Correction to "Unipotent flows and counting lattice points on homogeneous spaces."

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The purpose of this note is to make a correction in the proof of [EMS, Theorem 2.1], which is the main technical result in [EMS].

We follow that proof of [EMS, Theorem 2.1] up to and including [EMS, Equation (17), Page 274], which says that for any $\omega \in \Omega' = h\Theta(B) \cap G(\mathbb{Q})$,

$$\bar{p}_F \gamma_1 \cdot \rho_1(\omega) = \bar{p}_F \gamma_i \cdot \rho_i(\omega) \subset \Phi_i, \qquad \forall i \in \mathbb{N}. \tag{1}$$

Since $H(\mathbb{Q})$ is dense in H, we have that (1) holds for all $\omega \in h\Theta(B)$. Since the equality in (1) is a Zariski closed condition in ω and $h\Theta(B)$ is Zariski dense in H, we get

$$\bar{p}_F \gamma_1 \cdot \rho_1(w) = \bar{p}_F \gamma_i \cdot \rho_i(w) \subset \Phi_i, \qquad \forall i \in \mathbb{N}, \forall w \in H. \tag{2}$$

Now since $\cap_{i=0}^{\infty} \Phi_i = D \subset \mathcal{A}_{\mathcal{F}}$, we get $\bar{p}_F \gamma_1 \cdot \rho_1(H) \subset \mathcal{A}_F$. Therefore by [EMS, Proposition 3.3],

$$\gamma_1 \rho_1(H) \subset N(F, U) \tag{3}$$

Putting $\omega = e$ in (2) we get $\bar{p}_F \gamma_i = \bar{p}_F \gamma_1$ for all $i \in \mathbb{N}$. Therefore

$$\bar{p}_F \cdot (\gamma_1 \rho_i(w) \rho_1(w)^{-1} {\gamma_1}^{-1}) = \bar{p}_F, \qquad \forall w \in H, \forall i \in \mathbb{N}.$$

Thus $\rho_i(w){\rho_1}^{-1}(w) \subset {\gamma_1}^{-1} N^1(F) \gamma_1$, for all $i \in \mathbb{N}$ and all $w \in H$.

In view of this and (3), replacing F by ${\gamma_1}^{-1} F \gamma_1$, without loss of generality we obtain that

$$\rho_i(w){\rho_1}^{-1}(w) \in N^1(F), \qquad \forall w \in H, \tag{4}$$

and

$$\rho_1(H) \subset N(F, U). \tag{5}$$

Claim 1 *F is normal in G, and μ is F-invariant.*

It is this conclusion in the proof of [EMS, Theorem 2.1], stated in last line before the beginning of the last paragraph on [EMS, Page 274], which was obtained via an incomplete argument. Once we prove the claim below, the rest of the proof of [EMS, Theorem 2.1] requires no more changes.

Proof of Claim 1: Let L be the smallest connected real algebraic subgroup of G such that

$$\rho_i(w)\rho_1(w)^{-1} \in L, \qquad \forall w \in H, \forall i \in \mathbb{N}.$$

Since $H(\mathbb{Q})$ is Zariski dense in H and $\rho_i(H(\mathbb{Q})) \subset G(\mathbb{Q})$ for all i, we conclude that L is defined over $\mathbb{Q}$.

Let $g, w \in H$. Since ρ_i's are homogeneous functions,

$$\rho_1(g)(\rho_i(w)\rho_1(w)^{-1})\rho_1(g)^{-1} = (\rho_i(g)\rho_1(g)^{-1})^{-1}(\rho_i(gw)\rho_1(gw)^{-1}) \in L, \qquad \forall i \in \mathbb{N}.$$

This shows that $\rho_1(H) \subset N(L)$. Hence $L\rho_1(H)$ is a $\mathbb{Q}$-subgroup of G. Since $\rho_i(H) \subset L\rho_1(H)$ for all $i \in \mathbb{N}$, by the hypothesis (i) of [EMS, Theorem 2.1], we have that $G = L\rho_1(H)$.

By (4), $L \subset N^1(F)$. Therefore $LN(F,U) = N(F,U)$. Now by (5),

$$G = L\rho_1(H) \subset LN(F,U) = N(F,U).$$

Hence by definition of $N(F,U)$ (see [EMS, Page 263]), $gUg^{-1} \in F$ for all $g \in N(F,U) = G$. Let F' be the smallest real algebraic $\mathbb{Q}$-subgroup of G containing gUg^{-1} for all $g \in G$. Then F' is normal in G. Since F is defined over $\mathbb{Q}$, $F' \subset F$.

Note that $F' \in \mathcal{H}$ (see [EMS, Page 263], [S, Theorem 2.3, Proposition 3.2]). If $\dim F' < \dim F$ then

$$G = N(F') \subset N(F', U) \subset S(F, U).$$

On the other hand, we know that $N(F,U) \setminus S(F,U) \neq \emptyset$, which is a contradiction. Therefore $\dim F' = \dim F$ and hence $F = F'$. This proves that F is a normal subgroup of G.

Now since $\mu(\pi(S(F,U))) = 0$, by [EMS, Theorem 2.7], every U-ergodic component of μ is F-invariant. Hence μ is F-invariant. This completes the proof of the claim. □

Now since we have proved the Claim 1, we resume the proof of [EMS, Theorem 2.1] from the last paragraph on [EMS, Page 274], which begins with the sentence "Now we project everything onto $F\backslash G$, ... ". The rest of the proof is valid.

References

[EMS] A. Eskin, S. Mozes, and N. A. Shah: Unipotent flows and counting lattice points on homogeneous spaces. *Ann. of Math.* **143** (1996), 253–299.

[S] N. A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.* **289** (1991), 315–334.

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