CLOSURES OF TOTALLY GEODESIC IMMERSIONS IN MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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Abstract

Using techniques of Lie groups and ergodic theory, it can be shown that in a compact manifold of constant negative curvature, the closure of a totally geodesic, complete (immersed) submanifold of dimension atleast 2 is a totally geodesic immersed submanifold. The main purpose of this article is to illustrate some important ideas involved in this method, by giving a proof for the simplest case of a codimension-1 totally geodesic immersed submanifold.

1 Introduction

In this article we prove the following theorem :

Theorem A *Let M be a compact, connected, oriented riemannian manifold with constant negative curvature and dimension $n \geq 3$. Let D be a complete, oriented riemannian manifold, whose connected components are $(n-1)$-dimensional and simply connected and let $\phi : D \to M$ be a totally geodesic immersion. Then $\phi(D)$ is either compact or dense in M.*

Let $\mathcal{F}(D)$ be the oriented orthonormal $(n-1)$-frame bundle over D, $\mathcal{F}(M)$ be the oriented orthonormal n-frame bundle over M and $\phi_* : \mathcal{F}(D) \to \mathcal{F}(M)$ be the immersion induced from ϕ. Then $\phi_*(\mathcal{F}(D))$ is either compact or dense in $\mathcal{F}(M)$.

A riemannian immersion $\phi : D \to M$ is called *totally geodesic* if $\phi \circ \gamma$ is a geodesic in M for every geodesic γ in D.

We shall prove this theorem using Lie groups, discrete subgroups and ergodic transformations on homogeneous spaces. As we shall see in §2, the Theorem A can be reformulated in the group theoretic setup as follows :

Theorem B *Let $G = SO_0(1,n)$, $\Gamma \subset G$ be a discrete subgroup such that $\Gamma\backslash G$ is compact and let $H = SO_0(1,n-1)$, where $n \geq 3$. Then every H-invariant subset of $\Gamma\backslash G$ is either dense or it is a union of finitely many closed H-orbits.*

Certain techniques for studying the closures of orbits have been developed in [M2], [DM1], [DM2] and [DM3]. We shall give an elementry proof of Theorem B closely following the line of arguments in these references.

In the last section we shall discuss some related results of a more general nature.

2 Group theoretic interpretation

For convenience we recall some known facts about hyperbolic spaces and their groups of isometries; (see also [Fl, Preliminaries]).

2.1 The hyperbolic n-space and its isometry group

Let $SO_0(1,n)$ denote the connected component of the group of linear transformations of $\mathbb{R}^{n+1}$ preserving the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 - \sum_{i=1}^{n} x_i y_i.$$

$SO_0(1,n)$ acts on $\mathbb{R}^{n+1}$ in the standard way and its orbit through the point $\mathbf{f}_0 = {}^t(1,0,\ldots,0) \in \mathbb{R}^{n+1}$ is a sheet of the hyperboloid

$$\Sigma^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle = 1 \text{ and } x_0 > 0\}.$$

The bilinear form $-\langle \cdot, \cdot \rangle$ restricted to the tangent bundle $T(\Sigma^n) \subset \Sigma^n \times \mathbb{R}^{n+1}$ is positive definite. With this riemannian structure, Σ^n has the constant sectional curvature -1 and $SO_0(1,n)$ is the group of its oriented isometries.

It is a well-known fact that all equi-dimensional, simply connected, complete riemannian manifolds of a fixed constant sectional curvature are isometric. Hence we call any n-dimensional, simply connected, complete riemannian manifold with constant curvature -1 (for example, $(\Sigma^n, -\langle \cdot, \cdot \rangle)$); *the Hyperbolic n-space* and denote it by $\mathbb{H}^n$.

2.2 Identifications

The stabilizer of $\mathbf{f}_0$ in $SO_0(1,n)$ consists of matrices of the form

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & k \end{pmatrix}, \quad k \in SO(n),$$

where $\mathbf{0}_{i\times j}$ is an $i \times j$ matrix with all entries zero. We obtain the identification,

$$SO_0(1,n)/SO(n) \sim \mathbb{H}^n \tag{1}$$

given by $gSO(n) \sim g\mathbf{f}_0$, for all $g \in SO_0(1,n)$.

Notations. Let M be an oriented riemannian manifold of dimension n. The oriented orthonormal n-frame bundle over M is denoted by $\mathcal{F}(M)$ and the orhtonormal k-frame bundle over M is denoted by $\mathcal{F}^k(M)$, where $1 \leq k \leq n$.

Remark 2.1 Let E be an oriented n-dimensional euclidean vector space. Given any orthonormal $(n-1)$-frame $[\mathbf{v}_1,\dots,\mathbf{v}_{n-1}]$ in E, there exists unique $\mathbf{v}_n \in E$ such that $[\mathbf{v}_1,\dots,\mathbf{v}_n]$ is an oriented orthonormal n-frame in E. This shows that for M as above, there is a canonical isomorphism, $\mathcal{F}^{n-1}(M) \simeq \mathcal{F}(M)$.

For $1 \le i \le n$, let $\mathbf{f}_i = {}^t(0,\dots,1,\dots,0) \in \mathbb{R}^{n+1}$, with 1 in the $(i+1)^{\text{th}}$ co-ordinate and 0 in all the others; here ${}^tX$ denotes the transpose of a matrix X. The tangent space to Σ^n at $\mathbf{f}_0$, denoted by $\mathrm{T}_{\mathbf{f}_0}(\Sigma^n)$, is spanned by $\{\mathbf{f}_1,\dots,\mathbf{f}_n\}$. Now $SO(n)$ acts simply transitively on the set of all oriented orthonormal n-frames in $\mathrm{T}_{\mathbf{f}_0}(\Sigma^n)$. Hence $SO_0(1,n)$ acts simply transitively on $\mathcal{F}(\Sigma^n)$ and we have the identification,

$$SO_0(1,n) \sim \mathcal{F}(\Sigma^n) = \mathcal{F}(\mathbb{H}^n) \tag{2}$$

given by $g \sim [g\mathbf{f}_1,\dots,g\mathbf{f}_n]_{g\mathbf{f}_0} \subset \mathrm{T}_{g\mathbf{f}_0}(\Sigma^n)$, for all $g \in SO_0(1,n)$.

2.3 Totally geodesic submanifolds of $\mathbb{H}^n$

We want to describe all totally geodesic immersions in to $\mathbb{H}^n$. Observe that if L is a riemannian manifold and σ is an isometry of L, then each connected component of the σ-fixed set in L is a totally geodesic submanifold of L.

For $1 \le k \le n-1$, consider the standard inclusions

$$\Sigma^k \hookrightarrow \Sigma^n \quad \text{and} \quad SO_0(1,k) \hookrightarrow SO_0(1,n).$$

Using the above remark it is easy to verify that Σ^k is a totally geodsic submanifold of Σ^n.

Let Ψ be a simply connected, complete riemannian manifold of dimension k and $\phi : \Psi \to \mathbb{H}^n$ be a totally geodesic immersion. Then there exits an isometry $g \in SO_0(1,n)$ such that $\phi(\Psi) = g \cdot \Sigma^k$. In view of the identification 1, we have

$$g \cdot SO(1,k)\,(SO(n)) \sim \phi(\Psi) \subset \mathbb{H}^n. \tag{3}$$

Suppose Ψ as above has dimension $(n-1)$. The derivative $\mathrm{D}\phi : \mathrm{T}(\Psi) \to \mathrm{T}(\Sigma^n)$ induces the immersion $\phi_* : \mathcal{F}(\Psi) \to \mathcal{F}^{n-1}(\Sigma^n)$. Now there exits an isometry $g \in SO_0(1,n)$ such that $\phi(\Psi) = g\Sigma^{n-1}$ and $\phi_*(\mathcal{F}(\Psi)) = g\mathcal{F}(\Sigma^{n-1}) \hookrightarrow \mathcal{F}^{n-1}(\Sigma^n)$. In view of the identifications 1 and 2 and Remark 2.1, we have

$$\begin{aligned} \phi(\Psi) &\sim g \cdot SO_0(1,n-1)/SO(n-1) \hookrightarrow SO_0(1,n)/SO(n), \\ \phi_*(\mathcal{F}(\Psi)) &\sim g \cdot SO_0(1,n-1) \hookrightarrow SO_0(1,n). \end{aligned} \tag{4}$$

2.4 Totally geodesic immersions in manifolds of constant negative curvature

Let M be a connected, oriented, n-dimensional, complete riemannian manifold with constant sectional curvature -1. Then the universal covering space of M is isometric to $\mathbb{H}^n$. Now there exits a discrete group Γ consisting of oriented isometries acting properly discontinuously on $\mathbb{H}^n$ such that M is isometric to $\Gamma\backslash\mathbb{H}^n$. Since $\Gamma \subset SO_0(1,n)$, by identifications 1 and 2,

$$\begin{aligned} \Gamma\backslash SO_0(1,n)/SO(n) &\sim \Gamma\backslash\mathbb{H}^n \sim M, \\ \Gamma\backslash SO_0(1,n) &\sim \Gamma\backslash\mathcal{F}(\mathbb{H}^n) \sim \mathcal{F}(M). \end{aligned} \tag{5}$$

Let Ψ be a simply connected, complete, $(n-1)$-dimensional riemannian manifold and $\phi : \Psi \to M$ be a totally geodesic immersion. The derivative $\mathrm{D}\phi : \mathrm{T}(\Psi) \to \mathrm{T}(M)$ induces the immersion $\phi_* : \mathcal{F}(\Psi) \hookrightarrow \mathcal{F}(M)$, where $\mathcal{F}(M)$ is identified with $\mathcal{F}^{n-1}(M)$ by Remark 2.1.

Let $p : \mathbb{H}^n \to M$ be a locally isometric covering. Since Ψ is simply connected, there exits a totally geodesic immersion $\tilde{\phi} : \Psi \to \mathbb{H}^n$ such that $\phi = p \circ \tilde{\phi}$. Hence due to identifications 4 and 5, there exits an isometry $g \in SO_0(1,n)$ such that

$$\begin{aligned} \phi(\Psi) &\sim \Gamma g SO_0(1,n-1)SO(n) \subset \Gamma\backslash SO_0(1,n)/SO(n) \sim M, \\ \phi_*(\mathcal{F}(\Psi)) &\sim \Gamma g SO_0(1,n-1) \subset \Gamma\backslash SO_0(1,n) \sim \mathcal{F}(M). \end{aligned} \tag{6}$$

Using this dual language, Theorem A can be easily derived from Theorem B. The next four sections are devoted to giving a proof of Theorem B. Some notations and preliminaries are set up in §3. The main results needed to prove Theorem B are given in §4 and §5. And the proof of the theorem is completed in §6.

3 Some important subgroups of $SO_0(1,n)$

Let $B = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{Id}_{n\times n} \end{pmatrix}$. Then for all $\mathbf{v},\mathbf{w} \in \mathbb{R}^{n+1}$, $\langle \mathbf{v},\mathbf{w} \rangle = {}^t\mathbf{v}B\mathbf{w}$. Hence $G = SO_0(1,n)$ is the connected component of the identity of the group

$$\left\{ g \in GL(n+1,\mathbb{R}) : {}^tgBg = B \right\}$$

and its Lie algebra

$$\mathcal{G} = \left\{ X \in \underline{gl}(n+1,\mathbb{R}) : {}^tXB + BX = 0 \right\}.$$

There is a right Adjoint action Ad of G on $\mathcal{G}$ given by

$$X \cdot \mathrm{Ad}\, g = g^{-1}Xg \quad (X \in \mathcal{G},\ g \in G).$$

Let $D = SO_0(1,1) \subset G$ and $\mathcal{D} \subset \mathcal{G}$ be the associated Lie subalgebra. Let $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\exp t\alpha = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, for all $t \in \mathbb{R}$. Now

$$\begin{aligned} \mathcal{D} &= \left\{ d(t) \stackrel{\text{def}}{=} \begin{pmatrix} t\alpha & 0_{2\times n-1} \\ 0_{n-1\times 2} & 0_{n-1\times n-1} \end{pmatrix} : t \in \mathbb{R} \right\}, \\ D &= \left\{ d_t \stackrel{\text{def}}{=} \begin{pmatrix} \exp t\alpha & 0_{2\times n-1} \\ 0_{n-1\times 2} & \mathrm{Id}_{n-1\times n-1} \end{pmatrix} : t \in \mathbb{R} \right\}. \end{aligned}$$

With repect to the right Adjoint action of D, the Lie algebra $\mathcal{G}$ decomposes into the direct sum of simultaneous eigenspaces as $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^0 \oplus \mathcal{G}^-$, where

$$\begin{aligned} \mathcal{G}^+ &= \left\{ n^+(\mathbf{v}) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & {}^t\mathbf{v} \\ 0 & 0 & -{}^t\mathbf{v} \\ \mathbf{v} & \mathbf{v} & 0_{n-1\times n-1} \end{pmatrix} : \mathbf{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^{n-1} \right\}, \\ \mathcal{G}^- &= \left\{ n^-(\mathbf{v}) \stackrel{\text{def}}{=} {}^t n^+(\mathbf{v}) : \mathbf{v} \in \mathbb{R}^{n-1} \right\}, \\ \mathcal{G}^0 &= \mathcal{D} \oplus \mathcal{M}, \\ \mathcal{M} &= \left\{ m(A) \stackrel{\text{def}}{=} \begin{pmatrix} 0_{2\times 2} & 0_{2\times n-1} \\ 0_{n-1\times 2} & A_{n-1\times n-1} \end{pmatrix} :\ A + {}^tA = 0 \right\}. \end{aligned}$$

For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$ and $(n-1) \times (n-1)$ skew symmetric matrices A, we have following commutation relations:

$$\begin{aligned} n^+(\mathbf{v}) \cdot \mathrm{Ad}\, d_t &= n^+(e^t\mathbf{v}), \\ n^-(\mathbf{v}) \cdot \mathrm{Ad}\, d_t &= n^-(e^{-t}\mathbf{v}), \\ m(A) \cdot \mathrm{Ad}\, d_t &= m(A) \end{aligned} \tag{7}$$

$$\begin{aligned} \left[n^+(\mathbf{w}), n^+(\mathbf{v})\right] &= 0, \\ \left[m(A), n^+(\mathbf{v})\right] &= n^+(A \cdot \mathbf{v}), \\ \left[n^-(\mathbf{w}), n^+(\mathbf{v})\right] &= 2a({}^t\mathbf{w} \cdot \mathbf{v}) + 2m(\mathbf{w} \cdot {}^t\mathbf{v} - \mathbf{v} \cdot {}^t\mathbf{w}). \end{aligned} \tag{8}$$

Now $\mathcal{G}^+$, $\mathcal{G}^-$ and $\mathcal{M}$ are Lie subalgebrs of $\mathcal{G}$. Let N^+, N^- and M be the connected Lie subgroups associated to $\mathcal{G}^+$, $\mathcal{G}^-$ and $\mathcal{M}$ respectively.

Remark 3.1 The maps $\exp : \mathcal{G}^\pm \to N^\pm$ are group isomorphisms, hence $N^\pm$ are vector groups. Let $u = \exp(n^+(\mathbf{v})) \in N^+$. Then by Eq. 7, $d_t^{-1}ud_t \to 1$ as $t \to -\infty$. Similary if $v \in N^-$ then $d_t^{-1}vd_t \to 1$ as $t \to +\infty$.

The group M is isomorphic to $SO(n-1)$ and the group DM is the centralizer of D in G.

Remark 3.2 Due to Eq. 8, the Lie subalgebras $\mathcal{G}^+$ and $\mathcal{G}^-$ generate the Lie algebra $\mathcal{G}$. Hence the subgroup generated by N^+ and N^- is dense in G.

4 Ergodic properties of actions on homogeneous spaces

4.1 Ergodic transformations

Definition 4.1 Let X be a topological space and μ be a Borel measure on X. A measure preserving transformation T of (X,μ) is called *ergodic* if the following holds: for any measurable set $E \subset X$ if $\mu(T(E) \,\Delta\, E) = 0$ then either $\mu(E) = 0$ or $\mu(X \setminus E) = 0$, where $A \,\Delta\, B \stackrel{\text{def}}{=} A \cup B \setminus A \cap B$.

The following property of ergodic transformations makes the concept of ergodicity very useful for applications.

Lemma 4.1 *Let X be a second countable topological space and μ be a Borel measure on X such that $\mu(E) > 0$ for any non-empty open subset E of X. Let T be an ergodic transformation on (X,μ). Then for μ-almost all $x \in X$, the set $\{T^n x\}_{n\in\mathbb{N}}$ is dense in X.*

Proof. For a nonempty open subset E of X, define

$$X(E) = \bigcup_{n=0}^{\infty} T^{-n}(E).$$

Then $T(X(E)) \supset X(E)$. Now T preserves the measure μ, hence

$$\mu(T(X(E)) \,\Delta\, X(E)) = 0.$$

Since $\mu(E) > 0$, by the ergodicity of T-action $\mu(X(E)) = 1$.

Let $\mathcal{B}$ be a countable open base of X. Let

$$Y = \bigcap_{E \in \mathcal{B}\setminus\emptyset} X(E).$$

Then for all $y \in Y$ the set $\{T^n y\}_{y\in\mathbb{N}}$ is dense in X and $\mu(Y) = 1$. □

Remark 4.1 Let $X = \Gamma\backslash G$. Since Γ is discrete and X is compact, there exists a probability measure μ on X which is invariant under the right action of G on X (see [R, Chap. I]).

Lemma 4.2 (Mautner, cf. [M1]) *The right action of $d = d_1 \in D$ on $X = \Gamma\backslash G$ is an ergodic transformation on (X, μ).*

Proof. Since μ is finite and G-invariant, there is a continuous unitary representation ρ of G on the Hilbert Space $\mathcal{H} = \mathcal{L}^2(X, \mu)$, defined such that for all $\xi \in \mathcal{H}$, $g \in G$ and μ-almost all $x \in X$,

$$[\xi \cdot \rho(g)](x) = \xi(xg).$$

Suppose E is a measurable subset of X such that $\mu(E \cdot d \,\Delta\, E) = 0$. Let χ_E denote the charecteristic function of E. Then $\xi = \chi_E \in \mathcal{H}$ and for all $k \in \mathbb{Z}$,

$$\xi \cdot \rho(d^k) = \chi_{(E \cdot d^{-k})} = \chi_E = \xi.$$

Let $u \in N^+$. Since ρ is unitary, for all $k \in \mathbb{Z}$,

$$\langle \xi \cdot \rho(u), \xi \rangle = \langle \xi \cdot \rho(d^k)\rho(u), \xi \cdot \rho(d^k) \rangle = \langle \xi \cdot \rho(d^k u d^{-k}), \xi \rangle.$$

By Remark 3.1, $d^k u d^{-k} \to 1$ as $k \to +\infty$. Hence by continuity of ρ,

$$\langle \xi \cdot \rho(u), \xi \rangle = \langle \xi, \xi \rangle.$$

Thus $\xi \cdot \rho(u) = \xi$ for all $u \in N^+$. Similarly, we can show that $\xi \cdot \rho(w) = \xi$ for all $w \in N^-$. Now by Remark 3.2, $\xi \cdot \rho(g) = \xi$ for all $g \in G$. Thus $\chi_E = \xi$ is constant almost every where on X. Hence $\mu(E) = 1$ or 0. This shows that d acts ergodically on (X, μ). □

Lemma 4.2 and Lemma 4.1 imply that almost all orbits of D are dense in $\Gamma\backslash G$. For our purpose we will need its following consequence regarding individual orbits (see [D2, Preliminaries] for a general statement and references).

Lemma 4.3 *Every orbit of the subgroup N^+D acting on $X = \Gamma\backslash G$ is dense.*

Proof. Let $x, y \in X$. Since $d = d_1 \in D$ acts ergodically on (X, μ), by Lemma 4.1, there exist sequences $x_i \to x$, $x_i \in X$ and $n_i \to \infty$, $n_i \in \mathbb{N}$ such that $x_i d^{n_i} \to y$ as $i \to \infty$. Let the sequence $g_i \to 1$, $g_i \in G$ be such that $x_i = x g_i$. Since $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^0 \oplus \mathcal{G}^-$, for all large $i \in \mathbb{N}$ there exist $w_i \in N^-$, $v_i \in N^+$ and $z_i \in DM$, such that $g_i = v_i z_i w_i$ and $w_i, v_i, z_i \to 1$ as $i \to \infty$.

Now for all large $i \in \mathbb{N}$,

$$x_i d^{n_i} = x v_i d^{n_i} (d^{-n_i} z_i d^{n_i})(d^{-n_i} w_i d^{n_i}).$$

By Remark 3.1, as $i \to \infty$,

$$d^{-n_i} z_i d^{n_i} = z_i \to 1 \quad \text{and} \quad d^{-n_i} w_i d^{n_i} \to 1.$$

Therefore $x v_i d^{n_i} \to y$ as $i \to \infty$. Since x, y are arbitrary, this shows that for all $x \in X$ the orbit xN^+D is dense in X. □

4.2 Minimal closed invariant sets

It was shown by G.A. Margulis in [M2] that minimal closed invariant sets of the action of unipotent subgroups can be used very effectively for studying orbit closures in homogeneous spaces of Lie groups.

Definition 4.2 Let F be a semi-group acting on a topological space X by continuous transformations. If a closed subset Z of X is invariant under the action of F and no proper closed subset of Z is invariant under the F-action then Z is called *minimal closed F-invariant*. Thus, if Z is a minimal closed F-invariant set then every orbit of F in Z is dense.

Remark 4.2 Any compact F-invariant subset of X contains a minimal closed F-invariant subset. To see this, use Zorn's lemma along with the fact that the intersection of any totally ordered (with respect to set inclusion) family of compact sets is nonempty. This remark will be used in §6.

5 Orbits of unipotent groups under linear actions

Let $H = SO_0(1, n-1) \subset G$. Now $D = SO_0(1,1) \subset H$. Put $N_1 = N^+ \cap H$ and $M_1 = M \cap H$.

Let Y be a closed H-invariant subset of $\Gamma\backslash G$. Now H contains the subgroup $N_1 D$ and by Lemma 4.3 we know that every orbit of the subgroup N^+D is dense in $\Gamma\backslash G$. Let N_2 be a one-parameter subgroup of N^+ such that $N^+ = N_2 N_1$. In §6 we show that under certain 'local' condition, Y contains an orbit of N_2. This will imply that $Y = \Gamma\backslash G$. The next proposition is a crucial step for obtaining, under that condition, a N_2-invariant subset in Y. It will be convenient to introduce some notations to state and prove the proposition.

Let $\mathcal{H}$ be the Lie algebra corresponding to H. Let $\mathcal{H}^+ = \mathcal{G}^+ \cap \mathcal{H}$, $\mathcal{M}_1 = \mathcal{M} \cap \mathcal{H}$ and $\mathcal{H}^- = \mathcal{G}^- \cap \mathcal{H}$. Then $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{D} \oplus \mathcal{M}_1 \oplus \mathcal{H}^-$. Also $\mathcal{H}^+$ and $\mathcal{M}_1$ are the Lie subalgebras corresponding to N_1 and M_1 respectively.

Let $\mathcal{P}$ be the ortho-complement of $\mathcal{H}$ in $\mathcal{G}$ with respect to the symmetric bilinear form $Q : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$, defined by $Q(X, Y) = \mathrm{tr}(XY)$. Now Q is non-degenerate on $\mathcal{G}$ as well as on $\mathcal{H}$ and it is invariant under the right Adjoint action of G on $\mathcal{G}$. Therefore $\mathcal{G} = \mathcal{P} \oplus \mathcal{H}$ and $\mathcal{P}$ is invariant under the Adjoint action restricted to H. Let $\mathcal{P}^+ = \mathcal{P} \cap \mathcal{G}^+$, $\mathcal{P}^0 = \mathcal{P} \cap \mathcal{G}^0$ and $\mathcal{P}^- = \mathcal{P} \cap \mathcal{G}^-$. Then

$$\mathcal{P} = \mathcal{P}^+ \oplus \mathcal{P}^0 \oplus \mathcal{P}^-.$$

Let N_2 be the connected Lie subgroup corresponding to the Lie subalgebra $\mathcal{P}^+$. Now $\mathcal{G}^+ = \mathcal{P}^+ \oplus \mathcal{H}^+$ and $N^+ = N_2 N_1$.

Let $[\mathrm{e}_1, \ldots, \mathrm{e}_{n-1}]$ denote the standard ordered basis of $\mathbb{R}^{n-1}$. Then the set $\{n^+({}^t\mathrm{e}_k) : 1 \le k \le n-2\}$ is a basis of $\mathcal{H}^+$, $p^+ \stackrel{\text{def}}{=} n^+({}^t\mathrm{e}_{n-1})$ is a basis of $\mathcal{P}^+$, $p^- \stackrel{\text{def}}{=} n^-({}^t\mathrm{e}_{n-1})$ is a basis of $\mathcal{P}^-$ and the set

$$\left\{p_k^0 \stackrel{\text{def}}{=} m(X_k - {}^tX_k) : X_k = \begin{pmatrix} 0_{n-2\times n-1} \\ \mathbf{e}_k \end{pmatrix},\ 1 \le k \le n-2\right\}$$

is a basis of $\mathcal{P}^0$.

Proposition 5.1 (Margulis, cf. [DM1, Lemma 2.2]) *Let $\{q_i\}_{i\in\mathbb{N}} \subset \mathcal{P}\setminus\mathcal{P}^+$ be a sequence such that $q_i \to 0$ as $i\to\infty$. Then there exist a one-parameter subgroup $\{u_t\}_{t\in\mathbb{R}} \subset N_1$, a sequence $t_i\to\infty$ and a non-constant polynomial φ such that if $\{q_i\}_{i\in\mathbb{N}}$ is replaced by a suitable subsequence then for every $s\in\mathbb{R}$, as $i\to\infty$,*

$$q_i\cdot \mathrm{Ad}(u_{st_i}) \to \varphi(s)p^+.$$

Proof. For each $i\in\mathbb{N}$, let $\theta_i\in\mathbb{R}$, $\{\sigma_{k,i} : 1\le k\le n-2\}\subset\mathbb{R}$ and $\delta_i\in\mathbb{R}$ be such that

$$q_i = \theta_i\, p^- + \sum_{k=1}^{n-2}\sigma_{k,i}\, p_k^0 + \delta_i\, p^+.$$

Now as $i\to\infty$: $\theta_i\to 0$, $\delta_i\to 0$ and $\sigma_{k,i}\to 0$ for all $1\le k\le n-2$. Since $\{q_i\}_{i\in\mathbb{N}}\cap\mathcal{P}^+=\emptyset$, there exists $k\in\{1,\ldots,n-2\}$ such that replacing $\{q_i\}_{i\in\mathbb{N}}$ by a subsequence, we get $\theta_i\neq 0$ or $\sigma_{k,i}\neq 0$ for all $i\in\mathbb{N}$. Consider the one-parameter subgroup

$$\left\{u_t \stackrel{\text{def}}{=} \exp n^+(t\cdot{}^t\mathbf{e}_k) : t\in\mathbb{R}\right\}\subset N_1.$$

Then by Eq. 8,

$$\begin{aligned} q_i\cdot \mathrm{Ad}\,u_t &= q_i + t\cdot\left[q_i, n^+({}^t\mathbf{e}_k)\right] + (t^2/2)\cdot\left[[q_i, n^+({}^t\mathbf{e}_k)], n^+({}^t\mathbf{e}_k)\right] \\ &= q_i + (\theta_i\, t)\cdot p_k^0 + (\sigma_{k,i}\, t + \theta_i\, t^2/2)\cdot p^+. \end{aligned}$$

For each $i\in\mathbb{N}$, let $t_i>0$ be such that

$$\max\{|\sigma_{k,i}|\, t_i,\ |\theta_i|\, t_i^2\} = 1.$$

Replacing $\{q_i\}_{i\in\mathbb{N}}$ by a subsequence, there exist $\lambda_1,\lambda_2\in\mathbb{R}$ such that as $i\to\infty$, $\theta_i\, t_i^2\to\lambda_1$ and $\sigma_{k,i}\, t_i\to\lambda_2$. Note that $\max\{|\lambda_1|,|\lambda_2|\}=1$. Since $\theta_i\, t_i^2\to\lambda_1$ and $t_i\to\infty$, we have $\theta_i\, t_i\to 0$ as $i\to\infty$.

Let φ be a polynomial defined by $\varphi(s)=\lambda_1 s+\lambda_2 s^2$, $s\in\mathbb{R}$. Then φ is non-constant and for every $s\in\mathbb{R}$, as $i\to\infty$,

$$q_i\cdot\mathrm{Ad}\,u_{st_i}\to\varphi(s)p^+.$$

□

6 Proof of Theorem B

Let $X=\Gamma\backslash G$ and Y be the closure of the given H-invariant subset in X. Then Y is H-invariant. We want to show that either $Y=X$ or Y is a union of finitely many closed H-orbits.

Let Y_1 be a minimal closed H-invariant subset of Y and Z be a minimal closed N_1-invariant subset of Y_1. The existance of these sets follows from Remark 4.2.

Since $\mathcal{G}=\mathcal{P}\oplus\mathcal{H}$, there exist a neighbourhood Ψ of 0 in $\mathcal{G}$ and a neighbourhood Ω of 1 in G such that the map $(q,\mathbf{y})\mapsto\exp q\cdot\exp\mathbf{y}$, $(q\in\mathcal{P}\cap\Psi,\ \mathbf{y}\in\mathcal{H}\cap\Psi)$ is a diffeomorphism onto Ω.

Fix $z\in Z$ for rest of the proof. Let $g\in\Omega$ be such that $zg\in Y$. Write $g=(\exp q)h$ for some $q\in\mathcal{P}\cap\Psi$ and $h\in H$. Since Y is H-invariant, $z\exp q\in Y$. Define

$$\mathcal{Q}=\{q\in\mathcal{P}\cap\Psi : z\exp q\in Y\}.$$

If we choose Ψ small enough then one of the following possibilities occurs :

I. $0\in\overline{\mathcal{Q}\setminus\mathcal{P}^+}$.

II. $0\in\overline{\mathcal{Q}\setminus\{0\}}$ and $\mathcal{Q}\subset\mathcal{P}^+$.

III. $\mathcal{Q}=\{0\}$.

We shall prove that **a)** if Case I occurs then Y_1 is dense in $\Gamma\backslash G$,**b)** if Case III occurs then Y_1 is a closed H-orbit and it is a connected component of Y and **c)** the occurrence of Case II leads to a contradiction. This shows that either $Y=X$ or every connected component of Y is a closed H-orbit. Note that since Y is compact, it has only finitely many connected components. This will prove Theorem B.

Case I : (cf. [DM3, Prop. 8, Case a)])

In this case there exists a sequence $\{q_i\}_{i\in\mathbb{N}}\subset\mathcal{P}\setminus\mathcal{P}^+$ such that $q_i\to 0$, as $i\to\infty$ and $z\exp q_i\in Y$, for all $i\in\mathbb{N}$.

Step 1 Replacing $\{q_i\}_{i\in\mathbb{N}}$ by a suitable subsequence, there exist a one-parameter subgroup $\{u_t\}_{t\in\mathbb{R}}\subset N_1$, a sequence $t_i\to\infty$ and a non-constant polynomial φ such that for every $s\in\mathbb{R}$, as $i\to\infty$,

$$q_i\cdot\mathrm{Ad}\,u_{st_i}\to\varphi(s)p^+.$$

This is just a restatement of Proposition 5.1.

Step 2 (cf. [M2, Lemma 1]) *For every $s\in\mathbb{R}$, $Z\exp(\varphi(s)p^+)\subset Y$.*

$$\left\{p_k^0 \stackrel{\text{def}}{=} m(X_k - {}^tX_k) : X_k = \begin{pmatrix} 0_{n-2\times n-1} \\ \mathbf{e}_k \end{pmatrix},\ 1 \le k \le n-2\right\}$$

is a basis of $\mathcal{P}^0$.

Proposition 5.1 (Margulis, cf. [DM1, Lemma 2.2]) *Let $\{q_i\}_{i\in\mathbb{N}} \subset \mathcal{P} \setminus \mathcal{P}^+$ be a sequence such that $q_i \to 0$ as $i \to \infty$. Then there exist a one-parameter subgroup $\{u_t\}_{t\in\mathbb{R}} \subset N_1$, a sequence $t_i \to \infty$ and a non-constant polynomial φ such that if $\{q_i\}_{i\in\mathbb{N}}$ is replaced by a suitable subsequence then for every $s \in \mathbb{R}$, as $i \to \infty$,*

$$q_i \cdot \mathrm{Ad}(u_{st_i}) \to \varphi(s)p^+.$$

Proof. For each $i \in \mathbb{N}$, let $\theta_i \in \mathbb{R}$, $\{\sigma_{k,i} : 1 \le k \le n-2\} \subset \mathbb{R}$ and $\delta_i \in \mathbb{R}$ be such that

$$q_i = \theta_i\, p^- + \sum_{k=1}^{n-2} \sigma_{k,i}\, p_k^0 + \delta_i\, p^+.$$

Now as $i \to \infty$: $\theta_i \to 0$, $\delta_i \to 0$ and $\sigma_{k,i} \to 0$ for all $1 \le k \le n-2$. Since $\{q_i\}_{i\in\mathbb{N}} \cap \mathcal{P}^+ = \emptyset$, there exists $k \in \{1, \dots, n-2\}$ such that replacing $\{q_i\}_{i\in\mathbb{N}}$ by a subsequence, we get $\theta_i \ne 0$ or $\sigma_{k,i} \ne 0$ for all $i \in \mathbb{N}$. Consider the one-parameter subgroup

$$\left\{u_t \stackrel{\text{def}}{=} \exp n^+(t \cdot {}^t\mathbf{e}_k) : t \in \mathbb{R}\right\} \subset N_1.$$

Then by Eq. 8,

$$\begin{aligned} q_i \cdot \mathrm{Ad}\, u_t &= q_i + t \cdot \left[q_i, n^+({}^t\mathbf{e}_k)\right] + (t^2/2) \cdot \left[[q_i, n^+({}^t\mathbf{e}_k)], n^+({}^t\mathbf{e}_k)\right] \\ &= q_i + (\theta_i\, t) \cdot p_k^0 + (\sigma_{k,i}\, t + \theta_i\, t^2/2) \cdot p^+. \end{aligned}$$

For each $i \in \mathbb{N}$, let $t_i > 0$ be such that

$$\max\{|\sigma_{k,i}|\, t_i,\ |\theta_i|\, t_i^2\} = 1.$$

Replacing $\{q_i\}_{i\in\mathbb{N}}$ by a subsequence, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that as $i \to \infty$, $\theta_i\, t_i^2 \to \lambda_1$ and $\sigma_{k,i}\, t_i \to \lambda_2$. Note that $\max\{|\lambda_1|, |\lambda_2|\} = 1$. Since $\theta_i\, t_i^2 \to \lambda_1$ and $t_i \to \infty$, we have $\theta_i\, t_i \to 0$ as $i \to \infty$.

Let φ be a polynomial defined by $\varphi(s) = \lambda_1 s + \lambda_2 s^2$, $s \in \mathbb{R}$. Then φ is non-constant and for every $s \in \mathbb{R}$, as $i \to \infty$,

$$q_i \cdot \mathrm{Ad}\, u_{st_i} \to \varphi(s)p^+.$$

□

6 Proof of Theorem B

Let $X = \Gamma\backslash G$ and Y be the closure of the given H-invariant subset in X. Then Y is H-invariant. We want to show that either $Y = X$ or Y is a union of finitely many closed H-orbits.

Let Y_1 be a minimal closed H-invariant subset of Y and Z be a minimal closed N_1-invariant subset of Y_1. The existance of these sets follows from Remark 4.2.

Since $\mathcal{G} = \mathcal{P} \oplus \mathcal{H}$, there exist a neighbourhood Ψ of 0 in $\mathcal{G}$ and a neighbourhood Ω of 1 in G such that the map $(q, \mathrm{y}) \mapsto \exp q \cdot \exp \mathrm{y}$, $(q \in \mathcal{P} \cap \Psi, \mathrm{y} \in \mathcal{H} \cap \Psi)$ is a diffeomorphism onto Ω.

Fix $z \in Z$ for rest of the proof. Let $g \in \Omega$ be such that $zg \in Y$. Write $g = (\exp q)h$ for some $q \in \mathcal{P} \cap \Psi$ and $h \in H$. Since Y is H-invariant, $z \exp q \in Y$. Define

$$\mathcal{Q} = \{q \in \mathcal{P} \cap \Psi : z \exp q \in Y\}.$$

If we choose Ψ small enough then one of the following possibilities occurs :

I. $0 \in \overline{\mathcal{Q} \setminus \mathcal{P}^+}$.

II. $0 \in \overline{\mathcal{Q} \setminus \{0\}}$ and $\mathcal{Q} \subset \mathcal{P}^+$.

III. $\mathcal{Q} = \{0\}$.

We shall prove that **a)** if Case I occurs then Y_1 is dense in $\Gamma\backslash G$,**b)** if Case III occurs then Y_1 is a closed H-orbit and it is a connected component of Y and **c)** the occurrence of Case II leads to a contradiction. This shows that either $Y = X$ or every connected component of Y is a closed H-orbit. Note that since Y is compact, it has only finitely many connected components. This will prove Theorem B.

Case I : (cf. [DM3, Prop. 8, Case a)])

In this case there exists a sequence $\{q_i\}_{i\in\mathbb{N}} \subset \mathcal{P} \setminus \mathcal{P}^+$ such that $q_i \to 0$, as $i \to \infty$ and $z \exp q_i \in Y$, for all $i \in \mathbb{N}$.

Step 1 Replacing $\{q_i\}_{i\in\mathbb{N}}$ by a suitable subsequence, there exist a one-parameter subgroup $\{u_t\}_{t\in\mathbb{R}} \subset N_1$, a sequence $t_i \to \infty$ and a non-constant polynomial φ such that for every $s \in \mathbb{R}$, as $i \to \infty$,

$$q_i \cdot \mathrm{Ad}\, u_{st_i} \to \varphi(s)p^+.$$

This is just a restatement of Proposition 5.1.

Step 2 (cf. [M2, Lemma 1]) *For every $s \in \mathbb{R}$, $Z \exp(\varphi(s)p^+) \subset Y$.*

Proof. Fix $s \in \mathbb{R}$. Put $y_i = z\exp q_i \in Y$ and $z_i = zu_{st_i} \in Z$, for all $i \in \mathbb{N}$. Since Z is compact, by passing to subsequences, we may assume that as $i \to \infty$, $z_i \to z'$ for some $z' \in Z$. Now by Step 1, as $i \to \infty$,

$$y_i u_{st_i} = z_i \exp(u_{-st_i} q_i u_{st_i}) \to z' \exp(\varphi(s)p^+)$$

Put $v = \exp(\varphi(s)p^+) \in N_2$. Since $y_i u_{st_i} \in Y$ and Y is closed, $z'v \in Y$.

Since Z is minimal closed N_1-invariant, $z'N_1$ is dense in Z. Now N_2 normalizes N_1, therefore

$$Zv = \overline{z'N_1}v \subset \overline{Yv^{-1}N_1v} = \overline{YN_1} = Y.$$

Step 3 *Y contains an orbit of* N_2.

Proof. Note that $\varphi(0) = 0$. Hence we can choose $F = \{\exp(tp^+) : t \geq 0\}$ or $\{\exp(tp^+) : t \leq 0\}$, so that

$$F \subset \left\{\exp(\varphi(s)p^+) : s \in \mathbb{R}\right\}.$$

Now by Step 2, $ZF \subset Y$.

Since Y is compact, by Remark 4.2, $\overline{ZF}$ contains a minimal closed F-invariant subset Z_1. If $v \in N_2$ then there exists $w \in F$ such that $wv \in F$ and hence

$$Z_1 = Z_1(wv) = Z_1 v$$

Thus Z_1 is N_2 invariant. This proves Step 3.

Since Y is N_1D invariant, by Step 3, Y contains an orbit of $N_2(N_1D) = N^+D$. Now by Lemma 4.3, $Y = X = \Gamma\backslash G$, as we wanted to show in this case.

Case III : (cf. [DM3, Prop. 8, Case b)])

In this case zH contains a neighbourhood of z in Y and hence it is an open subset of Y. Now $Y_1 \setminus zH$ is a closed H-invariant subset of Y_1. Since Y_1 is closed minimal H-invariant, $Y_1 = zH$. Thus Y_1 is a closed orbit of H and is a connected component of Y. This is what wanted to show in this case.

Case II : (cf. [DM3, Prop. 8, Case c)])

In this case there exists neighbourhood Ω of 1 in G such that

$$Y \cap z\Omega \subset z(N_2H \cap \Omega) \tag{9}$$

and there exists a sequence $\{v_i\}_{i\in\mathbb{N}} \subset N_2 \setminus \{1\}$ such that $v_i \to 1$, as $i \to \infty$ and $zv_i \in Y$ for all $i \in \mathbb{N}$. Since Y is compact and Γ is discrete, we can choose Ω small enough so that $\Omega\Omega^{-1} \cap G_y = \{1\}$ for all $y \in Y$, where G_y denotes the stabilizer of y in G.

Step 1 (cf. [M2, Lemma 4]) *Given any compact set* $K \subset N_1$, *there exists* $u \in N_1 \setminus K$ *such that* $zu \in z\Omega$.

Proof. Since Z is compact and N_1 is non-compact, there exists a sequence $\{u_i\}_{i\in\mathbb{N}} \subset N_1$ such that as $i \to \infty$, $zu_i \to z' \in Z$ and $u_i \to \infty$. Since Z is minimal closed N_1-invariant, $z'N_1$ is dense in Z. Let $u' \in N_1$ be such that $z'u' \in z\Omega$. Hence for all large enough $i \in \mathbb{N}$, $(zu_i)u' \in z\Omega$ but $u_iu' \notin K$. This proves Step 1.

Step 2 $zN_1 \cap z\Omega \subset z(N^+DM_1 \cap \Omega)$.

Proof. Let $u \in N_1$ be such that $zu \in Z\Omega$. Then by Eq. 9 there exists $g \in N_2H \cap \Omega$ such that $zu = zg$.

Since $v_i \to 1$, there is $i_0 \in \mathbb{N}$ such that $gv_{i_0} \in \Omega$. Since $zv_{i_0} \in Y$, we have

$$zv_{i_0}u = zuv_{i_0} = zgv_{i_0} \in Y \cap z\Omega.$$

Therefore by Eq. 9 there exist $v \in N_2$ and $h \in H$ such that $vh \in \Omega$ and $zgv_{i_0} = zvh$. Hence $gv_{i_0} = vh$, because $\Omega\Omega^{-1} \cap G_z = \{1\}$.

Now according to the notations in §5,

$$p^+ \cdot \mathrm{Ad}(gv_{i_0}) = p^+ \cdot \mathrm{Ad}(vh) = p^+ \cdot \mathrm{Ad}\, h \in \mathcal{P}.$$

Put $q = p^+ \cdot \mathrm{Ad}\, g$. Since $q \in \mathcal{P}$ and $v_{i_0} \in N_2 \setminus \{1\}$, from Eq. 8 it follows that, $q \cdot \mathrm{Ad}\, v_{i_0} \in \mathcal{P}$ only if $q \in \mathcal{P}^+$. Writing $g = v'h'$ for suitable $v' \in N_2$ and $h' \in H$, we get $q = p^+ \cdot \mathrm{Ad}\, h'$. Since $\mathcal{P}^+$ is the fixed point space of $\mathrm{Ad}(N_1)$ in $\mathcal{P}$ and N_1DM_1 is the normalizer of N_1 in H, we have $q \in \mathcal{P}^+$ only if $h' \in N_1DM_1$. Hence $g \in N_2N_1DM_1 = N^+DM_1$. This completes the proof of Step 2.

By Steps 1 and 2, there exist $u \in N_1 \setminus \overline{\Omega}$ and $g \in (N^+DM_1 \cap \Omega)$ such that $zu = zg$. Then $\delta = gu^{-1} \in G_z \setminus \{1\}$. By Remark 3.1, there exists $t_0 > 0$ such that $d_{t_0}ud_{t_0}^{-1} \in \Omega$ and for all $t > 0$, $d_t(N^+DM_1 \cap \Omega)d_t^{-1} \subset \Omega$. Now $d_{t_0}G_zd_{t_0}^{-1} = G_{zd_{t_0}^{-1}}$. Hence

$$d_{t_0}\delta d_{t_0}^{-1} = (d_{t_0}gd_{t_0}^{-1})(d_{t_0}u^{-1}d_{t_0}^{-1}) \in (G_{zd_{t_0}^{-1}} \cap \Omega\Omega^{-1}) \setminus \{1\}.$$

This contradicts the choice of Ω, for $zd_{t_0}^{-1} \in Y$. Hence Case II does not occur.

This completes the proof of Theorem B. □

Remark 6.1 Theorem B is still valid if we assume that $\Gamma\backslash G$ admits a finite G-invariant measure, even though it need not be compact. In order to extend our proof in this case, we will need to show that any closed N_1-invariant subset of $\Gamma\backslash G$ contains a minimal compact N_1-invariant subset. A result due to S.G. Dani and G.A. Margulis achieves precisely this (see [DM1, Corollary 1.5] and [M3]). Now with the help of the proof of Proposition 8 in [DM3], the reader may be able to verify, without much difficulty, the Theorem B under the above assumption.

7 General Results

Using the ideas from [M2], [DM1] and [DM2] and using the method of the proof of the Main theorem in [Sh], the following result can be proved :

Theorem C *Let $G = SO_0(1,n)$ and Γ be a discrete subgroup of G such that $\Gamma\backslash G$ admits a finite G-invariant measure (i.e. Γ is a lattice in G). Let $H = SO_0(1,k)$ for some $2 \leq k \leq n$ and Y be a closed H-invariant subset of $\Gamma\backslash G$. Then Y has finitely many connected components; each of them is of the form $xLCC'$, where C' is a compact subset of $C_G(H)$, the centralizer of H in G, and $L = SO_0(1,m)$, $k \leq m \leq n$, C a compact subgroup of $C_G(L)$ and $x \in X$ are such that $\overline{xH} = xLC$.*

In particular, if Y is the closure of a single orbit of H then $Y = y(g^{-1}LCg)$, where $y \in Y$, $g \in C_G(H)$ and L and C are as above.

This theorem has the following geometric consequence.

Theorem D *Let M be a complete, connected riemannian manifold with constant negative curvature and finite riemannian volume. Let D be a complete riemmanian manifold, whose connected components are simply connected and of dimension $k \geq 2$ and let $\phi : D \to M$ be a totally geodesic immersion. Then there exists a complete riemannian manifold L and a totally geodesic immersion $\psi : L \to M$ such that the following holds :*

1. *L has finitely many components, possibly of different dimensions, and each one of them has finite riemannian volume.*
2. *$\psi(L)$ is the closure of $\phi(D)$ in M.*
3. *Let $\tilde{\phi} : D \to L$ be a riemannian immersion such that $\phi = \psi \circ \tilde{\phi}$. Let $\tilde{\phi}_* : \mathcal{F}^k(D) \to \mathcal{F}^k(L)$ be the immersion of the orthonormal k-frame bundles, which is induced from the derivative of $\tilde{\phi}$. Then $\tilde{\phi}_*(\mathcal{F}^k(D))$ is dense in $\mathcal{F}^k(L)$.*

Note that the closure of a geodesic in M need not be the image of a closed immersion. To give an example of such a geodesic, let $p : \widetilde{M} \to M$ be the universal cover of M and let $\tilde{\gamma}_-$ and $\tilde{\gamma}_+$ be two distinct geodesics in $\widetilde{M}$ such that $\gamma_\pm = p \circ \tilde{\gamma}_\pm$ are closed compact geodesics in M. Since $\widetilde{M}$ is isomorphic to the Hyperbolic n-space, there exists a geodesic $\tilde{\gamma}$ in $\widetilde{M}$ such that $\tilde{\gamma}(-\infty) = \tilde{\gamma}_-(-\infty)$ and $\tilde{\gamma}(+\infty) = \tilde{\gamma}_+(+\infty)$. Then the geodesic $\gamma = p \circ \tilde{\gamma}$ of M winds around γ_+ in one direction and γ_- in the opposite direction. Clearly, $\gamma_- \cup \gamma \cup \gamma_+$ is the closure of γ in M but it is not the image of a closed immersion into M.

In the group theoretic setup, the geodesics in M correspond to orbits of the subgroup $SO_0(1,1)$ in $\Gamma\backslash G$. Note that $SO_0(1,1)$ does not contain any unipotent element other than identity, while our proof of Theorem B depends crucially

on the behaviour of the orbits of nontrivial unipotent one-parameter subgroups contained in H. In fact, Theorem C proves a particular case of the following very general conjecture due to M.S. Raghunathan.

Conjecture 1 (Raghunathan) *Let G be a Lie group, Γ be a lattice in G and H be a subgroup generated by unipotent elements of G contained in it. Then for every $x \in \Gamma\backslash G$, there exists a closed subgroup L of G containing H such that, $\overline{xH} = xL$ and xL supports a finite L-invariant measure.*

An element $u \in G$ is called *unipotent* if the map $\mathrm{Ad}\, u$ is a unipotent automorphism of the Lie algebra of G. We note that if a connected subgroup H of G is semisimple and has no connected nontrivial compact normal subgroup, then H is generated by unipotent elements of G contained in it.

We refer the reader to the survey articles by S.G. Dani [D1] and G.A. Margulis [M1,M4] for various developements related to Raghunathan's conjecture. Recently, this conjecture has been proved by Marina Ratner. She first classiffied all finite ergodic invariant measures of H on $\Gamma\backslash G$ (see [Ra1]) and then proved the following stronger theorem, which implies Raghunathan's conjecture.

Theorem E (Ratner [Ra2], see also [DS,Sh]) *Let G be a Lie group, Γ be a lattice in G and $\{u_t : t \in \mathbb{R}\}$ be a unipotent one-parameter subgroup of G. Then for every $x \in \Gamma\backslash G$ there exists a closed subgroup L such that xL is closed, xL admits an L-invariant probability measure σ and the $\{u_t\}_{t\in\mathbb{R}}$-orbit through x is uniformly distributed with respect to σ; that is, for all bounded continuous functions f on $\Gamma\backslash G$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(xu_t)\,dt = \int_{xL} f\,d\sigma.$$

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