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On actions of epimorphic subgroups on homogeneous spaces

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Abstract. For an inclusion $F < G < L$ of connected real algebraic groups such that F is epimorphic in G, we show that any closed F-invariant subset of L/Λ is G-invariant, where Λ is a lattice in L. This is a topological analogue of a result due to S. Mozes, that any finite F-invariant measure on L/Λ is G-invariant.

This result is established by proving the following result. If in addition G is generated by unipotent elements, then there exists $a \in F$ such that the following holds. Let $U \subset F$ be the subgroup generated by all unipotent elements of F, $x \in L/\Lambda$, and λ and μ denote the Haar probability measures on the homogeneous spaces $\overline{Ux}$ and $\overline{Gx}$, respectively (cf. Ratner's theorem). Then $a^n\lambda \to \mu$ weakly as $n \to \infty$.

We also give an algebraic characterization of algebraic subgroups $F < \mathrm{SL}_n(\mathbb{R})$ for which all orbit closures on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ are finite-volume almost homogeneous, namely the smallest observable subgroup of $\mathrm{SL}_n(\mathbb{R})$ containing F should have no non-trivial algebraic characters defined over $\mathbb{R}$.

1. *Introduction*

Let L be a Lie group, $\mathfrak{L}$ its Lie algebra, $\mathrm{Ad} : L \to \mathrm{GL}(\mathfrak{L})$ its adjoint representation, Λ a discrete subgroup of L, and $\pi : L \to L/\Lambda$ the quotient map. Consider the action of L on the quotient space L/Λ via left translations: $g \cdot \pi(h) = \pi(gh)$, $\forall g, h \in L$. Assume that Λ is a lattice in L; that is, there exists an L-invariant Borel probability measure on L/Λ. From the point of view of applications to problems in number theory and geometry, it is of interest to find algebraic descriptions of the closures of individual F-orbits, and the F-ergodic F-invariant measures on L/Λ, where F is a subgroup of L.

A fundamental result in this regard is the following theorem due to Ratner [**R1, R2**]. Let F be a connected subgroup of L generated by Ad-unipotent elements (here $u \in L$ is

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called Ad-*unipotent* if $(\mathrm{Ad}\, u) - 1$ is a nilpotent linear transformation on $\mathfrak{L}$). Then for any $x \in L/\Lambda$, $\overline{Fx}$ is a *finite-volume homogeneous set*; that is, there exists a closed subgroup H of L such that $\overline{Fx} = Hx$ and Hx has a finite H-invariant measure. Also, any finite F-ergodic, F-invariant Borel measure, say μ, on L/Λ is a *homogeneous measure*; that is, there exists a closed subgroup H of G such that μ is H-invariant and $\mathrm{supp}(\mu) = Hx$ for some $x \in L/\Lambda$, where $\mathrm{supp}(\mu)$ denotes the support of the measure μ. The problem which motivates this paper is to understand that to what extent the assumptions on F in these theorems may be relaxed.

In [**S4**], the first-named author of this article obtained the same conclusions for the action of any subgroup F of L such that the subgroup generated by unipotent elements of $\mathrm{Ad}(F)$ is Zariski dense in $\mathrm{Ad}(F)$. Partial results indicate that a similar behaviour occurs when F is a higher-dimensional $\mathbb{R}$-split abelian subgroup and L is semisimple (see [**Moz1, KS**] for related results and conjectures).

At the other extreme, no such behaviour can be expected when F is a one-dimensional $\mathbb{R}$-split abelian subgroup of a semisimple group G contained in L. For example, Furstenberg and Weiss have shown (oral communication) that if $L = \mathrm{SL}(2, \mathbb{R})$, $\Lambda = \mathrm{SL}(2, \mathbb{Z})$ and F is the group of diagonal matrices in L, then for every $\alpha \in [1, 3]$ there is an F-orbit whose closure has Hausdorff dimension equal to α.

We now consider an example of the action of a subgroup F which is neither generated by unipotent elements, nor diagonalizable over the reals. Let G be a connected semisimple Lie group. Let $\{g_t\} \subset G$ be a one-parameter group of semisimple elements whose projection on any (non-trivial) factor of G is not contained in a compact subgroup. Let

$$U^+ = \{u \in G : g_{-t} u g_t \to e \text{ as } t \to \infty\},$$

be the expanding horospherical subgroup associated with $\{g_t\}$, and let F be the group generated by $\{g_t\}$ and U^+. If $L = G$, then any F-orbit on G/Λ is dense (see [**DR**]). More generally when $L \supset G$, using Ratner's theorems it was shown that $\overline{Fx} = \overline{Gx}$ and it is a finite-volume homogeneous set for all $x \in L/\Lambda$ (see [**S3**]). In this article we shall show that the dynamical property that $\overline{Fx} = \overline{Gx}$ is shared by a larger class of subgroups F of G which are described in terms of linear representations.

Epimorphic subgroups

Definition 1.1. Let G be a real algebraic group (that is, G is an open subgroup of the $\mathbb{R}$-points of a linear algebraic group defined over $\mathbb{R}$). A subgroup F of G is called *epimorphic* in G (notation: $F <_{\mathrm{epi}} G$) if any F-fixed vector is also G-fixed for any finite dimensional algebraic linear representation of G.

Epimorphic subgroups were introduced by Bergman [**Be**], and their in-depth study was made by Bien and Borel [**BB**]. We note some examples of epimorphic subgroups: (i) a parabolic subgroup of a semisimple group without compact factors; (ii) the subgroup F generated by $\{g_t\}$ and U^+, described above, is epimorphic in G (cf. [**S3**, Lemma 5.2]); and (iii) a Zariski dense subgroup of a real algebraic group. It may be noted that any non-compact simple algebraic group contains a three-dimensional algebraic epimorphic subgroup (see [**BB**]).

An ergodic-theoretic consequence of the representation-theoretic definition of an epimorphic subgroup was first obtained by Mozes in the following.

THEOREM. **[Moz2]** *Let L be a linear Lie group and Λ a discrete subgroup of L. Let G be a connected real algebraic group contained in L, and generated by unipotent one-parameter subgroups. Let F be a connected real algebraic epimorphic subgroup of G. Then any finite F-invariant Borel measure on L/Λ is also G-invariant. In particular, any F-invariant F-ergodic Borel probability measure on L/Λ is homogeneous.*

The same conclusion is valid for a connected epimorphic subgroup F of G of the form TU, where T is a non-algebraic subgroup diagonalizable over $\mathbb{R}$ and U is a unipotent subgroup normalized by T.

In [**MT**, Section 8], a similar result is proved for the actions on homogeneous spaces of products of real and p-adic Lie groups.

Orbit closures. Let the notation be as above. For $F <_{\text{epi}} G$, it is natural to ask: is it true that every F-invariant closed subset of L/Λ is also G-invariant? An example, due to Raghunathan (see [**W1**]), shows that this is not true for certain non-algebraic epimorphic subgroups F of G. However if F is a connected real algebraic epimorphic subgroup of G, we have the following result due to the second-named author of this article.

THEOREM. **[W1]** *Let G be a connected real algebraic group defined over $\mathbb{Q}$, and with no non-trivial $\mathbb{Q}$-characters. Let F be a connected real algebraic epimorphic subgroup of G. Then every F-orbit in $G/G(\mathbb{Z})$ is dense.*

In this article we extend this result to prove the following.

THEOREM 1.1. *Let L be a Lie group, Λ a lattice in L and* Ad *the adjoint representation of L on its Lie algebra $\mathfrak{L}$. Let $F < G$ be connected Lie subgroups of L such that* $\mathrm{Ad}(G)$ *and* $\mathrm{Ad}(F)$ *are real algebraic subgroups of* $\mathrm{GL}(\mathfrak{L})$. *Suppose that* $\mathrm{Ad}(F)$ *is epimorphic in* $\mathrm{Ad}(G)$. *Then* $\mathrm{Ad}(G) = \mathrm{Ad}(F[G, G])$*, and any closed F-invariant subset of L/Λ is $F[G, G]$-invariant, where $[G, G]$ denotes the commutator subgroup of G.*

In particular, if $G = [G, G]$, or if G intersects the center of L^0 in a discrete subgroup, then every closed F-invariant subset in L/Λ is G-invariant.

COROLLARY 1.2. *Let L, Λ, G and F be as in Theorem 1.1. If G is generated by* Ad-*unipotent one-parameter subgroups, then the closure of any F-orbit in L/Λ is a finite-volume homogeneous set.*

COROLLARY 1.3. *Let $F < G < L$ be an inclusion of connected real algebraic groups such that F is epimorphic in G. Then any closed F-invariant subset in L/Λ is G-invariant, where Λ is a lattice in L.*

Remark. Suppose that G is a connected real algebraic group generated by unipotent one-parameter subgroups. Let F be a connected real algebraic epimorphic subgroup of G. Then there exists a connected $\mathbb{R}$-split solvable real algebraic subgroup, say TU, of F such that $TU <_{\text{epi}} G$ (see [**BB**]), where U is a connected unipotent subgroup and T is an $\mathbb{R}$-split real algebraic torus normalizing U.

Thus most of our questions about the actions of algebraic epimorphic subgroups can be easily reduced to the case of $\mathbb{R}$-split solvable algebraic epimorphic subgroups.

Limiting distributions. In view of the above remark, we will deduce Theorem 1.1 from the following stronger result, which is the main result of this article.

THEOREM 1.4. *Let L be a Lie group and Λ a lattice in L. Let $G \subset L$ be a Lie subgroup such that $G = [G, G]$ and G has no non-trivial compact quotients. Let F be a Lie subgroup of G such that $\mathrm{Ad}(F)$ is a connected $\mathbb{R}$-split solvable algebraic epimorphic subgroup of $\mathrm{Ad}(G)$ of the form TU. Then there exists an open sub-semigroup $T^{++} \subset T$ such that $\overline{T^{++}}$ is noncompact and for a sequence $\{a_i\} \subset F$, if $\{\mathrm{Ad}\, a_i\} \subset T^{++}$ and it is divergent in $\overline{T^{++}}$ then the following holds. Let $\tilde{U}$ be a connected Lie subgroup of F such that $\mathrm{Ad}(\tilde{U}) = U$. Let ν be a Haar measure on $\tilde{U}$ and $\psi \in L^1(\tilde{U}, \nu)$ with $\|\psi\|_1 = 1$. Then for any $x \in L/\Lambda$ there exists a closed subgroup H of L containing G such that for any bounded continuous function f on L/Λ,*

$$\lim_{i \to \infty} \int_{\tilde{U}} f(a_i u x)\psi(u)\, d\nu(u) = \int_{L/\Lambda} f\, d\mu_H, \tag{1.1}$$

where Hx is closed, and μ_H is an H-invariant probability measure on Hx. In particular, for any non-empty open set $\Omega \subset \tilde{U}$,

$$\overline{Fx} = \overline{\cup_{i=1}^{\infty} a_i \Omega x} = Hx = \overline{Gx}.$$

Consider the standard representations of $\mathrm{Ad}(G)$ on $V = \oplus_{k=1}^{\dim \mathfrak{L}} \wedge^k \mathfrak{L}$, and on the quotient $\bar{V} = V/V^G$, where V^G is the subspace of all $\mathrm{Ad}(G)$-fixed vectors in V. The main consequence of the hypothesis, that $TU <_{\mathrm{epi}} \mathrm{Ad}(G)$ and T is real algebraic, is the following [**W1**, Lemma 1]: *there exists a non-empty open sub-semigroup $T^{++} \subset T$ such that if $\{g_i\}$ is a divergent sequence in $\overline{T^{++}}$ then for any U-fixed vector $v \in V \cup \bar{V}$, either v is $\mathrm{Ad}(G)$-fixed or $g_i v \to \infty$ as $i \to \infty$.*

It is this semigroup T^{++} which is involved in the statement of Theorem 1.4.

The proof of Theorem 1.4 uses Ratner's classification of ergodic invariant measures for actions of unipotent subgroups, and the techniques developed for analysing the behaviour of unipotent trajectories near the images of algebraic subvarieties of L on L/Λ (see the survey articles [**R3, D3, M3**]).

We obtain a variant of Theorem 1.4 by relaxing the conditions that $G = [G, G]$ and F is solvable (see Theorem 5.1).

We also obtain versions of Theorem 1.4 where one has uniform convergence in (1.1) as x varies over certain relatively compact open subsets of L/Λ (see Theorems 3.1 and 3.2); the following special case is of interest.

COROLLARY 1.5. *Let the notation be as in Theorem 1.4. Furthermore, suppose that G is a maximal proper connected subgroup of L. Then there exists an open sub-semigroup $T^{++} \subset T$ with the following property. Let a compact set $K \subset L/\Lambda$, a bounded continuous function f on L/Λ, and an $\epsilon > 0$ be given. Then there exist finitely many closed orbits $Gx_1, \ldots, Gx_r$ such that for any compact set $K_1 \subset K \setminus \cup_{i=1}^{r} Gx_j$, there exists a compact* *set $S \subset \overline{T^{++}}$ such that for any $a \in G$ with* $\mathrm{Ad}\, a \in \overline{T^{++}} \setminus S$ *the following holds:*

$$\left| \int_{\tilde{U}} f(aux)\psi(u)\, d\nu(u) - \int_{L/\Lambda} f\, d\mu_L \right| < \epsilon, \quad \forall x \in K_1,$$

where μ_L is the L-invariant probability measure on L/Λ.

Orbit closures and observable subgroups. Regarding the general question of algebraically describing orbit-closures the following concept is useful.

Definition 1.2. Let $G < L$ be an inclusion of connected real algebraic groups. If there exists an algebraic linear representation $\rho : L \to \mathrm{GL}(V)$ and a vector $\mathbf{v} \in V$ such that $G = \{g \in L : \rho(g)\mathbf{v} = \mathbf{v}\}$ then G is called an *observable subgroup* of L (see [**BHM**] for equivalent definitions).

Definition 1.3. Let $F < L$ be an inclusion of connected real algebraic groups. The *observable envelope* of F in L is defined to be the smallest (it exists) observable subgroup of L containing F.

Remark. The observable envelope of F in L is also the largest connected real algebraic subgroup of L in which F is epimorphic [**BB**, Proposition 1].

Now the following result is an immediate consequence of Theorem 1.1.

THEOREM 1.6. *Let L be a connected Lie group and Λ a lattice in L. Let F be a connected Lie subgroup of L such that* $\mathrm{Ad}(L)$ *and* $\mathrm{Ad}(F)$ *are real algebraic. Let G be the smallest closed connected subgroup of L containing F such that* $\mathrm{Ad}(G)$ *is observable in* $\mathrm{Ad}(L)$. *Then any closed F-invariant subset in L/Λ is G-invariant.*

Furthermore, if G is generated by Ad*-unipotent one-parameter subgroups then the closures of F-orbits are finite-volume homogeneous sets.*

Using Theorem 1.6 we intend to describe the class of algebraic subgroups for which any orbit-closure in any finite-volume homogeneous space is a finite-volume almost homogeneous set.

Definition 1.4. A closed subset of L/Λ will be called *finite-volume almost homogeneous* if it is of the form KS, where K is a compact subgroup of L and S is a finite-volume homogeneous set.

THEOREM 1.7. *Let L be a connected $\mathbb{R}$-split real algebraic semisimple group (for example, $L = \mathrm{SL}(n, \mathbb{R})$). Let F be a connected algebraic subgroup of L, and G be the observable envelope of F in L. Then the following statements are equivalent:*

(1) *G has no non-trivial character defined over $\mathbb{R}$;*
(2) *For any lattice Λ in L and any $x \in L/\Lambda$, the set $\overline{Fx}$ is finite-volume almost homogeneous.*

The implication (1) $\Rightarrow$ (2) of the theorem follows immediately from Theorem 1.6. For the converse we use a result of Sukhanov [**Su**] on the structure of observable subgroups.

Finite invariant measures. Regarding the invariant measures for the actions of epimorphic subgroups, we extend the result due to Mozes, mentioned above, to the actions of all (possibly non-algebraic) epimorphic subgroups of G.

THEOREM 1.8. *Let L be Lie group and Λ a discrete subgroup of L. Let G be a subgroup of L which is generated by* Ad*-unipotent one-parameter subgroups. Let F be a connected Lie subgroup of G such that* $\mathrm{Ad}(F)$ *is an epimorphic subgroup of* $\mathrm{Ad}(G)$*. Then* $\mathrm{Ad}(G) = \mathrm{Ad}(F[G, G])$*, and any finite F-invariant Borel measure on L/Λ is $F[G, G]$-invariant.*

In particular, if $G = [G, G]$, or if G intersects the center of L in a discrete subgroup, then any finite F-invariant Borel measure on L/Λ is G-invariant.

The proof of the above theorem uses Ratner's theorem and a generalized version of Borel's density theorem due to Dani [**D2**].

Locally finite invariant measures. We recall that a Borel measure which is finite on compact sets is called *locally finite*. Using a variant of Theorem 1.4 (see Theorem 5.1) we obtain the following result on locally finite F-invariant Borel measures.

THEOREM 1.9. *Let L be a Lie group and Λ a lattice in L. Let G be a Lie subgroup of L generated by* Ad*-unipotent one-parameter subgroups. Let F be a connected subgroup of L such that* $\mathrm{Ad}(F)$ *is a real algebraic epimorphic subgroup of* $\mathrm{Ad}(G)$*. Then* $\mathrm{Ad}(G) = \mathrm{Ad}(F[G, G])$*, and any locally finite F-invariant Borel measure μ on L/Λ is $F[G, G]$-invariant.*

Moreover, there exists a countable partition of L/Λ into $F[G, G]$-invariant Borel measurable subsets X_i ($i \in \mathbb{N}$) such that $\mu(X_i) < \infty$ for each i.

In particular, any locally finite F-ergodic F-invariant Borel measure on L/Λ is a finite $F[G, G]$-invariant homogeneous measure.

In other words, the subgroup action of F on a finite-volume homogeneous space of L has Property-(D) (see [**M1**] for a definition, and [**M2**, Theorem 15] for examples of subgroup actions with Property-(D)).

In Example 8.1 we show that Theorem 1.9 is not valid without the assumption that $\mathrm{Ad}(F)$ is real algebraic.

The article is organized as follows. First we obtain some results about representations of epimorphic subgroups, and recall some results on unipotent flows on homogeneous spaces in §2. The main theorem, Theorem 1.4 is proved in §3. The results on orbit closures of epimorphic subgroups are deduced in §4. A variant of Theorem 1.4 is obtained in §5. The results relating observable subgroups and orbit closures are obtained in §6. The finite invariant measures for epimorphic subgroup actions are studied in §7. The locally finite invariant measures are considered in §8.

2. *Basic results*

In this section we collect some results about linear representations of epimorphic subgroups, and on unipotent flows on homogeneous spaces.

2.1. *Epimorphic subgroups*. Let G be a connected real algebraic group which is generated by algebraic unipotent elements. Any connected real algebraic epimorphic subgroup of G contains a connected $\mathbb{R}$-split solvable algebraic epimorphic subgroup, which is a semidirect product of the form TU, where T is a connected $\mathbb{R}$-split torus and U is an algebraic unipotent subgroup normalized by T (see [**BB**]).

Let TU, as above, be an $\mathbb{R}$-split solvable epimorphic subgroup of G. Let $X(T)$ denote the group of algebraic characters on T defined over $\mathbb{R}$. Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic linear representation of G. Define

$$\begin{aligned} \mathcal{C}(\rho) = \{\chi \in X(T) \setminus \{1\} : \exists\, v \in V \text{ such that } \rho(U)v = v, \\ \text{and } \rho(t)v = \chi(t)v,\ \forall t \in T\}. \end{aligned} \tag{2.2}$$

A sequence $\{a_i\} \subset T$ is called $\mathcal{C}(\rho)$-*divergent*, if

$$\lim_{i\to\infty} \chi(a_i) = \infty, \quad \forall \chi \in \mathcal{C}(\rho). \tag{2.3}$$

The main results of this paper are based on the existence of $\mathcal{C}(\rho)$-divergent sequences.

LEMMA 2.1. [**W1**, Lemma 1] *Let the notation be as above. Then*

$$T^+ = T^+(\mathcal{C}(\rho)) = \{t \in T : \chi(t) > 1, \forall \chi \in \mathcal{C}(\rho)\}$$

is a non-empty open sub-semigroup (a 'cone') in T.

In particular $\{t^n\}$ is a $\mathcal{C}(\rho)$-divergence sequence for any $t \in T^+$.

It follows that there exists a non-empty open sub-semigroup $T^{++} \subset T^+$ such that any divergent sequence (that is, eventually escaping every compact set) in $\overline{T^{++}}$ is $\mathcal{C}(\rho)$-divergent. (See [**W2**] for another proof and some applications of this lemma.)

PROPOSITION 2.2. *Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic linear representation of G such that V has no non-zero G-fixed vectors. Let a sequence $\{v_i\} \subset V$ be such that $0 \notin \overline{\{v_i\}}$. Let Ω be a neighbourhood of the identity in U and $\{a_i\}$ be a $\mathcal{C}(\rho)$-divergent sequence in T. Then as $i \to \infty$,*

$$\sup_{\omega\in\Omega} \|\rho(a_i\omega)v_i\| \to \infty, \tag{2.4}$$

where $\|\cdot\|$ is any norm on V.

The proof uses the following.

LEMMA 2.3. [**S3**, Lemma 5.1] *Let V be a finite dimensional normed linear space over $\mathbb{R}$. Let N be a connected unipotent subgroup of $\mathrm{GL}(V)$. Let $W = \{\mathbf{v} \in V : N \cdot \mathbf{v} = \mathbf{v}\}$, and let Pr_W denote a projection onto W. Then for any neighbourhood Ω of the identity in N there exists $C > 0$ such that the following holds: for every $\mathbf{v} \in V$ there exists $\omega \in \Omega$ such that*

$$\|\mathbf{v}\| \le C \cdot \|\mathrm{Pr}_W(\omega \cdot \mathbf{v})\|.$$

Proof of Proposition 2.2. Let $W = \{v \in V : \rho(U)v = v\}$. By the Lie–Kolchin theorem, $W \neq 0$. Since TU is epimorphic in G and V has no G-fixed vectors, there is no non-zero T-fixed vector in W. Since W is T-invariant and T is $\mathbb{R}$-split,

$$W = \oplus_{\chi \in \mathcal{C}(\rho)} W^{\chi}, \tag{2.5}$$

were $W^{\chi} = \{v \in W : \rho(t)v = \chi(t)v, \forall t \in T\}$. Let Pr_W be a T-equivariant projection onto W.

Since $0 \notin \overline{\{v_i\}}$, by Lemma 2.3, there exists a sequence $\{\omega_i\} \in \Omega$ such that

$$0 \notin \overline{\{\mathrm{Pr}_W(\rho(\omega_i)v_i)\}}.$$

Therefore, by the definition of $\mathcal{C}(\rho)$-divergent sequences and (2.5), as $i \to \infty$,

$$\|\rho(a_i) \cdot \mathrm{Pr}_W(\rho(\omega_i)v_i)\| \to \infty,$$

and hence $\|\mathrm{Pr}_W(\rho(a_i\omega_i)v_i)\| \to \infty$. From this (2.4) follows. □

Using the same argument we obtain the following.

PROPOSITION 2.4. *Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic linear representation. Then given a neighbourhood Ω of e in U, there exists a constant $C > 0$ such that*

$$\sup_{\omega \in \Omega} \|\rho(a\omega)v\| \geq C\|v\|, \quad \forall v \in V,\ \forall a \in T^{+}(\mathcal{C}(\rho)).$$

When V has non-zero G-fixed vectors, we need additional conditions on G to obtain the stronger conclusion as in (2.4).

PROPOSITION 2.5. *Suppose further that $G = [G, G]$ (and recall that G has no non-trivial compact quotients). Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic linear representation. Let $\bar{\rho}$ be the corresponding representation of G on $\bar{V} = V/V^G$, where V^G denotes the space of G-fixed vectors on V. Let a sequence $\{v_i\} \subset V$ be such that $\overline{\{v_i\}} \cap V^G = \emptyset$. Let Ω be a neighbourhood of e in U and $\{a_i\}$ be a $\mathcal{C}(\rho \oplus \bar{\rho})$-divergent sequence in T. Then as $i \to \infty$,*

$$\sup_{\omega \in \Omega} \|\rho(a_i\omega)v_i\| \to \infty. \tag{2.6}$$

Proof. Since $G = [G, G]$, there are no non-trivial solvable quotients of G, and hence V/V^G has no non-trivial G-fixed vectors. Let $\{\bar{v}_i\}$ denote the image of $\{v_i\}$ on $\bar{V}$. Let Pr_{V^G} denote a T-equivariant projection onto V^G.

Suppose that (2.6) does not hold. Then $\sup_{\omega \in \Omega} \|\rho(a_i\omega)v_i\|$ is bounded along a subsequence. After passing to a subsequence, there are two cases:

(i) $\mathrm{Pr}_{V^G}(v_i) \to \infty$, or

(ii) $\mathrm{Pr}_{V^G}(v_i)$ is bounded.

If (i) holds, then

$$\mathrm{Pr}_{V^G}(\rho(a_i)v_i) = \rho(a_i)\mathrm{Pr}_{V^G}(v_i) \geq \mathrm{Pr}_{V^G}(v_i) \to \infty,$$

a contradiction. If (ii) holds, then by our hypothesis on $\{v_i\}$, we have that $0 \notin \overline{\{\bar{v}_i\}}$. Therefore by Proposition 2.2,

$$\sup_{\omega \in \Omega} \|\bar{\rho}(a_i\omega)\bar{v}_i\| \to \infty,$$

which is also a contradiction. □

2.2. *Flows on finite-volume homogeneous spaces*

Notation. For $d, m \in \mathbb{N}$, let $\mathcal{P}_{d,m}(L)$ denote the set of continuous maps $\Theta : \mathbb{R}^m \to L$ such that for all $c, a \in \mathbb{R}^m$ and $X \in \mathfrak{L}$, the map

$$t \in \mathbb{R} \mapsto \mathrm{Ad}(\Theta(tc + a))(X) \in \mathfrak{L}$$

is a polynomial of degree at most d in each coordinate of $\mathfrak{L}$ (with respect to any basis).

Let $V_L = \bigoplus_{k=1}^{\dim \mathfrak{L}} \wedge^k \mathfrak{L}$, the direct sum of exterior powers of $\mathfrak{L}$, and consider the linear action of L on V_L via the direct sum of the exterior powers of the Adjoint representation. Fix any norm on V_L.

For any non-trivial connected Lie subgroup W of L, and its Lie algebra $\mathfrak{W}$, we choose a non-zero vector $\mathbf{p}_W$ in the one-dimensional subspace $\wedge^{\dim \mathfrak{W}} \mathfrak{W} \subset V_L$.

The following result, essentially due to Dani and Margulis [**D1, DM2**], is one of the most important results for studying unipotent flows on *non-compact* finite-volume homogeneous spaces.

THEOREM 2.6. *Let Λ be a lattice in L and $\pi : L \to L/\Lambda$ be the natural quotient. Then there exist closed subgroups $W_1, \dots, W_r$ of L such that $\pi(W_i)$ is compact, $\Lambda \cdot \mathbf{p}_{W_i}$ is discrete for each $1 \leq i \leq r$, and the following holds. Given $d, m \in \mathbb{N}$ and $\alpha, \epsilon > 0$, there exists a compact set $K \subset L/\Lambda$ such that for any $\Theta \in \mathcal{P}_{d,m}(L)$, and any bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions is satisfied:*

(i) *there exists $\gamma \in \Lambda$ and $i \in \{1, \dots, r\}$ such that*

$$\sup_{t \in B} \|\Theta(t)\gamma.\mathbf{p}_{W_i}\| < \alpha.$$

(ii) $(1/|B|)|\{t \in B : \pi(\Theta(t)) \in K\}| \geq 1 - \epsilon$*, where $|\cdot|$ denotes a Lebesgue measure on* $\mathbb{R}^m$.

See [**S3**, Theorem 2.2] for the deduction of this result from the results of Dani and Margulis for semisimple groups.

Remark. In the above result, if L is a semisimple real algebraic group defined over $\mathbb{Q}$ and with no compact factors, and Λ is an arithmetic lattice in L with respect to the $\mathbb{Q}$-structure, then the subgroups W_i are the unipotent radicals of maximal $\mathbb{Q}$-parabolic subgroups of L. The number r of W_i's needed is at most the product of the $\mathbb{Q}$-rank of L and the number of 'cusps' in the fundamental domain of Λ.

2.3. *Singular sets*. For the rest of §2, let L be a connected Lie group, and Λ a discrete subgroup of L (here Λ need not be a lattice in L.)

Let $\mathcal{H} = \mathcal{H}_\Lambda$ denote the collection of all closed connected subgroups H of L such that $H \cap \Lambda$ is a lattice in H, and the subgroup generated by the one-parameter unipotent subgroups of L contained in H acts ergodically on $H/H \cap \Lambda$ with respect to the H-invariant probability measure.

THEOREM 2.7. [**R1**, Theorem 1.1] *The collection $\mathcal{H}$ is countable.*

Let W be a subgroup of L which is generated by one-parameter Ad-unipotent subgroups.

For any $H \in \mathcal{H}$, we define:

$$\begin{aligned} N(H, W) &= \{g \in L : W \subset gHg^{-1}\} \\ S(H, W) &= \bigcup \{N(H', W) : H' \in \mathcal{H}, H' \subset H, \dim H' < \dim H\} \\ N^*(H, W) &= N(H, W) \setminus S(H, W). \end{aligned}$$

We note that:

$$N(H, W) = N_L(W) N(H, W) N_L(H) \tag{2.7}$$

$$N(H, W)\gamma = N(\gamma^{-1} H \gamma, W), \quad \forall \gamma \in \Lambda \tag{2.8}$$

$$N^*(H, W) = N_L(W) N^*(H, W)(N_L(H) \cap \Lambda). \tag{2.9}$$

LEMMA 2.8. [**MS**, Lemma 2.4] *For any $g \in N^*(H, W)$, the group gHg^{-1} is the smallest closed subgroup of L which contains W and whose orbit through $\pi(g)$ is closed. In particular,*

$$\pi(N^*(H, W)) = \pi(N(H, W)) \setminus \pi(S(H, W)). \tag{2.10}$$

LEMMA 2.9. *The natural map*

$$N^*(H, W)/(N_L(H) \cap \Lambda) \to \pi(N^*(H, W))$$

is injective.

Proof. Let $g_1, g_2 \in N^*(H, W)$ be such that $\pi(g_1) = \pi(g_2)$. By Lemma 2.8, $g_i H g_i^{-1}$ is the smallest closed subgroup of L whose orbit through $\pi(g_i)$ is closed, for $i = 1, 2$. Therefore $g_1 H g_1^{-1} = g_2 H g_2^{-1}$. Hence $g_2^{-1} g_1 \in N_L(H) \cap \Lambda$. This completes the proof. □

2.4. *Ratner's theorem.* Using Ratner's description [**R1**] of the finite W-invariant W-ergodic Borel measures on L/Λ and Theorem 2.7, one can describe finite W-invariant measures as follows:

THEOREM 2.10. [**R1**] *Let L, Λ and W be as above. Let μ be a finite W-invariant measure on L/Λ. Then there exists $H \in \mathcal{H}$ such that*

$$\mu(\pi(N(H, W))) > 0 \quad \textit{and} \quad \mu(\pi(S(H, W))) = 0.$$

Moreover, almost every W-ergodic component of the restriction of μ to $\pi(N(H, W))$ is concentrated on $g\pi(H)$ for some $g \in N^*(H, W)$, and it is invariant under gHg^{-1}. In particular, if $\mu(\pi(S(L, W))) = 0$ then μ is L-invariant.

See [**MS**, Theorem 2.2] or [**D3**, Corollary 5.6] for its deduction.

2.5. *Linear presentation*. For $H \in \mathcal{H}$, let $V_L(H, W)$ denote the linear span of the set $N(H, W) \cdot \mathbf{p}_H$ in V_L, and let $\mathfrak{H}$ denote the Lie algebra of H. We note that (cf. [**DM1**, Proposition 3.2])

$$N(H, W) = \{g \in L : g \cdot \mathbf{p}_H \in V_L(H, W)\} \tag{2.11}$$

and

$$\begin{aligned} N_L^1(H) &\stackrel{\text{def}}{=} \{g \in N_L(H) : \det(\operatorname{Ad} g|_{\mathfrak{H}}) = 1\} \\ &= \{g \in L : g \cdot \mathbf{p}_H = \mathbf{p}_H\}. \end{aligned} \tag{2.12}$$

THEOREM 2.11. [**DM1**, Theorem 3.4] *For $H \in \mathcal{H}$, the orbit $\Lambda \cdot \mathbf{p}_H$ is discrete. In particular, $N_L^1(H)\Lambda$ is closed in L/Λ.*

The following result is one of the basic technical tools used for applying Ratner's measure classification to understand limiting distributions of 'polynomial like' trajectories [**DM1, S1, MS, S2, D3**].

THEOREM 2.12. [**S3**, Theorem 4.1] *Let $H \in \mathcal{H}$, $d, m \in \mathbb{N}$ and $\epsilon > 0$ be given. Then for any compact set $C \subset \pi(N^*(H, W))$, there exists a compact set $D \subset V_L(H, W)$ with the following property. For any neighbourhood Φ of D in V_L, there exists a neighbourhood Ψ of C in L/Λ, such that for any $\Theta \in \mathcal{P}_{d,m}(L)$, and a bounded open convex set $B \subset \mathbb{R}^m$, one of the following holds:*
(i) $\Theta(B)\gamma \cdot \mathbf{p}_H \subset \Phi$ *for some* $\gamma \in \Lambda$.
(ii) $(1/|B|)|\{t \in B : \pi(\Theta(t)) \in \Psi\}| < \epsilon$.

3. *Limit distributions of translates of measures*

In this section we will complete the proof of Theorem 1.4, and also obtain certain uniform versions of the theorem.

Proof of Theorem 1.4. We will prove the theorem for $x = \pi(e)$. The general case follows by replacing G with gGg^{-1} and F with gFg^{-1}, if $x = \pi(g)$, $g \in L$.

Let the representation $\rho : \operatorname{Ad}(G) \to \mathrm{GL}(V_L)$ be the direct sum of the exterior powers of the inclusion $\operatorname{Ad}(G) \subset \mathrm{GL}(\mathfrak{L})$. Let $\bar{\rho}$ be the corresponding representation of $\operatorname{Ad}(G)$ on $V_L/(V_L)^G$ as defined in Proposition 2.5. By Lemma 2.1, there exists an open subsemigroup T^{++} in T such that if $\{a_i\} \subset F$ is a sequence such that $\{\operatorname{Ad}(a_i)\}$ is divergent in $\overline{T^{++}}$, then

$$\{\operatorname{Ad} a_i\} \text{ is a } \mathcal{C}(\rho \oplus \bar{\rho})\text{-divergent sequence in } T. \tag{3.13}$$

Without loss of generality we may assume that ψ vanishes outside a compact subset of $\tilde{U}$. Let $\tilde{\lambda}$ be the Borel measure on $\tilde{U}$ such that $d\tilde{\lambda} = \psi\, d\nu$. We identify the Lie algebra $\tilde{\mathfrak{U}}$ of $\tilde{U}$ with $\mathbb{R}^m$, where $m = \dim \tilde{U}$. Without loss of generality we may assume that ν is the pushforward of the Lebesgue measure on $\mathbb{R}^m$ under the exponential map $\exp : \tilde{\mathfrak{U}}(= \mathbb{R}^m) \to \tilde{U}$, and that $\tilde{\lambda}$ is a probability measure. Let B be a ball in $\mathbb{R}^m$ centred at zero such that $\operatorname{supp}(\tilde{\lambda}) \subset \exp(B)$. Let λ denote the pushforward of $\tilde{\lambda}$ on L/Λ under π. To prove the theorem, it is enough to show that $a_i\lambda \to \mu_H$ as $i \to \infty$ (recall that

$a_i\lambda(E) = \lambda(a_i^{-1}E)$ for any Borel measurable subset E of L/Λ). Note that the subgroup H as in the conclusion of the theorem does not depend on the choice of the sequence $\{a_i\}$, because $\overline{Gx} = Hx$. Thus it is enough to prove the convergence for some subsequence of $\{a_i\}$.

For each $i \in \mathbb{N}$, define $\Theta_i : \mathbb{R}^m \to L$ as $\Theta_i(t) = a_i \exp(t)$, $\forall t \in \mathbb{R}^m$. Since $\tilde{\mathfrak{U}}$ is a nilpotent Lie algebra, there is $d \in \mathbb{N}$ such that $\Theta_i \in \mathcal{P}_{d,m}(L)$ for all $i \in \mathbb{N}$.

CLAIM 3.1. *Given $\delta > 0$ there exists a compact $K \subset L/\Lambda$ such that*

$$a_i\lambda(K) > 1 - \delta, \quad \forall i \in \mathbb{N}.$$

Suppose that the claim fails to hold. Since $d\tilde{\lambda} = \psi\, d\nu$, there exists $\epsilon > 0$ such that for any compact set $K \subset L/\Lambda$,

$$\frac{1}{|B|}|\{t \in B : \pi(\Theta_i(t)) \in K\}| < 1 - \epsilon,$$

for all i in a subsequence. For each i, we apply Theorem 2.6 for $\Theta = \Theta_i$ and $\alpha = 1/i$. Then, after passing to a subsequence, there is a non-zero $\mathbf{p} \in V_L$ such that the following holds. The orbit $\Lambda \cdot \mathbf{p}$ is discrete and for each $i \in \mathbb{N}$ there exists $v_i \in \Lambda \cdot \mathbf{p}$, such that

$$\sup_{u \in \exp B} \|a_i u \cdot v_i\| \to 0 \quad \text{as } i \to \infty. \tag{3.14}$$

After passing to a subsequence we may assume that for all $i \in \mathbb{N}$, v_i is not fixed by G. Since $\{v_i\}$ is a discrete set not containing zero, we apply Proposition 2.4 (with $\mathrm{Ad}(G)$ in place of G), to obtain a contradiction to (3.14). This proves the claim.

By Claim 3.1, the set of measures $\{a_i\lambda\}$ is relatively compact in the space of probability measures on L/Λ. Thus to show that $a_i\lambda \to \mu_H$ as $i \to \infty$, it suffices to show this for all convergent subsequences. So we pass to a subsequence, and assume $a_i\lambda \to \mu$ for a probability measure μ on L/Λ.

Define

$$W = \{w \in \tilde{U} : a_i^{-1} w a_i \to e \text{ as } i \to \infty\}. \tag{3.15}$$

Note that W is a connected Lie subgroup of F and consists of Ad-unipotent elements. By passing to a subsequence of $\{a_i\}$, we may assume that $\dim W$ does not change if we replace $\{a_i\}$ by any subsequence.

CLAIM 3.2. *The limit measure μ is W-invariant.*

Let $w \in W$. Then for all $i \in \mathbb{N}$,

$$w a_i \lambda = a_i w_i \lambda, \quad \text{where } w_i = a_i^{-1} w a_i. \tag{3.16}$$

For any bounded continuous function f on L/Λ, we have

$$\begin{aligned} &\left| \int f\, d[a_i w_i \lambda] - \int f\, d[a_i\lambda] \right| \\ &\quad = \left| \int_{\tilde{U}} f(a_i \pi(w_i u))\, d\tilde{\lambda}(u) - \int_{\tilde{U}} f(a_i \pi(u))\, d\tilde{\lambda}(u) \right| \end{aligned}$$

$$= \left| \int_{\tilde{U}} f(a_i\pi(w_iu))\psi(u)\,d\nu(u) - \int_{\tilde{U}} f(a_i\pi(u))\psi(u)\,d\nu(u) \right|$$
$$= \left| \int_{\tilde{U}} f(a_i\pi(u))\psi(w_i^{-1}u)\,d\nu(u) - \int_{\tilde{U}} f(a_i\pi(u))\psi(u)\,d\nu(u) \right|$$
$$\leq \|f\|_\infty \cdot \int_{\tilde{U}} |\psi(w_i^{-1}u) - \psi(u)|\,d\nu(u).$$

Now since $w_i \to e$ and the left regular representation of $\tilde{U}$ on $L^1(\tilde{U}, \nu)$ is continuous,

$$\left| \int f\,d[a_iw_i\lambda] - \int f\,d[a_i\lambda] \right| \to 0, \quad \text{as } i \to \infty.$$

Therefore, since $a_i\lambda \to \mu$, we have $a_iw_i\lambda \to \mu$. Therefore by (3.16), $w\mu = \mu$, completing the proof of the claim.

It will be proved (Claim 3.3) that W is non-trivial. In view of Claim 3.2, we apply Theorem 2.10 to obtain that there exists a closed subgroup $H \in \mathcal{H}$, such that $\mu(\pi(S(H, W))) = 0$ and $\mu(\pi(N(H, W))) > 0$. Let a compact set $C \subset \pi(N^*(H, W))$ be such that $\mu(C) > 0$.

Since $d\tilde{\lambda} = \psi\,d\nu$, there exists $\epsilon > 0$ such that for any Borel measurable $E \subset \operatorname{supp}(\lambda) \subset L/\Lambda$,

$$\frac{1}{|B|}|\{t \in B : \pi(\exp(t)) \in E\}| < \epsilon \Rightarrow \lambda(E) < \mu(C)/2. \tag{3.17}$$

We apply Theorem 2.12 for $\epsilon > 0$, $d \in \mathbb{N}$, $m \in \mathbb{N}$, $\Theta = \Theta_i$, $B \subset \mathbb{R}^m$ chosen as above. For the compact set C as above there exists a compact set $D \subset V_L(H, W)$ such that the following holds. For each $i \in \mathbb{N}$, let Φ_i be a relatively compact neighbourhood of D in V_L such that $\Phi_{i+1} \subset \Phi_i$ and $\cap_{i=1}^{\infty}\Phi_i = D$. Then there exists an open neighbourhood Ψ_i of C in L/Λ such that one of the following conditions holds:
(i) there exists $v_i \in \Lambda \cdot \mathbf{p}_H$ such that $a_i \exp(B)v_i \subset \Phi_i$;
(ii) $(1/|B|)|\{t \in B : \pi(a_i \exp(t)) \in \Psi_i\}| < \epsilon$.
Since $a_i\lambda \to \mu$, and Ψ_i's are neighbourhoods of C, there exists $i_0 \in \mathbb{N}$ such that $\lambda(a_i^{-1}\Psi_i) > \mu(C)/2$ for all $i \geq i_0$. Therefore by (3.17), condition (ii) does not hold, and hence condition (i) must hold for all $i \geq i_0$; that is

$$a_i \exp(B) \cdot v_i \subset \Phi_i \subset \Phi_1, \quad \forall i \in \mathbb{N}. \tag{3.18}$$

Note that $\overline{\Phi_1}$ is compact and $\{v_i\} \subset \Lambda \cdot \mathbf{p}_H$ is discrete (by Theorem 2.11). Therefore by Proposition 2.5, after passing to a subsequence, we conclude that $G \cdot v_i = v_i$ for all $i \in \mathbb{N}$. Now since $\Lambda \cdot \mathbf{p}_H$ is discrete and $\cap_i \Phi_i = D$, after passing to a subsequence we have that $v_i = v_1 \in D$. Let $\gamma \in \Lambda$ be such that $v_1 = \gamma \cdot \mathbf{p}_H$. Then $G\gamma \cdot \mathbf{p}_H = \gamma \cdot \mathbf{p}_H$ and $\gamma \cdot \mathbf{p}_H \in D$. Therefore by (2.11) $\gamma \in N(H, W)$. In view of (2.8), replacing H by $\gamma H \gamma^{-1}$, without loss of generality we may assume that $\gamma = e$. Therefore $G \cdot \mathbf{p}_H = \mathbf{p}_H$, and hence by (2.12) $G \subset N_L^1(H)$.

Since $\operatorname{Ad} F \subset \operatorname{Ad}(G)$, we have $F \subset N_L^1(H)$. By Theorem 2.11, $\pi(N_L^1(H))$ is closed. Therefore $\operatorname{supp}(\mu) \subset \pi(N_L^1(H))$. Also since $e \in N(H, W)$, we have $W \subset H$. Therefore by Theorem 2.10, μ is H-invariant.

If we can prove that H contains G, then μ is G-invariant. Since $F \subset G$, we have that $\mathrm{supp}(\mu) \subset \overline{\pi(F)} \subset \pi(H)$. Since μ is H-invariant, we have that $\mu = \mu_H$. Thus the proof will be complete once we prove the following.

CLAIM 3.3. *If H is a subgroup of L such that $W \subset H$ and $G \subset N_L(H)$ then $G \subset H$. In particular, W is non-trivial.*

To prove this claim, let H' be the subgroup generated by all Ad-unipotent one-parameter subgroups of H. Then H' satisfies the hypothesis of the claim, and it is enough to prove that $G \subset H'$. Therefore replacing H by H' without loss of generality we may assume that $\mathrm{Ad}(H)$ is a connected real algebraic group.

Consider the action of T on the Lie algebra $\mathfrak{F}$ of F. Since T is $\mathbb{R}$-split, there is a set $\mathcal{D}$ of $\mathbb{R}$-rational characters on T such that

$$\mathfrak{F} = \oplus_{\chi \in \mathcal{D}} \mathfrak{F}^{\chi},$$

where $\mathfrak{F}^{\chi} = \{v \in \mathfrak{F} : tv = \chi(t)v,\ \forall t \in T\}$. There exists $M > 0$ such that if we define

$$\mathcal{D}^{+} = \{\chi \in \mathcal{D} : \chi(\mathrm{Ad}\, a_i) \to \infty\} \quad \text{and} \quad \mathcal{D}^{0} = \{\chi \in \mathcal{D} : \chi(\mathrm{Ad}\, a_i) \leq M, \forall i \in \mathbb{N}\}, \tag{3.19}$$

then, after passing to a subsequence, $\mathcal{D} = \mathcal{D}^{+} \cup \mathcal{D}^{0}$. Let

$$\mathfrak{W} = \oplus_{\chi \in \mathcal{D}^{+}} \mathfrak{F}^{\chi}. \tag{3.20}$$

From (3.15) and the remark following it, we conclude that $\mathfrak{W}$ is the Lie algebra of W.

Let $H_1 = FH$. Let $\mathfrak{H}$ and $\mathfrak{H}_1$ denote the Lie algebras of H and H_1, respectively. Let $\mathfrak{E}$ be the T-invariant linear complement of $\mathfrak{H}$ in $\mathfrak{H}_1$. Since $W \subset H$, by (3.20)

$$\mathfrak{E} \subset \oplus_{\chi \in \mathcal{D}^{0}} \mathfrak{F}^{\chi}. \tag{3.21}$$

Let $\mathbf{q}_{\mathfrak{E}}$ be a non-zero vector in V_L associated to $\mathfrak{E}$. Then

$$\mathbf{p}_{H_1} = \mathbf{q}_{\mathfrak{E}} \wedge \mathbf{p}_H. \tag{3.22}$$

Let $\chi_0 \in X(T)$ such that

$$t \cdot \mathbf{q}_{\mathfrak{E}} = \chi_0(t)\mathbf{q}_{\mathfrak{E}}, \quad \forall t \in T. \tag{3.23}$$

By (3.21) and (3.19),

$$\chi_0(\mathrm{Ad}\, a_i) \leq M^{\dim \mathfrak{E}}, \quad \forall i \in \mathbb{N}. \tag{3.24}$$

Since $\mathrm{Ad}(G)$ is generated by unipotent elements and $G \subset N_L(H)$, we have $G \in N_L^1(H)$. In particular, $t\mathbf{p}_H = \mathbf{p}_H$ for all $t \in T$. Therefore by (3.22) and (3.23),

$$t \cdot \mathbf{p}_{H_1} = \chi_0(t)\mathbf{p}_{H_1}, \quad \forall t \in T. \tag{3.25}$$

Note that

$$U \cdot \mathbf{p}_{H_1} = \mathbf{p}_{H_1}. \tag{3.26}$$

Now by (2.2), (2.3), (3.13), (3.24), (3.25) and (3.26), we conclude that $\mathbf{p}_{H_1}$ is a TU-fixed vector. Since TU is epimorphic in $\mathrm{Ad}(G)$, $\mathbf{p}_{H_1}$ is $\mathrm{Ad}(G)$-fixed. Therefore G normalizes H_1.

Thus $\mathrm{Ad}(G)$ normalizes $\mathrm{Ad}(H_1)$. Note that $\mathrm{Ad}(H_1) = \mathrm{Ad}(F)\,\mathrm{Ad}(H)$ is a real algebraic group. Let $M = (\mathrm{Ad}(G) \cap \mathrm{Ad}(H_1))^0$. Since $TU \subset M$, M is a connected real algebraic normal epimorphic subgroup of $\mathrm{Ad}(G)$. Hence $M = \mathrm{Ad}(G)$. Thus $\mathrm{Ad}(G) \subset \mathrm{Ad}(H_1)$.

The image of $\mathrm{Ad}(G)$ in $\mathrm{Ad}(H_1)/\mathrm{Ad}(H) \cong \mathrm{Ad}(F)/(\mathrm{Ad}(F) \cap \mathrm{Ad}(H))$ is solvable. Since $G = [G, G]$, we have that $\mathrm{Ad}(G) \subset \mathrm{Ad}(H)$. Therefore $G \subset ZH$, where Z is the centre of L. The image of G in ZH/H is abelian. Therefore $G = [G, G] \subset H$, completing the proof of the claim, and hence the proof of the theorem. □

The same proof as above, with obvious modifications, yields the following uniform version of Theorem 1.4.

THEOREM 3.1. *Let the notation be as in Theorem 1.4. Suppose $x \in L/\Lambda$ is such that Hx is not closed for any proper closed subgroup H of L containing G. Let $g_i \to e$ be a sequence in L. Then there exists an open sub-semigroup $T^{++} \subset T$ with the following property. Given a bounded continuous function f on L/Λ and $\epsilon > 0$, there exists a compact set $S \subset \overline{T^{++}}$ and $j \in \mathbb{N}$ such that for any $a \in F$ with $\mathrm{Ad}\, a \in \overline{T^{++}} \setminus S$*

$$\left| \int_{\tilde{U}} f(aug_i x)\psi(u)\, d\nu(u) - \int_{L/\Lambda} f\, d\mu_L \right| < \epsilon, \quad \forall i \geq j,$$

where μ_L denotes the L-invariant probability measure on L/Λ.

Proof. We start arguing by contradiction, and obtain a sequence $\{a_i\} \subset F$ such that $\{\mathrm{Ad}\, a_i\}$ is divergent in $\overline{T^{++}}$. To adapt the proof of Theorem 1.4, the elements $v_i \in \Lambda \cdot \mathbf{p}_H$ are replaced by elements of the form $g_i\gamma_i \cdot \mathbf{p}_H$, where $\gamma_i \in \Lambda$. Note that since $g_i \to e$, any accumulation point of $\{g_i\gamma_i\mathbf{p}_H\}$ is contained in the discrete set $\{\Lambda \cdot \mathbf{p}_H\}$. In view of this the proof of Theorem 1.4 goes through. □

The following refined uniform version of the above theorems can be obtained by arguing as in the proof of [**DM1**, Theorem 3], and using Theorem 3.1. The result will not be used later in the article, and we shall omit its proof.

THEOREM 3.2. *Let the notation be as in Theorem 1.4. Then there exists an open sub-semigroup $T^{++} \subset T$ with the following property. Let a compact set $K \subset L/\Lambda$, a bounded continuous function f on L/Λ, and an $\epsilon > 0$ be given. Then there exist finitely many closed subgroups $H_1, \ldots, H_r \in \mathcal{H}_\Lambda$, and compact sets $C_j \subset N(H_j, G)$ $(1 \leq j \leq r)$ such that the following holds. For any compact set $K_1 \subset K \setminus \bigcup_{j=1}^r \pi(C_j)$, there exists a compact set $S \subset \overline{T^{++}}$ such that for any $a \in F$ with $\mathrm{Ad}\, a \in \overline{T^{++}} \setminus S$,*

$$\left| \int_{\tilde{U}} f(aux)\psi(u)\, d\nu(u) - \int_{L/\Lambda} f\, d\mu_L \right| < \epsilon, \quad \forall x \in K_1, \tag{3.27}$$

where μ_L denotes the L-invariant probability measure on L/Λ.

Proof of Corollary 1.5. Note that

$$H \neq L,\, g \in N(H, G) \Rightarrow G = gHg^{-1},\; N(H, G) = Gg.$$

Now the corollary follows from Theorem 3.2. □

4. *Closures of orbits of epimorphic subgroups*

First we obtain a consequence of the proof of Theorem 1.4.

COROLLARY 4.1. *Let L, Λ and G be as in Theorem 1.4. Let F be a connected Lie subgroup of L (not necessarily contained in G) such that* $\operatorname{Ad} F$ *is a real algebraic epimorphic subgroup of* $\operatorname{Ad}(G)$. *Then any closed F-invariant subset of L/Λ is G-invariant.*

Proof. It is enough to prove that for all $x \in L/\Lambda$, $Gx \subset \overline{Fx}$. Conjugating, it suffices to show $\pi(G) \subset \overline{\pi(F)}$.

There exists an $\mathbb{R}$-split solvable subgroup TU of $\operatorname{Ad}(F)$, which is epimorphic in $\operatorname{Ad}(G)$ [**BB**]. Therefore, without loss of generality we may assume that $\operatorname{Ad}(F) = TU$. We argue just as in the proof of Theorem 1.4, for any sequence $\{a_i\}$ satisfying the conditions of Theorem 1.4. The only difference here is that we do not assume $F \subset G$. The additional assumption that $F \subset G$ is used only in the paragraph preceding the proof of Claim 3.3 in the proof of Theorem 1.4, and nowhere else.

In the notation of the proof above, without using the condition that $F \subset G$, we obtain that μ is H-invariant and $G \subset H$. In particular, we conclude that $\overline{\pi(F)}$ contains an H-invariant subset, namely $\operatorname{supp}(\mu)$.

Let $Z = (\operatorname{Ad}^{-1}(e))^0$. Since $\operatorname{Ad}(F) \subset \operatorname{Ad}(G) \subset \operatorname{Ad}(H)$, we have that $F \subset ZH$. Now since $H \in \mathcal{H}$, we have that $\operatorname{Ad}(H \cap \Gamma)$ is Zariski dense in $\operatorname{Ad}(H)$ [**S1**, Theorem 2.3]. Therefore $\pi(Z_L(H))$ is closed, where $Z_L(H)$ denotes the centralizer of H in L [**S4**, Lemma 2.3]. Therefore $\overline{\pi(Z)} = \pi(Z_1)$ for a closed connected subgroup $Z_1 \subset Z_L(H)$, and $\pi(Z_1)$ has a finite Z_1-invariant measure. Thus $\pi(Z_1 H)$ has a finite $Z_1 H$-invariant measure, and by [**S4**, Lemma 2.2], $Z_1 H$ and $\pi(Z_1 H)$ are closed. Since $\operatorname{Ad}(Z_1 \cap \Lambda)$ is Zariski dense in $\operatorname{Ad} Z_1$, we have that $\operatorname{Ad}(Z_1 H \cap \Lambda)$ is Zariski dense in $\operatorname{Ad}(Z_1 H)$. Let Z_2 be the center of $Z_1 H$. Then $\pi(Z_2)$ is closed [**S4**, Lemma 2.3]. Since $Z \subset Z_2$, we have that $Z_1 \subset Z_2$. Thus $Z_1 H/H$ is an abelian group.

Thus

$$\operatorname{supp}(\mu) \subset \overline{\pi(F)} \subset \overline{\pi(ZH)} \subset \pi(Z_1 H). \tag{4.28}$$

Since $H\Lambda$ is closed and Λ is countable, $H(Z_1 \cap \Lambda)$ is closed. Put $X = Z_1 H/H(Z_1 \cap \Lambda)$. Then X is a compact abelian group. Let x_0 denote the identity in X. By (4.28) there exists $z \in Z_1 H$ such that $\pi(z) \in \operatorname{supp}(\mu)$. Then $y = zx_0 \in \overline{Fx_0}$. Since X is an abelian group, there exists a sequence $\{f_i\} \subset F$ such that $f_i y \to x_0$ as $i \to \infty$. Therefore there exists a sequence $\{h_i\} \subset H$ such that $\pi(f_i z h_i) \to \pi(e)$ as $i \to \infty$. Now $zh_i = h_i z$, $\pi(z) \in \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu)$ is H-invariant. Therefore $\pi(e) \in \overline{F \operatorname{supp}(\mu)}$. Now since $HF = FH$, and $\operatorname{supp}(\mu)$ is H-invariant, we conclude that

$$\pi(H) \subset \overline{HF \operatorname{supp}(\mu)} = \overline{FH \operatorname{supp}(\mu)} = \overline{F \operatorname{supp}(\mu)} \subset \overline{\pi(F)}.$$

In particular, $\pi(G) \subset \overline{\pi(F)}$. This completes the proof. □

Next we note some more results about algebraic epimorphic subgroups of real algebraic groups.

PROPOSITION 4.2. [**W1**, Theorem 5] *Let G be a connected real algebraic group and F be a connected real algebraic epimorphic subgroup of G. Let G_1 be the (real algebraic) subgroup of G generated by all one-parameter (algebraic) unipotent subgroups of G. Then $FG_1 = G$ and $(F \cap G_1)^0 <_{\text{epi}} G_1$.*

PROPOSITION 4.3. *Let G be a connected real algebraic group and F be a connected real algebraic epimorphic subgroup of G. Let G_0 be the subgroup of G generated by all connected semisimple subgroups of G without compact factors. Then $FG_0 = G$ and $(F \cap G_0)^0 <_{\text{epi}} G_0$.*

Proof. Proposition 4.2 reduces the proposition to the case when G is generated by one-parameter (algebraic) unipotent subgroups.

The projection of F onto G/G_0 is an epimorphic subgroup of G/G_0. Since G/G_0 is a unipotent group, it has no proper epimorphic subgroups. Therefore $FG_0 = G$. Put $F_0 = F \cap G_0$. Since G_0 is connected, to prove that $(F \cap G_0)^0 <_{\text{epi}} G_0$ it suffices to show that $F_0 <_{\text{epi}} G_0$. Since $F/F_0 \cong G/G_0$ is unipotent and connected, we conclude that F/F_0 is unipotent, and hence it has no non-trivial rational characters.

Consider an algebraic linear representation $\sigma_0 : G_0 \to GL(V_0)$. Since G_0 is a normal subgroup of G, by [**BHM**] there exists an algebraic representation $\sigma : G \to GL(V)$ such that σ extends σ_0, that is, V_0 is a $\sigma(G)$-invariant subspace of V and σ_0 is the restriction of σ to (G_0, V_0). Let $W = \{v \in V : \sigma(F_0)v = v\}$.

Since F_0 is normal in F, W is $\sigma(F)$-invariant. Let $\mathbf{v} \in \wedge^{\dim W} W \setminus \{0\}$. Then there exists a rational character χ on F such that $\sigma(f)\mathbf{v} = \chi(f)\mathbf{v}$ for all $f \in F$. Since $\sigma(F_0)\mathbf{v} = \mathbf{v}$, and F/F_0 has no non-trivial rational characters, $\sigma(F)\mathbf{v} = \mathbf{v}$. Since $F <_{\text{epi}} G$, $\sigma(G)\mathbf{v} = \mathbf{v}$. Hence W is $\sigma(G)$-invariant.

Let $\sigma_1 : G \to \mathrm{GL}(W)$ denote the restriction of σ to W. Since $\sigma_1(F_0) = 1$ and F/F_0 is unipotent, we have that $\sigma_1(F)$ is unipotent. Since $\sigma_1(F) <_{\text{epi}} \sigma_1(G)$ and proper unipotent subgroups are never epimorphic, we have that $\sigma_1(F) = \sigma_1(G)$ is a unipotent group. Since G_0 is generated by semisimple subgroups, $\sigma_1(G_0) = 1$. Thus G_0 acts trivially on W. This shows that all F_0-invariant vectors in V are G_0-invariant, proving that $F_0 <_{\text{epi}} G_0$. □

Proof of Theorem 1.1. Without loss of generality, it is enough to prove that $\pi([G, G]) \subset \overline{\pi(F)}$.

Let G_0 be the subgroup of G generated by all connected semisimple subgroups without compact factors. Then by Proposition 4.3 applied to $\mathrm{Ad}(G)$, $\mathrm{Ad}(F)$ and $\mathrm{Ad}(G_0)$, we get that $\mathrm{Ad}(G) = \mathrm{Ad}(F)\,\mathrm{Ad}(G_0) = \mathrm{Ad}(FG_0)$ and $(\mathrm{Ad}(F) \cap \mathrm{Ad}(G_0))^0 <_{\text{epi}} \mathrm{Ad}(G_0)$. We see now that $\mathrm{Ad}(G) = \mathrm{Ad}(F[G, G])$, and since $[G_0, G_0] = G_0$, by Corollary 4.1, $\pi(G_0) \subset \overline{\pi(F)}$. Therefore $\pi(FG_0) \subset \overline{\pi(F)}$. Now G/FG_0 is an abelian group. Hence $[G, G] \subset FG_0$. This completes the proof of the theorem. □

Proof of Corollary 1.2. As in Theorem 1.1, $\mathrm{Ad}(F[G, G]) = \mathrm{Ad}(G)$. Therefore $F[G, G]$ is generated by Ad-unipotent one-parameter subgroups. Therefore by Ratner's theorem the closure of any $F[G, G]$-orbit is a finite-volume homogeneous set. Now the conclusion of the corollary follows from Theorem 1.1. □

Proof of Corollary 1.3. Note that in a connected real algebraic group, there is no proper connected normal real algebraic epimorphic subgroup. Therefore F projects onto

$G/[G, G]$, i.e., $G = F[G, G]$. Now the conclusion of the corollary follows from Theorem 1.1. □

5. *A variant of Theorem 1.4.*

Using Proposition 4.3, we will obtain the following variant of Theorem 1.4 by relaxing the hypotheses that $G = [G, G]$ and F is solvable. The result will be used later in the proof of Theorem 1.9.

THEOREM 5.1. *Let L be a Lie group and Λ a lattice in L. Let G be a subgroup of L generated by one-parameter* Ad*-unipotent subgroups. Let $F \subset G$ be a connected Lie subgroup such that $(G \cap \mathrm{Ad}^{-1}(e))^0 \subset F$, and $\mathrm{Ad}(F)$ is a real algebraic epimorphic subgroup of $\mathrm{Ad}(G)$. Let W be the subgroup generated by all* Ad*-unipotent one-parameter subgroups of F. Then there exists $a \in F$ such that the following holds. Let λ be a W-invariant W-ergodic probability measure on L/Λ, and $x \in \mathrm{supp}(\lambda)$ such that $\overline{Wx} = \mathrm{supp}(\lambda)$ (such an x exists by ergodicity). Then in the space of probability measures on L/Λ,*

$$a^n \lambda \to \mu, \quad \text{as } n \to \infty,$$

where μ is a (unique) G-invariant G-ergodic (and hence homogeneous) probability measure with $\overline{Gx} = \mathrm{supp}(\mu)$.

Later the above result is used only in the case when $\mathrm{Ad}(F)$ is an $\mathbb{R}$-split solvable group. However the theorem easily reduces to this case due to the following.

LEMMA 5.2. *Let the notation be as in Theorem 5.1. Then there exists a connected Lie subgroup F_1 of F such that:*

(i) $\mathrm{Ad}(F_1)$ *is an $\mathbb{R}$-split solvable epimorphic subgroup of* $\mathrm{Ad}(F)$*;*

(ii) *if U is the maximal connected* Ad*-unipotent subgroup of F_1 then (W, U) is a Mautner Pair (that is, for any continuous unitary representation of W, any U-fixed vector is also W-fixed);*

(iii) $(F \cap \mathrm{Ad}^{-1}(e))^0 \subset U$.

Proof. There exists a connected Lie subgroup F_1 of F such that (i) holds [**BB**]. Enlarging F_1 if necessary, we may assume that the radical of W (which is Ad-unipotent and normal in F) is contained in F_1, and hence (iii) holds. Now (ii) follows from Mautner's phenomenon [**Mo**] if we prove the following.

CLAIM 5.1. *There is no proper closed normal subgroup of W containing U.*

By Proposition 4.2, if we put $F_2 = (F_1 \cap W)^0$ then $\mathrm{Ad}(F_2)$ is an epimorphic subgroup of $\mathrm{Ad}(W)$. Suppose V is a proper closed connected normal subgroup of W containing U. Since $\mathrm{Ad}(U)$ contains all unipotent elements of $\mathrm{Ad}(F_2)$, the image of $\mathrm{Ad}(F_2)$ in $\mathrm{Ad}(W)/\mathrm{Ad}(V)$ is an epimorphic subgroup with no algebraic unipotent elements. Since $\mathrm{Ad}(W)/\mathrm{Ad}(V)$ is generated by unipotent one-parameter subgroups, this leads to a contradiction, unless $\mathrm{Ad}(V) = \mathrm{Ad}(W)$. Hence the claim follows in view of (iii). □

Proof of Theorem 5.1. Let F_1 be a subgroup of F subgroup F_1 of F satisfying the conclusion of Lemma 5.2. Let $\tilde{U}$ denote the maximal connected Ad-unipotent subgroup of F_1. Since $(W, \tilde{U})$ is a Mautner pair and λ is a finite W-invariant W-ergodic measure on L/Λ, we conclude that λ is $\tilde{U}$-ergodic. Therefore to prove the theorem, without loss of generality we may assume that $F_1 = F$ and $W = \tilde{U}$.

Write $\mathrm{Ad}(F) = TU$, where $U = \mathrm{Ad}(\tilde{U})$. Let G_0 be the subgroup of G generated by connected semisimple subgroups. Since $\mathrm{Ad}(G)/\mathrm{Ad}(G_0)$ is unipotent, we have that $T \subset \mathrm{Ad}(G_0)$. Now since $(G \cap \mathrm{Ad}^{-1}(e))^0 \subset \tilde{U}$, we have $F \subset \tilde{U} G_0$. By Proposition 4.3, we have that $T(U \cap \mathrm{Ad}(G_0))^0$ is an epimorphic subgroup of $\mathrm{Ad}(G_0)$ and $\mathrm{Ad}(G) = \mathrm{Ad}(F)\,\mathrm{Ad}(G_0)$. Hence $G \subset FG_0 = \tilde{U} G_0$. Note that $G_0 = [G_0, G_0]$.

By conjugation, without loss of generality we may assume that $x = \pi(e)$. By Ratner's theorem, $\overline{\pi(G)} = \pi(H)$ for a closed subgroup H of L containing G such that $H \cap \Lambda$ is a lattice in H. Therefore without loss of generality, we may replace L by H and assume that $\overline{\pi(G)} = L/\Lambda$. Let μ_L denote the unique L-invariant probability measure on L/Λ. Now to prove the theorem it is enough to show that for any subsequence $\{a_i\} \subset \{a^n\}_{n \in \mathbb{N}}$, we have

$$a_i \lambda \to \mu_L \quad \text{as } i \to \infty.$$

We will argue as in the proof of Theorem 1.4, and use the notations introduced there, with $\tilde{U}$ as above and G_0 in place of G there. Let $a \in (F \cap G_0)$ such that $\mathrm{Ad}\, a \in T^{++} \setminus \{e\}$.

Note that at any stage in the proof there is no loss of generality in passing to a subsequence of $\{a_i\}$.

By [**S2**, Corollary 1.2-3], the orbit $\tilde{U}x$ is uniformly distributed with respect to λ in the following sense: for any open set $E \subset L/\Lambda$, and $\epsilon > 0$, there exists $R > 0$ such that for any ball B in $\mathbb{R}^m = \mathrm{Lie}(\tilde{U})$ about zero with radius $\geq R$,

$$\left| \lambda(E) - \frac{1}{|B|} |\{t \in B : \pi(\exp(t)) \in E\}| \right| < \epsilon.$$

Using this remark and the argument as in the proof of Claim 3.1, we deduce that given $\epsilon > 0$, there exists a compact set $K \subset L/\Lambda$ such that $a_i \lambda(K) > 1 - \epsilon$ for all i. Therefore, by passing to a subsequence, we may assume that $a_i \lambda \to \mu$ in the space of probability measures on L/Λ.

Since $\tilde{U}$ is normal in F and λ is $\tilde{U}$-invariant, we have that μ is $\tilde{U}$-invariant. This observation replaces Claim 3.2, and we use $\tilde{U}$ in place of W in the rest of the proof. Again we let $h \in \mathcal{H}_\Lambda$ be such that $\mu(\pi(N(H, \tilde{U}))) > 0$ and $\mu(\pi(S(H, \tilde{U}))) = 0$.

Using the same arguments as in the proof of Theorem 1.4 we get that $G_0 \subset N_L^1(H)$, μ is H-invariant and $\tilde{U} \subset H$. Using Claim 3.3 we conclude that $G_0 \subset H$. Thus $G = G_0 \tilde{U} \subset H$. Hence $\mathrm{supp}(\mu) \subset \overline{\pi(F)} \subset \pi(H)$. Since μ is H-invariant, we have $\mathrm{supp}(\mu) = \pi(H)$, so $\pi(H)$ is a closed orbit containing $\pi(G)$. Thus $H = L$ and $\mu = \mu_L$, completing the proof of the theorem. □

6. *Orbit-closures which are not almost homogeneous*

We will need the following theorem from [**Su**] about the structure of observable subgroups.

THEOREM 6.1. [**Su**] *Let L be a connected real algebraic group which is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-split. Let G be a connected real algebraic subgroup. Then G is observable in L*

if and only if there exists a conjugate G' of G in L and a connected $\mathbb{Q}$-split real algebraic subgroup H defined over $\mathbb{Q}$ with the following properties: $G' \subset H$, H is observable in L, and the unipotent radical of G' is contained in the unipotent radical of H.

Sukhanov's theorem provides more information about the structure of H, and does not state explicitly that H is a $\mathbb{Q}$-split real algebraic subgroup defined over $\mathbb{Q}$. However, Theorem 6.1 can be verified by examining Sukhanov's construction of H.

Proof of Theorem 1.7. (1) $\Rightarrow$ (2) Let $x_0 \in L$ and a lattice Λ of L be given, and let $x = \pi(x_0)$. By Theorem 1.1, $\overline{Fx} = \overline{Gx}$.

Since G has no $\mathbb{R}$-rational characters, $G = KG_0$, where G_0 is the normal subgroup of G generated by one-parameter unipotent subgroups of L contained in G and K is a compact subgroup of G. Since K is compact, $\overline{Gx} = K\overline{G_0x}$, and by Ratner's orbit closure theorem, $\overline{G_0x} = G_1x$ is a finite-volume homogeneous set. Therefore $\overline{Fx} = KG_1x$.

(2) $\Rightarrow$ (1) Let us suppose that G has non-trivial algebraic characters defined over $\mathbb{R}$. We will construct a lattice Λ, and $x \in L/\Lambda$ such that $\overline{Fx} = \overline{Gx}$ is not a finite-volume almost homogeneous set.

Replacing G with a conjugate merely permutes the orbits, so we may conjugate G by elements of L.

For a connected real algebraic group E we will write $E = T_E S_E U_E$, where U_E is the unipotent radical of E, S_E a maximal connected semisimple subgroup, and T_E a connected algebraic torus centralizing S_E. Note that while U_E is determined by E, we are free to choose T_E, S_E as long as T_E centralizes S_E and $T_E S_E$ is a maximal connected reductive real algebraic subgroup. If E is defined over $\mathbb{Q}$ then U_E is defined over $\mathbb{Q}$ and T_E, S_E can be chosen so they are defined over $\mathbb{Q}$, and we will do so without further comment. For a subgroup G of E, we will denote the centralizer of G in E by $Z_E(G)$.

Since L is $\mathbb{R}$-split, it is isomorphic as a real algebraic group to a $\mathbb{Q}$-split real algebraic group defined over $\mathbb{Q}$ [**O**, Proposition 1.4.2]. So let us assume L is $\mathbb{Q}$-split and defined over $\mathbb{Q}$, and define $\Lambda = L_{\mathbb{Z}}$. Since L is semisimple, Λ is a lattice.

By our assumption, G is a real algebraic subgroup of L with $\mathbb{R}$-rational characters, and is observable in L. By Theorem 6.1, after conjugation there exists an observable $\mathbb{Q}$-split subgroup H of L defined over $\mathbb{Q}$, containing G such that $U_G \subset U_H$.

Let T be a maximal $\mathbb{R}$-split torus in T_G. Since H is $\mathbb{Q}$-split, there exists a $\mathbb{Q}$-split torus T_1 in H defined over $\mathbb{Q}$ which is also a maximal $\mathbb{R}$-split torus in H. Therefore there exists $h \in H$ such that $hTh^{-1} \subset T_1$. Hence replacing G by hGh^{-1} we may assume that $T \subset T_1$. In particular, T is a $\mathbb{Q}$-split $\mathbb{Q}$-torus. Let $G_1 = Z_{T_H S_H}(T)U_H$, so G_1 is a subgroup of H defined over $\mathbb{Q}$ containing G. Since $Z_{T_H S_H}(T)$ is reductive, $U_{G_1} = U_H$ and therefore, by Theorem 6.1, G_1 is observable in H and hence in L. Also, $G \subset G_1$. By [**W1**, Proposition 1], this implies that $G_1\pi(e)$ is closed, and contains $\overline{G\pi(e)}$.

Let χ be a non-trivial $\mathbb{R}$-character on G. Then χ restricted to T is a non-trivial $\mathbb{Q}$-character on T. Since T is a $\mathbb{Q}$-split $\mathbb{Q}$-torus contained in the centre of G_1, there exists a $\mathbb{Q}$-character χ_1 on G_1 whose restriction to T is χ. Since χ_1 is a $\mathbb{Q}$-character, $G_1 \cap \Lambda \subset \ker(\chi_1)$, and so the function $\phi(g_1(G_1 \cap \Lambda)) = \chi_1(g_1)$ is well defined on $G_1/(G_1 \cap \Lambda) \cong G_1\pi(e)$ and continuous.

Suppose that $\overline{G\pi(e)}$ is a finite-volume almost homogeneous set. Then $\overline{G\pi(e)} = KG_2\pi(g)$, where $g^{-1}G_2g \cap \Lambda$ is a lattice in $g^{-1}G_2g$ and K is compact. Therefore $\phi(g^{-1}G_2g\pi(e)) \subset \{1, -1\}$, and hence $\phi(KG_2\pi(g))$ is compact. This contradicts the fact that $\phi(G\pi(e)) = \chi_1(G)$ is non-compact. Hence $\overline{G\pi(e)}$ is not a finite-volume almost homogeneous set. As noted before, this completes the proof. □

Question. In the above proof, the orbit-closure we construct may just be a closed orbit of a subgroup admitting an infinite measure invariant under the action of the subgroup; i.e., a 'homogeneous set of infinite volume'. It would be interesting to know whether one can construct, for subgroups F whose observable envelopes have $\mathbb{R}$-rational characters, the orbit-closures with non-integer Hausdorff dimensions.

7. *Finite invariant measures*

In this section we will prove Theorem 1.8.

LEMMA 7.1. *Let L be a connected Lie group. Let G be a subgroup of L generated by* Ad-*unipotent one-parameter subgroups. Let F be a connected subgroup of G such that for any connected Lie subgroup H of L,*

$$F \subset N_L^1(H) \Longrightarrow G \subset N_L(H). \tag{7.29}$$

Then for any closed connected normal subgroup V of G, if V contains all Ad-*unipotent one-parameter subgroups of F then $[G, G] \subset \overline{FV}$. In particular, if $V = \{e\}$ then $[G, G] \subset \overline{F}$.*

Proof. Let U denote the subgroup generated by all Ad-unipotent one-parameter subgroups of F. Let R denote the solvable radical of F. Then $[R, R] \subset U \subset V$. Therefore the radical of FV/V is abelian. Therefore FV/V is unimodular, or in other words, the conjugation by elements of F on FV/V has determinant one on the Lie algebra (note that FV is a connected Lie subgroup of L). Also $F \subset G \subset N_L^1(V)$. Therefore, $F \subset N_L^1(FV)$. Hence, by (7.29), we have that FV is a normal subgroup of G. Therefore FV is generated by Ad-unipotent one-parameter subgroups. Now by condition (7.29) we get that every connected one-dimensional subgroup of $G/\overline{FV}$ is normal. Therefore $G/\overline{FV}$ is abelian, and hence $[G, G] \subset \overline{FV}$. □

Theorem 1.8 will be deduced from the following:

THEOREM 7.2. *Let L be a connected Lie group and Λ be a discrete subgroup of L. Let G be a connected Lie subgroup of L such that* $\mathrm{Ad}(G)$ *is generated by one-parameter unipotent subgroups. Let F be a subgroup of G, and let F_1 be the smallest connected normal* cocompact *real algebraic subgroup of the Zariski closure of* $\mathrm{Ad}(F)$. *Suppose that for any Lie subalgebra $\mathfrak{H}$ of $\mathfrak{L}$,*

$$F_1 \cdot \mathfrak{H} = \mathfrak{H}, \det(F_1|_{\mathfrak{H}}) = 1 \Rightarrow \mathrm{Ad}(G)\mathfrak{H} = \mathfrak{H}. \tag{7.30}$$

Then any F-invariant finite Borel measure on L/Λ is $F[G, G]$-invariant. In particular, any finite F-invariant F-ergodic Borel measure on L/Λ is a homogeneous measure.

Proof. Let μ be a Borel probability measure on L/Λ which is F-invariant. Using ergodic decomposition, it is enough to show that finite ergodic F-invariant Borel measures on L/Λ are $[G, G]$-invariant. Hence we assume that μ is F-ergodic. Also, since μ is a probability measure it is invariant under $\overline{F}$.

Let U be the subgroup of F generated by all Ad-unipotent one-parameter subgroups of F. Equation (7.30) implies (7.29), and therefore Lemma 7.1 applies. Thus if $U = \{e\}$ then $\overline{F} \supset [G, G]$, and the result is trivial.

If U is non-trivial, then by Ratner's description of finite U-ergodic U-invariant measures and Theorem 2.10, there exists a closed connected subgroup $H \in \mathcal{H}_\Lambda$ such that

$$\mu(\pi(N(H, U))) > 0 \quad \text{and} \quad \mu(\pi(S(H, U))) = 0. \tag{7.31}$$

Since $F \subset N_L(U)$, by equations (2.9) and (2.10), $\pi(N^*(H, U))$ is F-invariant. By ergodicity, μ is concentrated on $\pi(N^*(H, U))$. By Lemma 2.9, the map

$$N^*(H, U)/N_\Lambda(H) \to \pi(N^*(H, U))$$

is injective, where $N_\Lambda(H) = N_L(H) \cap \Lambda$. Therefore we can lift μ to an F-invariant measure on $N^*(H, U)/N_\Lambda(H)$, say $\tilde{\mu}$.

Let L_1 be the Zariski closure of $\mathrm{Ad}(L)$ in $\mathrm{GL}(\mathfrak{L})$. Let $\mathfrak{H}$ denote the Lie algebra of H, and

$$N_1 = \{b \in L_1 : b \cdot \mathfrak{H} = \mathfrak{H}, \det(b|_{\mathfrak{H}})^2 = 1\}. \tag{7.32}$$

Then N_1 is a real algebraic subgroup of L_1. Since $\mathrm{Vol}(\pi(H)) < \infty$, if $g \in N_L(H)$ is such that $g\pi(H) = \pi(H)$, then $\det((\mathrm{Ad}\, g)|_{\mathfrak{H}}) = \pm 1$. Therefore

$$\mathrm{Ad}(N_\Lambda(H)) \subset N_1.$$

Let $\bar{\mu}$ be the image of $\tilde{\mu}$ on L_1/N_1 under the map $gN_\Lambda(H) \mapsto \mathrm{Ad}(g)N_1, \forall g \in N^*(H, U)$.

Let

$$T = \{b \in L_1 : b\bar{\mu} = \bar{\mu}\}$$
$$S = \{b \in L_1 : bx = x,\ \forall x \in \mathrm{supp}(\bar{\mu})\}.$$

Then by a theorem due to Dani [**D2**, Corollary 2.6], T is real algebraic, S is a real algebraic normal subgroup of T, and T/S is compact. Note that $\mathrm{Ad}(F) \subset T$. Since T is algebraic, $\mathrm{Zcl}(\mathrm{Ad}(F)) \subset T$. Since T/S is a compact algebraic group, by the definition of F_1, we have

$$F_1 \subset S. \tag{7.33}$$

Since μ is F-ergodic, by (7.31), there exists $g \in N^*(H, U)$ such that $\mathrm{supp}(\mu) = \overline{F\pi(g)}$. Then $\mathrm{Ad}(g)N_1 \in \mathrm{supp}(\bar{\mu})$. Put $H' = gHg^{-1}$ and $\mathfrak{H}' = \mathrm{Ad}(g)\mathfrak{H}$. Then $U \subset H'$. By (7.33), we get $F_1(\mathrm{Ad}(g)N_1) = \mathrm{Ad}(g)N_1$. Therefore by (7.32), $F_1 \cdot \mathfrak{H}' = \mathfrak{H}'$ and $\det(F_1|_{\mathfrak{H}'}) = 1$. Now by (7.30), $G \subset N_L^1(H')$. In particular, $F \subset N_L^1(H')$.

By Theorem 2.11 $\pi(N_L^1(H))$ is closed. Therefore $N_L^1(H')\pi(g)$ is closed. Hence

$$\mathrm{supp}(\mu) = \overline{F\pi(g)} \subset N_L^1(H')\pi(g) = \pi(gN_L^1(H)).$$

Therefore by Theorem 2.10 and (7.31), almost every U-ergodic component of μ is concentrated on $gh\pi(H)$ for some $h \in N_L^1(H)$, and is $H' = ghH(gh)^{-1}$-invariant. Thus μ is H'-invariant.

Note that $(G \cap H')^0$ is a closed normal subgroup of G containing U. Also (7.30) implies (7.29). Therefore by Lemma 7.1, $[G, G] \subset \overline{F(G \cap H')^0} \subset \overline{FH'}$. Since μ is FH'-invariant, we have that μ is $\overline{FH'}$-invariant. Therefore μ is $[G, G]$-invariant. This proves the theorem. □

Proof of Theorem 1.8. Clearly $\mathrm{Zcl}(\mathrm{Ad}(F)) <_{\mathrm{epi}} \mathrm{Ad}(G)$. Let F_1 be the smallest connected normal cocompact real algebraic subgroup of $\mathrm{Zcl}(\mathrm{Ad}(F))$. Then $F_1 <_{\mathrm{epi}} \mathrm{Ad}(G)$ [**W1**, Proposition 6]. Therefore condition (7.30) in the statement of Theorem 7.2 is satisfied, and hence any finite F-invariant Borel measure on L/Λ is $F[G, G]$-invariant.

Since the image of $\mathrm{Ad}(F)$ in $\mathrm{Ad}(G)/\mathrm{Ad}(F[G, G])$ is epimorphic, we deduce that $\mathrm{Ad}(G) = \mathrm{Ad}(F[G, G])$. This completes the proof of the theorem. □

The following example shows that if we relax a condition on $\mathrm{Ad}(G)$ in Theorem 1.8, allowing it to be any real algebraic group (say having non-trivial real characters, or non-trivial algebraic compact factors), then the conclusion of the theorem does not hold in general if $\mathrm{Ad}(F)$ is non-algebraic.

Example 7.1. Let $L = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, let $\Lambda = \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$, let $G = \{(u(t), a(s)) : t, s \in \mathbb{R}\}$ and $F = \{(u(t), a(t)) : t \in \mathbb{R}\}$, where $\{u(t)\}$ is a non-trivial one-parameter unipotent subgroup and $\{a(t)\}$ is a non-trivial one-parameter semisimple (i.e. diagonalizable or compact) subgroup such that $(u(1), a(1)) \in \Lambda$. Then G is algebraic and F is Zariski dense in G, and therefore $F <_{\mathrm{epi}} G$. On the other hand the compact orbit $F\pi(e)$ supports an F-invariant measure which is not G-invariant.

8. *Locally finite invariant measures*

In this section we obtain the proof of Theorem 1.9.

Proof of Theorem 1.9. Note that since μ is locally finite, by the dominated convergence theorem, μ is $\overline{F}$-invariant. Thus without loss of generality we may assume that F is a closed connected subgroup of L.

Due to ergodic decomposition, it is enough to prove the main part of the theorem under the additional assumption that μ is F-ergodic. Without loss of generality we may assume that $\mathrm{Ad}(F)$ is an $\mathbb{R}$-split solvable epimorphic subgroup of $\mathrm{Ad}(G)$ [**BB**].

First we will assume that

$$F \supset (G \cap \mathrm{Ad}^{-1}(e))^0. \tag{8.34}$$

Let $\tilde{U}$ be the maximal connected Ad-unipotent subgroup of F. Because μ is locally finite, by a result due to Dani [**D1**, Theorem 4.3], there exists a measurable $\tilde{U}$-invariant subset X_1 of L/Λ such that $0 < \mu(X_1) < \infty$. Let μ_1 denote the restriction of μ to X_1. Clearly, μ_1 is $\tilde{U}$-invariant. Consider the integral decomposition of μ_1 into $\tilde{U}$-ergodic components, and apply Theorem 5.1 to each of them. Then there exists $a \in F$ and a finite G-invariant measure σ on L/Λ such that $a^n\mu_1 \to \sigma$ as $n \to \infty$.

Since μ is F-invariant, for any measurable set $E \subset L/\Lambda$ and $n \in \mathbb{N}$,

$$a^n\mu_1(E) = \mu_1(a^{-n}E) \leq \mu(a^{-n}E) = a^n\mu(E) = \mu(E);$$

and since $a^n\mu_1 \to \sigma$ and μ is locally finite, we have $\sigma(E) \leq \mu(E)$. Therefore there exists a function $f \in L^1(L/\Lambda, \mu)$ such that $d\sigma = f\,d\mu$ and $f(x) \leq 1$ for μ-almost every $x \in L/\Lambda$. Since σ is G-invariant, and μ is F-ergodic, we have that f is constant almost everywhere. Thus $\sigma = \mu$, and hence μ is finite and G-invariant.

For the general case (i.e. without assuming (8.34)), we will argue by induction on the dimension of L. Since μ is F-ergodic, there exists $x \in \mathrm{supp}(\mu)$ such that $\overline{Fx} = \mathrm{supp}(\mu)$. By conjugation, without loss of generality we may assume that $x = \pi(e)$. Let L' be the smallest closed connected subgroup of L containing G such that $\pi(L')$ is closed and $L' \cap \Lambda$ is a lattice in L'. If the dimension of L' is strictly less than the dimension of L then we are done by the induction hypothesis. Thus $L' = L^0$. Without loss of generality we may assume that $L = L^0 = L'$. Now $\mathrm{Ad}(L \cap \Lambda)$ is Zariski dense in $\mathrm{Ad}(L)$ [**S1**, Section 2]. Then Zy is compact for all $y \in L/\Lambda$ [**S4**, Lemma 2.3], where Z denotes the centre of L. Consider the quotient homomorphism $\psi : L \to L/Z$. Then $\psi(\Lambda)$ is a lattice in $\psi(L)$, and the L-equivariant projection $q : L/\Lambda \to \psi(L)/\psi(\Lambda)$ is a proper map. In particular, the pushforward of μ, denoted by $q_*(\mu)$, on $\psi(L)/\psi(\Lambda)$ is a locally finite $\psi(F)$-ergodic $\psi(F)$-invariant Borel measure.

Suppose $\dim(Z) = 0$. Then (8.34) holds, and the theorem is proved above.

Now we may assume that $\dim(L/Z) < \dim(L)$, in which case by the induction hypothesis, $q_*(\mu)$ is finite. Therefore μ is a finite F-invariant measure. Now we apply Theorem 1.8 to conclude that μ is $F[G, G]$-invariant.

Note that since $\mathrm{Ad}(F[G, G])$ is normal in $\mathrm{Ad}(G)$ and $\mathrm{Ad}(G)$ is generated by unipotent elements, $\mathrm{Ad}(F[G, G])$ is also generated by unipotent elements. Therefore Ratner's theorem is applicable for $\mathrm{Ad}(F[G, G])$.

The rest of the conclusions follow from [**M2**, Theorem 15]. □

The following example shows that Theorem 1.9 is not valid in general without the assumption that $\mathrm{Ad}(F)$ is real algebraic.

Example 8.1. Let $L = G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, $\Lambda = \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$. Let U be the two-dimensional upper triangular unipotent subgroup of G. Let

$$T = \left\{ \left(\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}, \begin{bmatrix} a^{-\alpha} & \\ & a^{\alpha} \end{bmatrix} \right) : a > 0 \right\},$$

where $\alpha > 0$ an irrational number. Let $A = \left\{ \left[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right] : a > 0 \right\}$. Then T is Zariski dense in $D = A \times A$. Since DU is epimorphic in G, we have that $F = TU$ is epimorphic in G. Observe that $\pi(F)$ is closed, and there exists an *infinite* locally finite F-invariant measure on $\pi(F)$.

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