \bigcup_{k=1}^{\infty} \alpha^k(\Omega) = \Psi.$$

Then each $\alpha^k(\Omega)$ is abelian and centralizes G^0, and hence it follows that Ψ is abelian and centralizes G^0. Since $G = G^0\Psi$, this shows that Ψ is contained in the centre of G.

Now $\Psi_1, \ldots, \Psi_q$ are closed normal $\mathfrak{H}$-invariant abelian subgroups of G, and $G = G^0\Psi_1 \cdots \Psi_q$. To complete the proof it suffices to show that each Ψ_i contains a closed $\mathfrak{H}$-invariant subgroup, say Φ_i, such that Ψ_i is a direct product of K and Φ_i.

Let $1 \le i \le r$ be fixed. Let $X = \hat{\Psi}_i$ be the character group of Ψ_i. The annihilator of K, say A, is a closed subgroup of X isomorphic to the dual of $\Psi_i/K \cong F_i$. Therefore, $A \cong \hat{F}_i \cong F_i$ (the character group $\hat{F}_i$ of F_i is identified with F in the usual way, by associating to $\xi \in F_i$ the character $\eta \mapsto \chi(\xi\eta)$ for all $\eta \in F$, where χ is a fixed non-trivial character on F). In particular, A is divisible, and hence there exists a subgroup B of X such that $X = A \oplus B$ [**7**, Corollary A1.36(i)]. Also, $X/A \cong \hat{K} \cong \mathbb{Z}^d$ as topological groups, where $d = \dim(K)$. Therefore, B is a discrete, and hence a closed, subgroup of X isomorphic to $\mathbb{Z}^d$. We shall use the addition notation for the group operation on X; the multiplication with respect to the field structure on A will be set out by juxtaposition.

The $\mathfrak{H}$-action on Ψ_i induces a dual action on X. The subgroup A is invariant under the $\mathfrak{H}$-action; in fact, it can be seen that the action of $h \in \mathfrak{H}$ on $F_i = \Psi_i/K$ is given by multiplication by $\lambda \in F_i^*$, so then the h-action on $A \cong \hat{F}_i \cong F_i$ is also given by multiplication by λ, under the identification of A with F_i. Let $h_0 \in \mathfrak{H}$ be such that the factor action of h_0 on $F_i = \Psi_i/K$ is by multiplication by an element $\lambda \in F_i^*$ with absolute value $|\lambda|$ greater than 1; clearly such an element h_0 exists. For any $a \in A$ and $b \in B$ we have $h_0(a + b) = \lambda a + \varphi(b) + \gamma(b)$, where φ is a homomorphism of $\mathbb{Z}^d$ into A and γ is an automorphism of B.

Now let P be the space of all homomorphisms of $\mathbb{Z}^d$ into A. It has the natural structure of a d-dimensional vector space over A. For $\psi \in P$ let B_ψ denote the subgroup of X defined by $B_\psi = \{\psi(x) + x : x \in \mathbb{Z}^d\}$. We note that every discrete subgroup B' such that $X = A + B'$ is of the form B_ψ for some $\psi \in P$, and that the subgroup B_ψ is h_0-invariant if and only if $\lambda\psi(x) + \varphi(x) = \psi(\gamma(x))$ for all $x \in \mathbb{Z}^d$.

To produce a Φ_i as desired we obtain a $\mathfrak{H}$-invariant B_ψ as follows. Let q be the map defined by $q(\psi) = \psi \circ \gamma$ for all $\psi \in P$. It is an A-linear map. As γ is an automorphism of $B \cong \mathbb{Z}^d$, it is given by a matrix from $\mathrm{GL}(d, \mathbb{Z})$, say M_γ. It can be seen that with respect to a suitable basis of P the matrix of the map q is given by the transpose of M_γ. In particular, the eigenvalues of q are the same as those of M_γ. As $M_\gamma \in \mathrm{GL}(d, \mathbb{Z})$ its eigenvalues are units in the ring of algebraic integers, and hence their absolute value (in the appropriate finite extension field of F_i containing them) is 1; this may also be seen, alternatively, by noting that $\mathrm{GL}(d, \mathbb{Z})$ preserves a compact open subgroup of A^d. Since by choice λ is of absolute value greater than 1, it now follows that the map $\psi \mapsto q(\psi) - \lambda\psi$ is invertible. Therefore, for φ as above, there exists a unique $\psi \in P$ such that $\varphi = q(\psi) - \lambda\psi = \psi \circ \gamma - \lambda\psi$. Thus there exists a unique h_0-invariant discrete subgroup B', namely B_ψ for that choice of ψ, such that $X = A + B'$. Now, as A is $\mathfrak{H}$-invariant, for any $h \in \mathfrak{H}$, $h(B')$ is also a h_0-invariant discrete subgroup and $X = A + h(B')$. Therefore, by the uniqueness condition as above we get that B' is $\mathfrak{H}$-invariant. Thus X is a direct sum of A with a $\mathfrak{H}$-invariant discrete subgroup, and hence there exists a closed $\mathfrak{H}$-invariant subgroup Φ_i of Ψ_i, namely the annihilator of B', such that $\Psi_i = K\Phi_i$, a direct product.

Clearly $\Phi_i \cong F_i$ as topological groups, under the map $\xi \mapsto \xi K$ for all $\xi \in \Phi_i$. This proves the theorem. □

7. *An example*

In the context of Theorems 1.1 and 1.2 one may wonder if it is possible to have a locally compact group G, other than a direct product of finitely many locally compact fields of characteristic zero and a compact abelian subgroup, admitting a finitely generated abelian group of automorphisms acting with a dense orbit. In this section we give such an example.

7.1. *Choice of $W = G/K$ and K, and the $\mathfrak{H}$-actions on them.* Let F be a locally compact field of characteristic zero. Let $W = F \oplus F$, a vector space of dimension two over F, and $\{e_1, e_2\}$ be the standard basis of W. Let $\mathfrak{H}$ be a finitely generated abelian subgroup of $\mathrm{GL}(W)$ whose action on W has a dense orbit, and $\{h \in \mathfrak{H} : \det h = 1\}$ is finite; for instance, we may consider a finitely generated dense subgroup of $F^* \times F^*$ whose intersection with the anti-diagonal subgroup (λ, λ^{-1}) is trivial, and consider the component-wise action. Since $\mathfrak{H}$ has a dense orbit on W, it follows that $\{\det h : h \in \mathfrak{H}\}$ is dense in F. Let A be the (additive) subgroup of F generated by $\{\det h : h \in \mathfrak{H}\}$. Then A is a countable abelian group. Let $K = \hat{A}$, the character group of A equipped with the discrete topology. Then K is a compact connected second countable abelian group. We identify the character group $\hat{F}$ of F with F, as usual. The inclusion map of A into F then induces by duality a (continuous) map $j : F \to K$, such that the image of j is dense in K. Since A is dense in F the map j is injective. Thus F is realized as a dense subgroup of K. We note that A is invariant under multiplication by $\det h$ for all $h \in \mathfrak{H}$. The $\mathfrak{H}$-action on A, where $h \in \mathfrak{H}$ acts by multiplication by $\det h$, induces by duality an $\mathfrak{H}$-action on K. We denote by $\mu(h)$ the automorphism of K corresponding to $h \in \mathfrak{H}$. It can be verified, bearing in mind the identification of F with $\hat{F}$, that

$$\mu(h)(j(\xi)) = j((\det h)\xi) \quad \text{for all } h \in \mathfrak{H} \text{ and } \xi \in F. \tag{23}$$

7.2. *Construction of G as an extension of W by K.* We identify $\wedge^2 W$, the second exterior power of W as a F-vector space, with F via the correspondence $\xi(e_1 \wedge e_2) \mapsto \xi$ for all $\xi \in F$. Let $e : W \times W \to F$ be the map defined by $e(w, w') = \frac{1}{2}(w \wedge w') \in F$, with the identification as above. We now define a topological group G, with the underlying space chosen as $W \times K$, with the product topology, and the multiplication given by

$$(w, k) \cdot (w', k') = (w + w', kk' j(e(w, w'))) \quad \text{for all } w, w' \in W \text{ and for all } k, k' \in K.$$

It can be seen that G is a locally compact second countable topological group. We identify W and K as subsets of G canonically ($w \in W$ identified with (w, e) and $k \in K$ identified with $(0, k)$, where e denotes the identity in K and 0 denotes the zero of W). Then K is a subgroup of G, and in fact coincides with the center of G. We note that, on the other hand, W, viewed as a subset of G as above, is not a subgroup, and in fact G has no subgroup isomorphic to W. It may also be observed that two elements of W do not commute with each other unless they are scalar multiples of each other.

For any $h \in \mathfrak{H}$ let $T_h : G \to G$ be the map defined by $T_h(w, k) = (h(w), \mu(h)(k))$ for all $w \in W$ and $k \in K$. For any $h \in \mathfrak{H}$, since $e \circ (h \times h) = (\det h)e$, by (23) we have that T_h is an automorphism of G. Also $h \mapsto T_h$ defines an action of $\mathfrak{H}$ on G.

7.3. *Orbits of the $\mathfrak{H}$-action.* First we note that the $\mathfrak{H}$-action on K is mixing with respect to the Haar measure on K; this follows from the fact that for any sequence $\{h_i\}$ in $\mathfrak{H}$ consisting of distinct elements, and any non-zero $a \in A$, the set $\{(\det h_i)a : i \in \mathbb{N}\}$ is infinite.

We now show that the $\mathfrak{H}$-action on G has a dense orbit. Since G is second countable it suffices to show that for every non-empty open set Ω of G the set $\{h\omega : h \in \mathfrak{H}, \omega \in \Omega\}$ is dense in G. Furthermore, the set Ω may be chosen to be of the form $U \times N$ where U and N are non-empty open subsets of W and K, respectively. Let $g = (w_0, k_0) \in G$ be given. Since the $\mathfrak{H}$-action on W admits a dense orbit there exists $u \in U$ whose $\mathfrak{H}$-orbit is dense, and furthermore we may choose u so that u and w_0 are not contained in the same $\mathfrak{H}$-orbit. Then there exists a sequence $\{h_i\}$ consisting of distinct elements such that $h_i(u) \to w_0$. Since the $\mathfrak{H}$-action on K is mixing, it follows that $\{h_i(k) : i \in \mathbb{N}\}$ is dense in K, for almost all $k \in K$, with respect to the Haar measure. Since N is a non-empty open subset of K, in particular we get that there exists $y \in N$ such that the closure of $\{h_i(y) : i \in \mathbb{N}\}$ contains k_0. Therefore, (w_0, k_0) is contained in the closure of $\{T_{h_i}(u, y)\}$. Since (w_0, k_0) was an arbitrary element of G this shows that $\{h\omega : h \in \mathfrak{H}, \omega \in \Omega\}$ is dense in G. As noted above this shows that the $\mathfrak{H}$-action on G has a dense orbit.

Remark 7.1. We note that K as in the above example would be finite dimensional if and only if $A \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space (see [**7**, Theorem 8.22]). The subgroup $\mathfrak{H}$ can be chosen so that this holds and thus we get an example of a finite-dimensional group with the property as above.

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