EQUIDISTRIBUTION AND COUNTING FOR ORBITS OF GEOMETRICALLY FINITE HYPERBOLIC GROUPS

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Abstract. Let G be the identity component of $\mathrm{SO}(n,1)$, $n \geq 2$, acting linearly on a finite dimensional real vector space V. Consider a vector $w_0 \in V$ such that the stabilizer of w_0 is a symmetric subgroup of G or the stabilizer of the line $\mathbb{R}w_0$ is a parabolic subgroup of G. For any non-elementary discrete subgroup Γ of G with its orbit $w_0\Gamma$ discrete, we compute an asymptotic formula (as $T \to \infty$) for the number of points in $w_0\Gamma$ of norm at most T, provided that the Bowen-Margulis-Sullivan measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ and the Γ-skinning size of w_0 are finite.

The main ergodic ingredient in our approach is the description for the limiting distribution of the orthogonal translates of a totally geodesically immersed closed submanifold of $\Gamma\backslash\mathbb{H}^n$. We also give a criterion on the finiteness of the Γ-skinning size of w_0 for Γ geometrically finite.

Contents

2010 *Mathematics Subject Classification.* Primary 11N45, 37F35, 22E40; Secondary 37A17, 20F67.

Key words and phrases. Geometrically finite hyperbolic groups, mixing of geodesic flow, totally geodesic submanifolds, Patterson-Sullivan measure.

H. Oh was supported in part by NSF Grant #0629322 and #1068094.

N. Shah was supported in part by NSF Grant #1001654.

1. Introduction

1.1. **Motivation and Overview.** Let G denote the identity component of the special orthogonal group $\mathrm{SO}(n,1)$, $n \geq 2$, and V a finite dimensional real vector space on which G acts linearly from the right.

A discrete subgroup of a locally compact group with finite co-volume is called a lattice. For $v \in V$ and a subgroup H of G, let $H_v = \{h \in H : vh = v\}$ denote the stabilizer of v in H.

A subgroup H of G is called *symmetric* if there exists a nontrivial involutive automorphism σ of G such that the identity component of H is same as the identity component of $G^\sigma = \{g \in G : \sigma(g) = g\}$.

Theorem 1.1 (Duke-Rudnick-Sarnak [9])**.** *Fix $w_0 \in V$ such that G_{w_0} is symmetric. Let Γ be a lattice in G such that Γ_{w_0} is a lattice in G_{w_0}. Then for any norm $\|\cdot\|$ on V,*

$$\lim_{T\to\infty} \frac{\#\{w \in w_0\Gamma : \|w\| < T\}}{\mathrm{vol}(B_T)} = \frac{\mathrm{vol}(\Gamma_{w_0}\backslash G_{w_0})}{\mathrm{vol}(\Gamma\backslash G)}$$

where $B_T := \{w \in w_0 G : \|w\| < T\}$ and the volumes on G_{w_0}, G and $w_0 G \simeq G_{w_0}\backslash G$ are computed with respect to the right invariant measures chosen compatibly.

Eskin and McMullen [10] gave a simpler proof of Theorem 1.1 based on the mixing property of the geodesic flow of a hyperbolic manifold with finite volume. It may be noted that this approach for counting via mixing was used earlier by Margulis in his 1970 thesis [20]. We also refer to [3] for a quantitative version of Theorem 1.1.

The group G can be considered as the group of orientation preserving isometries of the n-dimensional hyperbolic space $\mathbb{H}^n$. The main achievement of this paper lies in extending Theorem 1.1 to a suitable class of discrete subgroups Γ of infinite covolume in G; namely, the groups Γ with finite Bowen-Margulis-Sullivan measure m^{BMS} on $\Gamma\backslash\mathbb{H}^n$. In particular, this class contains all geometrically finite subgroups of G. The analogue of $\mathrm{vol}(\Gamma_{w_0}\backslash G_{w_0})$ turns out to be a very interesting quantity, which we will call the 'skinning size' of w_0 relative to Γ and denote by $\mathrm{sk}_\Gamma(w_0)$. In fact, $\mathrm{sk}_\Gamma(w_0)$ will be the total mass of a Patterson-Sullivan type measure on the unit normal bundle of a closed immersed submanifold of $\Gamma\backslash\mathbb{H}^n$ associated to G_{w_0}. One of the important components of this work is to completely determine when $\mathrm{sk}_\Gamma(w_0)$ is finite (Theorem 1.5). In particular, $\mathrm{sk}_\Gamma(w_0) < \infty$ for any geometrically finite Γ whose critical exponent δ is greater than the codimension of the associated submanifold.

The main ergodic theoretic ingredient in the proof is the description for the limiting distribution of the evolution of the smooth measure on the unit normal bundle of a closed totally geodesically immersed submanifold of $\Gamma\backslash\mathbb{H}^n$ under the geodesic flow. The corresponding equidistribution statement (Theorem 1.8) is applicable to many other problems; for example, in [23, 24], it has been applied to the study of the asymptotic distributions in circle packings in the Euclidean plane or a sphere, invariant under a non-elementary group of Mobius transformations.

1.2. **Statement of main result.** Our generalization of Theorem 1.1 for discrete subgroups which are not necessarily lattices involves terms which can be best explained in the language of the hyperbolic geometry. Let $\Gamma < G$ be a torsion-free discrete subgroup which is non-elementary, that is, Γ has no abelian subgroup of finite index. This is a standing assumption on Γ throughout the whole paper. Now Γ acts properly discontinuously on $\mathbb{H}^n$. Let $0 < \delta \leq n-1$ be the critical exponent of Γ (see §3.1.1). Let $\{\nu_x\}_{x\in\mathbb{H}^n}$ be a Γ-invariant conformal density of dimension δ on the geometric boundary $\partial\mathbb{H}^n$ (see (2.11)) which exists by Patterson [26] and Sullivan [36]. Let m^{BMS} denote the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ associated to $\{\nu_x\}$ (see (3.2)).

For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we denote by $u^{\pm} \in \partial\mathbb{H}^n$ the forward and the backward endpoints of the geodesic determined by u respectively and by $\pi(u) \in \mathbb{H}^n$ the base point of u. Let $\mathbf{p} : \mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ be the canonical quotient map.

Let V be a finite dimensional vector space on which G acts linearly. Let $w_0 \in V$ be such that G_{w_0} is a symmetric subgroup or the stabilizer $G_{\mathbb{R}w_0}$ of the line $\mathbb{R}w_0$ is a parabolic subgroup. We define a subset $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ associated to the orbit $w_0\Gamma$ in each case.

When G_{w_0} is a symmetric subgroup associated to an involution σ, choose a Cartan involution θ of G which commutes with σ, and let $o \in \mathbb{H}^n$ be such that its stabilizer G_o is the fixed group of θ. Then $\tilde{S} := G_{w_0}.o$ is an isometric imbedding of $\mathbb{H}^k$ in $\mathbb{H}^n$ for some $0 \leq k \leq n-1$, where the embeddings of $\mathbb{H}^0$ and $\mathbb{H}^1$ mean a point and a complete geodesic respectively. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ be the unit normal bundle of $\tilde{S}$.

In the case when $G_{\mathbb{R}w_0}$ is parabolic, we fix any $o \in \mathbb{H}^n$. If N is the unipotent radical of $G_{\mathbb{R}w_0}$, then $\tilde{S} := N.o$ is a horosphere. We set $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ to be the unstable horosphere over $\tilde{S}$.

Now in either case, we define the following Borel measure on $\tilde{E}$:

$$d\mu^{\mathrm{PS}}_{\tilde{E}}(v) := e^{\delta\beta_{v^+}(x,\pi(v))}\, d\nu_x(v^+)$$

for $x \in \mathbb{H}^n$ and $\beta_\xi(x_1, x_2)$ denotes the value of the Busemann function, that is, the signed distance between the horospheres based at ξ, one passing through x_1 and the other through x_2 (see (2.2)). This definition of $\mu^{\mathrm{PS}}_{\tilde{E}}$ is independent of the choice of $x \in \mathbb{H}^n$. Due to the Γ-invariance property of the conformal density $\{\nu_x\}$, it induces a measure on $E := \mathbf{p}(\tilde{E})$ which we denote by μ^{PS}_E.

Fix any $X_0 \in \tilde{E}$ based at o, and let $A = \{a_r : r \in \mathbb{R}\}$ be a one-parameter subgroup of G consisting of $\mathbb{R}$-diagonalizable elements such that $r \mapsto a_r.X_0$ is a unit speed geodesic. Note that A is contained in a copy of $\mathrm{SO}(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$ such that a_r corresponds to $d_r = \mathrm{diag}(e^{r/2}, e^{-r/2})$. Any irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ is given by the standard action of $\mathrm{SL}(2,\mathbb{R})$ on homogeneous polynomials of degree k in two variables such that the action of $-I$ is trivial, so k is even and the largest eigenvalue of d_r is $e^{(k/2)r}$. Therefore if λ denotes the log of the largest eigenvalue of a_1 on $\mathbb{R}\text{-}\mathrm{span}(w_0G)$, then $\lambda \in \mathbb{N}$. We set

$$w_0^\lambda := \lim_{r\to\infty} e^{-\lambda r} w_0 a_r \neq 0, \quad \text{by [13, Lemma 4.2]}.$$

Theorem 1.2. *Let $\Gamma < G$ be a non-elementary discrete subgroup with $|m^{\mathrm{BMS}}| < \infty$. Suppose that $w_0\Gamma$ is discrete and that its skinning size $\mathrm{sk}_\Gamma(w_0) := |\mu^{\mathrm{PS}}_E|$ is finite. Then for any G_o-invariant norm $\|\cdot\|$ on V, we have*

$$\lim_{T\to\infty} \frac{\#\{w \in w_0\Gamma : \|w\| < T\}}{T^{\delta/\lambda}} = \frac{|\nu_o| \cdot \mathrm{sk}_\Gamma(w_0)}{\delta \cdot |m^{\mathrm{BMS}}| \cdot \|w_0^\lambda\|^{\delta/\lambda}}\ . \tag{1.1}$$

Remark 1.3. (1) If Γ is convex cocompact, $\mathrm{sk}_\Gamma(w_0) < \infty$. In the case when $G_{\mathbb{R}w_0}$ is parabolic, $\mathrm{sk}_\Gamma(w_0) < \infty$ as well. A finiteness criterion for $\mathrm{sk}_\Gamma(w_0)$ is provided in the section 1.4.

(2) Since $w_0\Gamma$ is infinite, $\mathrm{sk}_\Gamma(w_0) > 0$ (Proposition 6.7), and hence the limit (1.1) is strictly positive.

(3) The description of the limit changes if we do not assume the G_o-invariance of the norm $\|\cdot\|$; see Theorem 7.8, Remark 7.9(3-5), and Theorem 7.10.

(4) If G_{w_0} is symmetric and Γ is Zariski dense in G, then the condition $|\mu^{\mathrm{PS}}_E| < \infty$ implies that $w_0\Gamma$ is discrete, for by Theorem 2.21 and Remark 2.22, $w_0\Gamma$ is closed in w_0G, and by [13, Lemma 4.2], w_0G is closed in V. Therefore $w_0\Gamma$ is closed and hence discrete in V.

(5) If $G_{\mathbb{R}w_0}$ is parabolic, then the condition $|\mu_E^{\rm PS}| < \infty$ implies that $w_0\Gamma$ is discrete. To see this, note that if the horosphere $\tilde{S}$ is based at ξ, then $\partial\tilde{S} = \{\xi\}$ and by Theorem 2.21, $\Gamma\tilde{S}$ is closed in $\mathbb{H}^n$ and $w_0\Gamma$ is closed in $w_0G = \overline{w_0G} \smallsetminus \{0\}$. If $w_0\Gamma$ were not closed in V, $w_0\gamma_i \to 0$ for a sequence $\{\gamma_i\} \subset \Gamma$. Then $\gamma_i o \to \xi$ and ξ is a horospherical limit point of Γ. Since $|m^{\rm BMS}|$ is finite, the geodesic flow is mixing (Theorem 3.2) and hence by [7, Thm.A & Prop.B], $\Gamma\tilde{S}$ is dense in $\pi(\{u : u^- \in \Lambda(\Gamma)\})$, a contradiction to $\Gamma\tilde{S}$ being closed. Therefore $w_0\Gamma$ is closed and hence discrete in V.

Thanks are due to the referee for the last two remarks.

A discrete group Γ is called *geometrically finite*, if the unit neighborhood of its convex core[1] has finite Riemannian volume (see also Theorem 4.6). Any discrete group admitting a finite sided polyhedron as a fundamental domain in $\mathbb{H}^n$ is geometrically finite.

Sullivan [36] showed that $|m^{\rm BMS}| < \infty$ for all geometrically finite Γ. However Theorem 1.2 is not limited to those, as Peigné [27] constructed a large class of geometrically infinite groups admitting a finite Bowen-Margulis-Sullivan measure.

We will provide a general criterion on the finiteness of $\mathrm{sk}_\Gamma(w_0)$ in Theorem 1.14. For the sake of concreteness, we first describe the results for the standard representation of G.

1.3. **Standard representation of G.** Let Q be a real quadratic form of signature $(n, 1)$ for $n \geq 2$ and G the identity component of the special orthogonal group $\mathrm{SO}(Q)$. Then G acts on $\mathbb{R}^{n+1}$ by the matrix multiplication from the right, i.e., the standard representation. For any non-zero $w_0 \in \mathbb{R}^{n+1}$, up to conjugation and commensurability, G_{w_0} is $\mathrm{SO}(n-1, 1)$ (resp. $\mathrm{SO}(n)$) if $Q(w_0) > 0$ (resp. if $Q(w_0) < 0$). If $Q(w_0) = 0$, the stabilizer of the line $\mathbb{R}w_0$ is a parabolic subgroup. Therefore Theorem 1.2 is applicable for any non-zero $w_0 \in \mathbb{R}^{n+1}$, provided $\mathrm{sk}_\Gamma(w_0) < \infty$ (in this case, $\lambda = 1$).

An element $\gamma \in \Gamma$ is called *parabolic* if there exists a unique fixed point of γ in $\partial\mathbb{H}^n$. For $\xi \in \partial\mathbb{H}^n$, we denote by Γ_ξ the stabilizer of ξ in Γ and call ξ a *parabolic fixed point* of Γ if ξ is fixed by a parabolic element of Γ.

Noting that G_{w_0} is the isometry group of the codimension one totally geodesic subspace, say, $\tilde{S}_{w_0}$, when $Q(w_0) > 0$, we give the following:

Definition 1.4. Let $w_0\Gamma$ be discrete. Then $w_0 \in \mathbb{R}^{n+1}$ is said to be *externally* Γ-*parabolic* if $Q(w_0) > 0$ and there exists a parabolic fixed

[1]The *convex core* $C_\Gamma \subset \Gamma\backslash\mathbb{H}^n$ of Γ is the image of the minimal convex subset of $\mathbb{H}^n$ which contains all geodesics connecting any two points in the limit set of Γ.

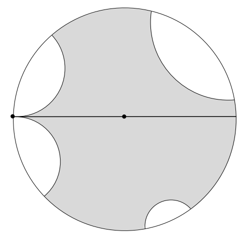

FIGURE 1. An externally Γ-parabolic vector

point $\xi \in \partial\tilde{S}_{w_0}$ for Γ such that $G_{w_0} \cap \Gamma_\xi$ is trivial, where $\partial\tilde{S}_{w_0} \subset \partial\mathbb{H}^n$ denotes the boundary of $\tilde{S}_{w_0}$ in $\overline{\mathbb{H}^n}$.

For $n = 2$, $w_0 \in \mathbb{R}^3$ with $Q(w_0) > 0$ is externally Γ-parabolic if and only if the projection of the geodesic $\tilde{S}_{w_0}$ in $\Gamma\backslash\mathbb{H}^n$ is divergent in both directions, and at least one end of $\tilde{S}_{w_0}$ goes into a cusp of a fundamental domain of Γ in $\mathbb{H}^2$ (see Fig. 1).

Theorem 1.5 (On the finiteness of $\mathrm{sk}_\Gamma(w_0)$)**.** *Let Γ be geometrically finite and $w_0\Gamma$ discrete.*

(1) *If $\delta > 1$, then $\mathrm{sk}_\Gamma(w_0) < \infty$.*
(2) *If $\delta \le 1$, then $\mathrm{sk}_\Gamma(w_0) = \infty$ if and only if w_0 is externally Γ-parabolic.*

Corollary 1.6. *Let Γ be geometrically finite and $w_0\Gamma$ discrete. If either $\delta > 1$ or w_0 is not externally Γ-parabolic, then* (1.1) *holds.*

Remark 1.7. (1) For geometrically finite Γ, if the Riemannian volume of E is finite, then $\mathrm{sk}_\Gamma(w_0) < \infty$ (Corollary 1.15).

(2) It can be proved that if $\delta \le 1$ and w_0 is externally Γ-parabolic, the asymptotic count is of the order $T \log T$ if $\delta = 1$ and of the order T if $\delta < 1$, instead of T^δ (cf. [25]).

(3) When $Q(w_0) < 0$, the orbital counting with respect to the hyperbolic metric balls was obtained by Lax and Phillips [19] for Γ geometrically finite with $\delta > (n-1)/2$, by Lalley [18] for convex cocompact subgroups and by Roblin [31] for all groups with finite Bowen-Margulis-Sullivan measure.

(4) When $Q(w_0) = 0$ and Γ is geometrically finite with $\delta > (n-1)/2$, a version of Theorem 1.2 was obtained in [17].

1.4. **Equidistribution of expanding submanifolds.** In this section, we will describe the main ergodic theoretic ingredients used in the proof of Theorem 1.2. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ be one of the following:

(1) an unstable horosphere over a horosphere $\tilde{S}$ in $\mathbb{H}^n$;

(2) the unit normal bundle of a complete proper connected totally geodesic subspace $\tilde{S}$ of $\mathbb{H}^n$; that is, $\tilde{S}$ is an isometric imbedding of $\mathbb{H}^k$ in $\mathbb{H}^n$ for some $0 \le k \le n-1$.

Let Γ be a discrete subgroup of G, and set $E := \mathbf{p}(\tilde{E})$ for the projection $\mathbf{p} : \mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

Recall that $\{\nu_x : x \in \mathbb{H}^n\}$ denotes a Patterson-Sullivan density of dimension δ. Let $\{m_x : x \in \mathbb{H}^n\}$ denote a G-invariant conformal density of dimension $(n-1)$. We consider the following locally finite Borel measures on $\tilde{E}$:

$$d\mu^{\mathrm{Leb}}_{\tilde{E}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))} dm_o(v^+), \quad d\mu^{\mathrm{PS}}_{\tilde{E}}(v) = e^{\delta\beta_{v^+}(o,\pi(v))} d\nu_o(v^+),$$

where $o \in \mathbb{H}^n$. Note that $\mu^{\mathrm{Leb}}_{\tilde{E}}$ is the measure associated to the Riemannian volume form on $\tilde{E}$.

The measures $\mu^{\mathrm{PS}}_{\tilde{E}}$ and $\mu^{\mathrm{Leb}}_{\tilde{E}}$ are invariant under $\Gamma_{\tilde{E}} = \{\gamma \in \Gamma : \gamma(\tilde{E}) = \tilde{E}\}$ and hence induce measures on $\Gamma_{\tilde{E}}\backslash\tilde{E}$. We denote by μ^{Leb}_{E} and μ^{PS}_{E} respectively the projections of these measures on E via the projection map $\Gamma_{\tilde{E}}\backslash\tilde{E} \to E$ induced by $\mathbf{p}$.

Let m^{BR} denote the Burger-Roblin measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ associated to the conformal densities $\{\nu_x\}$ in the backward direction and $\{m_x\}$ in the forward direction ([6], [31], see (3.3)).

Let $\{\mathcal{G}^t\}$ denote the geodesic flow on $\mathrm{T}^1(\mathbb{H}^n)$.

Theorem 1.8. *Suppose that $|m^{\mathrm{BMS}}| < \infty$ and $|\mu^{\mathrm{PS}}_{E}| < \infty$. Let $F \subset E$ be a Borel subset with $\mu^{\mathrm{PS}}_{E}(\partial F) = 0$. For any $\psi \in C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$,*

$$\lim_{t\to+\infty} e^{(n-1-\delta)t} \cdot \int_F \psi(\mathcal{G}^t(v))\, d\mu^{\mathrm{Leb}}_{E}(v) = \frac{\mu^{\mathrm{PS}}_{E}(F)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BR}}(\psi). \tag{1.2}$$

In particular, this holds for $F = E$.

See Theorem 3.6 for a version of Theorem 1.8 without the finiteness assumption on $|\mu^{\mathrm{PS}}_{E}|$.

Remark 1.9. Theorem 1.8 applies to F with $\mu^{\mathrm{Leb}}_{E}(F) = \infty$ as well, provided $|\mu^{\mathrm{PS}}_{E}| < \infty$. The proof for this generality requires greater care since it cannot be deduced from the cases of F bounded. It is precisely this general nature of our equidistribution theorem which enabled us to state Theorem 1.2 for general groups Γ only assuming the finiteness of the skinning size $\mathrm{sk}_\Gamma(w_0) = |\mu^{\mathrm{PS}}_{E}|$ for a suitable E.

When E is a horosphere and F is bounded, Theorem 1.8 was obtained earlier by Roblin [31, P.52]. We were motivated to formulate and prove the result from an independent view point; our attention is especially on the case of $\pi(E)$ being a totally geodesic immersion. This case involves many new features, observations, and applications (cf. [23], [24]).

The main key to our proof is the transversality theorem 3.5, which was influenced by the work of Schapira [34]. The transversality theorem provides a precise relation between the transversal intersections of geodesic evolution of F with a given piece, say T, of a weak stable leaf and the transversal measure corresponding to the m^{BMS} measure on T.

For Γ Zariski dense, we generalize Theorem 1.8 to $\psi \in C_c(\Gamma\backslash G)$. To state the generalization, we fix $o \in \mathbb{H}^n$ and $X_0 \in \tilde{E}$ based at o. Then, for $K = G_o$ and $M = G_{X_0}$, we may identify $\mathbb{H}^n$ and $\mathrm{T}^1(\mathbb{H}^n)$ with G/K and G/M respectively. Let $A = \{a_r\}$ be the one-parameter subgroup such that the right translation action by a_r on G/M corresponds to $\mathcal{G}^r$. Let $\bar{m}^{\mathrm{BR}}$ denote the measure on $\Gamma\backslash G$ which is the M-invariant extension of m^{BR} via the natural projection map $\Gamma\backslash G \to \Gamma\backslash G/M = \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. Let $H = G_{\tilde{E}}$, and let dh denote the invariant measure on $\Gamma_H\backslash H$ whose projection to E coincides with μ_E^{Leb}.

Theorem 1.10. *Let Γ be a Zariski dense discrete subgroup of G such that $|m^{\mathrm{BMS}}| < \infty$ and $|\mu_E^{\mathrm{PS}}| < \infty$. Then for any $\psi \in C_c(\Gamma\backslash G)$,*

$$\lim_{r\to\infty} e^{(n-1-\delta)r} \int_{h\in\Gamma_H\backslash H} \psi(\Gamma h a_r)\, dh = \frac{|\mu_E^{\mathrm{PS}}|}{|m^{\mathrm{BMS}}|} \bar{m}^{\mathrm{BR}}(\psi).$$

When Γ is a lattice in G and E is of finite Riemannian volume, Theorem 1.10 is due to Sarnak [33] for horocycles in $\mathbb{H}^2$, Randol [28] for unit normal vectors based at a point in the cocompact lattice case in $\mathbb{H}^2$, Duke-Rudnick-Sarnak [9] and Eskin-McMullen [10] in general (also see [15, Appendix]).

In Section 7, we deduce Theorem 1.2 from Theorem 1.8. The standard techniques of orbital counting via equidistribution results require significant modifications due to the fact that m^{BR} is not G-invariant.

1.5. **On finiteness of μ_E^{PS} for geometrically finite Γ.** An important condition for the application of Theorem 1.8 is to determine when μ_E^{PS} is finite. In this subsection we assume that Γ is geometrically finite. Letting $\tilde{E}$ and $\tilde{S} = \pi(\tilde{E})$ be as in section 1.4, *suppose further that the natural imbedding $\Gamma_{\tilde{S}}\backslash\tilde{S} \to \Gamma\backslash\mathbb{H}^n$ is proper*; in particular, $\mathbf{p}(\tilde{S})$ is closed in $\Gamma\backslash\mathbb{H}^n$, where $\Gamma_{\tilde{S}} = \{\gamma \in \Gamma : \gamma\tilde{S} = \tilde{S}\}$.

When $\tilde{S}$ is a point or a horosphere, μ_E^{PS} is compactly supported (Theorem 4.9).

Theorem 1.11 (Theorem 4.7). *If $\tilde{S}$ is totally geodesic, then $\Gamma_{\tilde{S}}$ is geometrically finite.*

Definition 1.12 (Parabolic-Corank)**.** Let $\Lambda_{\rm p}(\Gamma)$ denote the set of parabolic fixed points of Γ in $\partial\mathbb{H}^n$. For any $\xi \in \Lambda_{\rm p}(\Gamma)$, Γ_ξ is a virtually free abelian group of rank at least one. Define

$$\text{pb-corank}(\Gamma_{\tilde{S}}) = \max_{\xi\in\Lambda_{\rm p}(\Gamma)\cap\partial(\tilde{S})} \left(\text{rank}(\Gamma_\xi) - \text{rank}(\Gamma_\xi \cap \Gamma_{\tilde{S}})\right).$$

If $\Lambda_{\rm p}(\Gamma)\cap\partial(\tilde{S}) = \emptyset$, we set pb-corank$(\Gamma_{\tilde{S}}) = 0$. In particular, the parabolic co-rank of $\Gamma_{\tilde{S}}$ is always zero when Γ is convex cocompact.

Lemma 1.13 (Lemma 6.2)**.** *If $\tilde{S}$ is totally geodesic, then*

$$\text{pb-corank}(\Gamma_{\tilde{S}}) \le \text{codim}(\tilde{S}).$$

Theorem 1.14 (Theorems 6.3 and 6.4)**.** *We have:*

(1) $\text{supp}(\mu_E^{\rm PS})$ *is compact if and only if* pb-corank$(\Gamma_{\tilde{S}}) = 0$.
(2) $|\mu_E^{\rm PS}| < \infty$ *if and only if* pb-corank$(\Gamma_{\tilde{S}}) < \delta$.

Note that by [8, Prop. 2], $\delta > \frac{1}{2}\max_{\xi\in\Lambda_{\rm p}(\Gamma)} \text{rank}(\Gamma_\xi)$. As a consequence of Theorem 1.14, we get:

Corollary 1.15 (Corollary 6.5)**.** *Suppose that* $\dim(\tilde{S}) \ge (n+1)/2$. *If* $|\mu_E^{\rm Leb}| < \infty$*, then* $|\mu_E^{\rm PS}| < \infty$.

1.6. **Finiteness of $\mu_E^{\rm PS}$ or $\mu_E^{\rm Leb}$ and closedness of E.** Let $\tilde{E}$ and E be as in the subsection 1.4. In [29], it is shown that $|\mu_E^{\rm Leb}| < \infty$ implies that E is a closed subset of $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. We prove an analogous statement for $\mu_E^{\rm PS}$.

Theorem 1.16 (Theorem 2.21)**.** *Let Γ be a discrete Zariski dense subgroup of G. If $|\mu_E^{\rm PS}| < \infty$, then the natural embedding $\Gamma_{\tilde{S}}\backslash\tilde{S} \to \Gamma\backslash\mathbb{H}^n$ is proper.*

1.7. **Integrability of ϕ_0 and a characterization of a lattice.** Define $\phi_0 \in C(\Gamma\backslash\mathbb{H}^n)$ by

$$\phi_0(x) := |\nu_x| \quad \text{for } x \in \Gamma\backslash\mathbb{H}^n.$$

The function ϕ_0 is an eigenfunction of the hyperbolic Laplace operator with eigenvalue $-\delta(n-1-\delta)$ [36]. Sullivan [37] showed that if $\delta > \frac{n-1}{2}$, then $\phi_0 \in L^2(\Gamma\backslash\mathbb{H}^n, d\,\text{Vol}_{\text{Riem}})$ if and only if $|m^{\rm BMS}| < \infty$. The following theorem, which is a new application of Ratner's theorem [30], relates the integrability of ϕ_0 with the finiteness of $\text{Vol}_{\text{Riem}}(\Gamma\backslash\mathbb{H}^n)$:

Theorem 1.17 (§3.6)**.** *For any discrete subgroup Γ, the following statements are equivalent:*

(1) $\phi_0 \in L^1(\Gamma\backslash\mathbb{H}^n, d\,\text{Vol}_{\text{Riem}})$;
(2) $|m^{\rm BR}| < \infty$;
(3) Γ *is a lattice in* G.

Although m^{BR} depends on the choice of the base point o, its finiteness is independent of the choice. If Γ is a lattice, then $\delta = n-1$ and hence ϕ_0 is a constant function by the uniqueness of the harmonic function [38].

Acknowledgements. We thank Thomas Roblin for useful comments on an earlier version of this paper. We thank the referee for carefully reading the paper and asking many pertinent questions, which led us to proving more general results and improving the overall presentation of the paper.

2. Transverse measures

2.1. Let $(\mathbb{H}^n, d)$ denote the hyperbolic n-space and $\partial\mathbb{H}^n$ its geometric boundary. Let G denote the identity component of the isometry group of $\mathbb{H}^n$. We denote by $\mathrm{T}^1(\mathbb{H}^n)$ the unit tangent bundle of $\mathbb{H}^n$ and by π the natural projection from $\mathrm{T}^1(\mathbb{H}^n) \to \mathbb{H}^n$. By abuse of notation, we use d to denote a left G-invariant metric on $\mathrm{T}^1(\mathbb{H}^n)$ such that $d(\pi(u), \pi(v)) = \min\{d(u_1, v_1) : \pi(u_1) = \pi(u), \pi(v_1) = \pi(v)\}$. For a subset A of $\mathrm{T}^1(\mathbb{H}^n)$ or $\mathbb{H}^n$ or $\partial\mathbb{H}^n$ and a subgroup H of G, we denote by H_A the stabilizer subgroup $\{g \in H : g(A) = A\}$ of A in H.

Denote by $\{\mathcal{G}^r : r \in \mathbb{R}\}$ the geodesic flow on $\mathrm{T}^1(\mathbb{H})$. For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we set

$$u^+ := \lim_{r\to\infty} \mathcal{G}^r(u) \quad \text{and} \quad u^- := \lim_{r\to-\infty} \mathcal{G}^r(u), \tag{2.1}$$

which are the endpoints in $\partial\mathbb{H}^n$ of the geodesic defined by u. Note that $(g(u))^\pm = g(u^\pm)$ for $g \in G$. The map $\mathrm{Viz} : \mathrm{T}^1(\mathbb{H}^n) \to \partial\mathbb{H}^n$ given by $\mathrm{Viz}(u) = u^+$ is called the *visual* map.

2.2. The *Busemann function* $\beta : \partial\mathbb{H}^n \times \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ is defined as follows: for $\xi \in \partial\mathbb{H}^n$ and $x, y \in \mathbb{H}^n$,

$$\beta_\xi(x, y) = \lim_{r\to\infty} d(x, \xi_r) - d(y, \xi_r) \tag{2.2}$$

where ξ_r is any geodesic ray tending to ξ as $r \to \infty$; and the limiting value is independent of the choice of the ray ξ_r.

Note that β is differentiable and invariant under isometries; that is, for $g \in G$ and $x, y \in \mathbb{H}^n$, $\beta_\xi(x, y) = \beta_{g(\xi)}(g(x), g(y))$.

For $u \in \mathrm{T}^1(\mathbb{H}^n)$, the *unstable horosphere* based at u^- is the set

$$\mathcal{H}_u^+ = \{v \in \mathrm{T}^1(\mathbb{H}^n) : v^- = u^-, \beta_{u^-}(\pi(u), \pi(v)) = 0\},$$

and the *stable horosphere* based at u^+ is the set

$$\mathcal{H}_u^- = \{v \in \mathrm{T}^1(\mathbb{H}^n) : v^+ = u^+, \beta_{u^+}(\pi(u), \pi(v)) = 0\}.$$

The *weak stable manifold* corresponding to u is

$$\tilde{W}_u^s = \mathrm{Viz}^{-1}(u^+) = \{v \in \mathrm{T}^1(\mathbb{H}^n) : v^+ = u^+\}.$$

$$v_1, v_2 \in \mathcal{H}_u^+,\ r \in \mathbb{R} \Rightarrow d(\mathcal{G}^r(v_1), \mathcal{G}^r(v_2)) = e^r d(v_1, v_2). \tag{2.3}$$

$$v_1, v_2 \in \tilde{W}_u^s,\ r \geq 0 \Rightarrow d(\mathcal{G}^r(v_1), \mathcal{G}^r(v_2)) \leq d(v_1, v_2). \tag{2.4}$$

The image under π of a stable or an unstable horosphere $\mathcal{H}$ in $\mathrm{T}^1(\mathbb{H}^n)$ based at ξ is called a *horosphere in* $\mathbb{H}^n$ *based at* ξ. Hence $\pi(\mathcal{H}) = \{y \in \mathbb{H}^n : \beta_\xi(x, y) = 0\}$ for $x \in \pi(\mathcal{H})$.

2.3. Let $\tilde{S}$ be one of the following: a horosphere or a complete connected totally geodesic submanifold of $\mathbb{H}^n$ of dimension k for $0 \leq k \leq n-1$. Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ denote the unstable horosphere with $\pi(\tilde{E}) = \tilde{S}$ if $\tilde{S}$ is a horosphere, and the unit normal bundle over $\tilde{S}$ if $\tilde{S}$ is totally geodesic.

Lemma 2.1. *The visual map* Viz *restricted to* $\tilde{E}$ *is a diffeomorphism onto* $\partial(\mathbb{H}^n) \smallsetminus \partial(\tilde{S})$.

Proof. The conclusion is obvious if $\tilde{S}$ is a point or a horosphere.

Now suppose that $\tilde{S}$ is a totally geodesic subspace of dimension $1 \leq k \leq n-1$. Consider the upper-half space model for $\mathbb{H}^n$:

$$\mathbb{H}^n = \{x + jy : x \in \mathbb{R}^{n-1},\ y > 0,\ j = (0, \dots, 0, 1)\}, \tag{2.5}$$

and $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1} \cup \{\infty\}$. Without loss of generality, we may assume that $\infty \in \partial(\tilde{S})$ and hence $\partial\tilde{S} \smallsetminus \{\infty\}$ is a $(k-1)$-dimensional affine subspace, say F, of $\mathbb{R}^{n-1}$. For any $x \in \mathbb{R}^{n-1} \smallsetminus L$, let x_1 be the orthogonal projection of x on L. Let $x_2 = x_1 + \|x - x_1\| \cdot j \in \mathbb{H}^n$. Let $v \in \mathrm{T}^1(\mathbb{H}^n)$ be the unit vector based at x_2 in the same direction as $x - x_1$. Then $v \in \tilde{E}$ and $v^+ = x$. Now the conclusion of the lemma is straightforward to deduce. □

2.3.1. *Maps between* $\tilde{E}$ *and* $\mathcal{H}_v^+$. For $v \in \mathrm{T}^1(\mathbb{H}^n)$, $-v$ is the vector with the same base point as v but in the opposite direction. For $v \in \tilde{E}$, let $\xi_v : \mathcal{H}_v^+ \smallsetminus \mathrm{Viz}^{-1}(\partial\tilde{S}) \to \tilde{E} \smallsetminus \{-v\}$ be the map given by

$$\xi_v(u) = \mathrm{Viz}^{-1}(u^+) \cap \tilde{E}. \tag{2.6}$$

Then ξ_v is a diffeomorphism. Its inverse, $q_v : \tilde{E} \smallsetminus \{-v\} \to \mathcal{H}_v^+ \smallsetminus \mathrm{Viz}^{-1}(\partial\tilde{S})$ is the map given by

$$q_v(w) = \mathrm{Viz}^{-1}(w^+) \cap \mathcal{H}_v^+. \tag{2.7}$$

Proposition 2.2. *There exist* $C_1 > 0$ *and* $\epsilon_0 > 0$ *such that:*

(1) *if* $v, w \in \tilde{E}$ *and* $d(v, w) < \epsilon_0$, *then*

$$|\beta_{w^+}(\pi(q_v(w)), \pi(w))| \leq d(q_v(w), w) < C_1 d(w, v);$$

(2) *if* $v \in \tilde{E}$ *and* $w \in \mathcal{H}_v^+$ *with* $d(v,w) < \epsilon_0$, *then*
$$|\beta_{w^+}(\pi(\xi_v(w)), \pi(w))| \le d(\xi_v(w), w) < C_1 d(v,w).$$

Proof. In each of the two statements, the first inequality follows directly from the definition of Busemann function, so we only need to prove the second inequality.

Consider the upper half space model of $\mathbb{H}^n$ given by (2.5). By applying an isometry $g \in G$, since $q_{g(v)}(gw) = g(q_v(w))$, we may assume that v is the unit vector based at j so that $v^+ = \{\infty\}$.

Since $f(u) := d(q_v(u), u)$ is a differentiable function of $u \in \tilde{E}$, there exist $\epsilon_0 > 0$ and $C_1' > 0$ such that $\|Df(u)\| \le C_1'$ for any u with $d(v,u) < \epsilon_0$. Therefore, since $f(v) = 0$, there exists $C_1 > 0$ such that $|f(u)| = |f(u) - f(v)| \le C_1 \cdot d(v,u)$ for all $u \in \tilde{E}$ with $d(u,v) < \epsilon_0$. This proves (1). And (2) can be proved similarly. □

Remark 2.3. The following stronger form of statements in Proposition 2.2 holds: *There exist* $\epsilon_0 > 0$ *and* $C_1 > 0$ *such that*
$$|\beta_{w^+}(\pi(q_v(w)), \pi(w))| \le C_1 d(v,w)^2, \text{ for all } w \in \tilde{E} \text{ with } d(w,v) < \epsilon_0;$$
$$|\beta_{w^+}(\pi(\xi_v(w), \pi(w)))| \le C_1 d(v,w)^2, \text{ for all } w \in \mathcal{H}_v^+ \text{ with } d(w,v) < \epsilon_0.$$
We omit a proof as the stronger version will not be used in this article.

Notation 2.4. Let Γ be a non-elementary torsion-free discrete subgroup of G and set $X := \Gamma\backslash\mathbb{H}^n$. Both the natural projection maps $\mathbb{H}^n \to X$ and $\mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(X)$ will be denoted by $\mathbf{p}$.

2.4. **Boxes, Plaques and Transversals.** Let $u \in \mathrm{T}^1(\mathbb{H}^n)$. Consider a relatively compact open set P containing u in $\mathcal{H}_u^+$, and a relatively compact open neighborhood T of u in $\mathrm{Viz}^{-1}(u^+)$. For each $t \in T$ and $p \in P$, the horosphere $\mathcal{H}_t^+$ intersects $\mathrm{Viz}^{-1}(p^+)$ at a unique vector: we define
$$tp := \mathcal{H}_t^+ \cap \mathrm{Viz}^{-1}(p^+) \in T^1(\mathbb{H}^n).$$
The map $(t,p) \to tp$ provides a local chart of a neighborhood of u in $\mathrm{T}^1(\mathbb{H}^n)$. Since $u \in P$, in this notation $tu = t$. We call the set
$$B(u) = \{tp \in \mathrm{T}^1(\mathbb{H}^n) : t \in T, p \in P\}$$
a *box around* u if some neighborhood of $B(u)$ *injects* into $\mathrm{T}^1(X)$ under $\mathbf{p}$. We write $B = B(u) = TP$.

Note that P (resp. T) may be disconnected and of 'large' diameter, and then the corresponding T (resp. P) will be chosen to be of small diameter in order to achieve the required injectivity of $\mathbf{p}$ on a neighborhood of $B(u)$.

For any $t \in T$, the set

$$tP := \{tp : p \in P\} \subset \mathcal{H}_t^+$$

is called a *plaque at* t; and for any $p \in P$, the set

$$Tp := \{tp : t \in T\} \subset \mathrm{Viz}^{-1}(p^+)$$

is called a *transversal at* p. The *holonomy map* between the transversals Tp and Tp' is given by $tp \mapsto tp'$ for all $t \in T$.

Remark 2.5. If $v = tp \in B$, then $tP \subset \mathcal{H}_v^+$, $Tp \subset \mathrm{Viz}^{-1}(v^+)$ and $B(v) = (Tp)(tP)$ is a box about v and $TP = (Tp)(tP)$. Also $B(u)$ and $B(v)$ have the same collections of plaques and transversals.

For small $\epsilon > 0$, let

$$T_{\epsilon+} = \{s \in \mathrm{Viz}^{-1}(u^+) : d(s, T) < \epsilon\},$$
$$T_{\epsilon-} = \{t \in T : d(t, \partial T) > \epsilon\}, \text{ and } B_{\epsilon\pm} = T_{\epsilon\pm}P.$$

Note that for any $\gamma \in G$, $\gamma P \subset \mathcal{H}_{\gamma u}^+$, $\gamma T \subset \mathrm{Viz}^{-1}((\gamma u)^+)$, $\gamma(tp) = (\gamma t)(\gamma p)$ for any $(t,p) \in T \times P$, $\gamma(TP) = \gamma(B(u)) = B(\gamma u) = (\gamma T)(\gamma P)$, $\gamma(tP)$ is a plaque at γt and $\gamma(Tp)$ is a transversal at γp. Also $\gamma B_{\epsilon\pm} = (\gamma T)_\pm(\gamma P)$.

For $r \in \mathbb{R}$, $\mathcal{G}^r(B(u)) = B(\mathcal{G}^r(u)) = (\mathcal{G}^r(T))(\mathcal{G}^r(P))$.

2.5. For the rest of this section, let $B = TP \subset \mathrm{T}^1(\mathbb{H}^n)$ denote a box such that B_{ϵ_0+} injects into $\mathrm{T}^1(X)$ for some $\epsilon_0 > 0$. By choosing a smaller ϵ_0 if necessary, let $C_1 > 0$ be such that Proposition 2.2 holds. Let

$$C_2 = \max\{d(tp_1, tp_2) : t \in T_{\epsilon_0+},\, p_1, p_2 \in P\}. \tag{2.8}$$

In this section we will develop auxiliary results to understand the intersection of $\mathcal{G}^r(E)$ with $\mathbf{p}(B)$ for $r \gg 1$. First we will show that for any $\gamma \in \Gamma$ if $\mathcal{G}^r(\gamma\tilde{E}) \cap B$ is nonempty, then there exists a unique $t \in T_{\epsilon_0+} \cap \mathcal{G}^r(\gamma\tilde{E})$ and the sets $\mathcal{G}^r(\gamma\tilde{E})$ and $\mathcal{G}^r(tP)$ are contained in $C_1C_2e^{-r}$-tubular neighborhoods of each other.

Lemma 2.6. *Let $r \in \mathbb{R}$ and $\gamma \in \Gamma$. Suppose that $\mathcal{G}^r(\gamma\tilde{E}) \ni tp$ for some $t \in T$, $p \in P$. Let $v = \mathcal{G}^{-r}(\gamma^{-1}tp) \in \tilde{E}$. Let $p_1 \in P$, $y = \mathcal{G}^{-r}(\gamma^{-1}tp_1) \in \mathcal{H}_v^+$, and $w = \xi_v(y) \in \tilde{E}$. Then $w^+ = y^+$,*

$$d(v, y) \le C_2e^{-r} \text{ and } d(y, w) \le C_1C_2e^{-r}.$$

Proof. By (2.6), $w^+ = y^+$. Since $tp, tp_1 \in \mathcal{H}_t^+$, by (2.3) and (2.8),

$$d(v,y) = d(\mathcal{G}^{-r}(\gamma^{-1}tp), \mathcal{G}^{-r}(\gamma^{-1}tp_1)) = d(\mathcal{G}^{-r}(tp), \mathcal{G}^{-r}(tp_1))$$
$$\le d(tp, tp_1)e^{-r} \le C_2e^{-r}.$$

By Proposition 2.2, $d(y, w) = d(y, \xi_v(y)) \le C_1 d(v, y) \le C_1C_2e^{-r}$. ∎

Lemma 2.7. *For any $r \in \mathbb{R}$ and $\gamma \in \Gamma$,*

$$\#\big(T \cap \mathcal{G}^r(\gamma \tilde{E})\big) = \#(\mathcal{G}^{-r}(\gamma^{-1}T) \cap \tilde{E}) \leq 1.$$

Proof. Since $\mathrm{Viz}(\mathcal{G}^{-r}(\gamma^{-1}T)) = \gamma^{-1}\,\mathrm{Viz}(T)$ is a singleton set and Viz restricted to $\tilde{E}$ is injective, the conclusion follows. □

Notation 2.8. For $r \in \mathbb{R}$ and $\gamma \in \Gamma$, in view of Lemma 2.7, define

$$\tilde{E}_{r,\gamma} = \begin{cases} \xi_{\mathcal{G}^{-r}(\gamma^{-1}t)}(\mathcal{G}^{-r}(\gamma^{-1}tP)) \subset \tilde{E} & \text{if } T \cap \mathcal{G}^r(\gamma\tilde{E}) = \{t\} \\ \emptyset & \text{if } T \cap \mathcal{G}^r(\gamma\tilde{E}) = \emptyset. \end{cases} \tag{2.9}$$

Proposition 2.9. *For any $0 < \epsilon \leq \epsilon_0$, $r > r_\epsilon := \log(C_1(C_1+1)C_2/\epsilon)$ and $\gamma \in \Gamma$, we have*

$$\mathcal{G}^{-r}(\gamma^{-1}B_{\epsilon-}) \cap \tilde{E} \subset \tilde{E}_{r,\gamma} \subset \mathcal{G}^{-r}(\gamma^{-1}B_{\epsilon+}) \cap \tilde{E}. \tag{2.10}$$

Proof of first inclusion in (2.10). Let $\gamma \in \Gamma$ and $t \in T_{\epsilon-}$ and $p \in P$ be such that $v := \mathcal{G}^{-r}(\gamma^{-1}tp) \in \tilde{E}$. Let $y = \mathcal{G}^{-r}(\gamma^{-1}t)$ and $w = \xi_v(y) \in \tilde{E}$. By Lemma 2.6,

$$d(y,w) \leq C_1C_2e^{-r} < \epsilon/(C_1+1) < \epsilon.$$

Let $t_1 = \mathcal{G}^r(\gamma w)$. Since $t = \mathcal{G}^r(\gamma y)$ and $w^+ = y^+$, $t_1^+ = t^+$. By (2.4),

$$d(t,t_1) = d(\mathcal{G}^r(\gamma y)), \mathcal{G}^r(\gamma w)) \leq d(\gamma y, \gamma w) = d(y,w) < \epsilon.$$

Therefore $t_1 \in T$, for $t \in T_{\epsilon-}$. Since $(t_1p)^+ = (tp)^+$, we have

$$\mathcal{G}^{-r}(\gamma^{-1}t_1p)^+ = \mathcal{G}^{-r}(\gamma^{-1}tp)^+ = v^+.$$

Since $w = \mathcal{G}^{-r}(\gamma^{-1}t_1)$, $\mathcal{G}^{-r}(\gamma^{-1}t_1p) \in \mathcal{H}^+_w$. Also $w, v \in \tilde{E}$. Therefore by (2.6),

$$v = \xi_w(\mathcal{G}^{-r}(\gamma^{-1}t_1p)) \in \tilde{E}_{r,\gamma}.$$

□

Proof of second inclusion in (2.10). By Lemma 2.7, let $\{t\} = T \cap \mathcal{G}^r(\gamma\tilde{E})$ for some $\gamma \in \Gamma$. Let $v = \mathcal{G}^{-r}(\gamma^{-1}t) \in \tilde{E}$, $p \in P$, $y = \mathcal{G}^{-r}(\gamma^{-1}tp)$, and $w = \xi_v(y) \in \tilde{E}_{r,\gamma}$. By Lemma 2.6,

$$d(v,w) \leq d(v,y) + d(y,w) \leq C_2e^{-r} + C_1C_2e^{-r} \leq \epsilon/C_1.$$

Put $v_1 = q_w(v) \in \mathcal{H}^+_w$. By (2.7), $v_1^+ = v^+$, and by Proposition 2.2(1),

$$d(v,v_1) \leq C_1 d(v,w) \leq \epsilon.$$

Put $t_1 = \mathcal{G}^r(\gamma v_1) \in \mathcal{H}^+_{\mathcal{G}^r(\gamma w)}$. Since $t = \mathcal{G}^r(\gamma v)$, $t_1^+ = t^+$. By (2.4),

$$d(t,t_1) = d(\mathcal{G}^r(\gamma v), \mathcal{G}^r(\gamma v_1)) \leq d(\gamma v, \gamma v_1) \leq \epsilon.$$

Hence $t_1 \in T_{\epsilon+}$. Now $\mathcal{G}^r(\gamma w), t_1p \in \mathcal{H}^+_{t_1}$. Since $w^+ = y^+$,

$$(\mathcal{G}^r(\gamma w))^+ = (\mathcal{G}^r(\gamma y))^+ = (tp)^+ = (t_1p)^+.$$

Since Viz is injective on $\mathcal{H}^+_{t_1}$, $\mathcal{G}^r(\gamma w) = t_1 p$. Hence $w \in \mathcal{G}^{-r}(\gamma^{-1}B_{\epsilon+})$. □

2.6. **Measure on E corresponding to a conformal density on $\partial\mathbb{H}^n$.** Let $\{\mu_x : x \in \mathbb{H}^n\}$ be a Γ-*invariant conformal density* of dimension $\delta_\mu > 0$ on $\partial\mathbb{H}^n$. That is, for each $x \in \mathbb{H}^n$, μ_x is a positive finite Borel measure on $\partial\mathbb{H}^n$ such that for all $y \in \mathbb{H}^n$, $\xi \in \partial\mathbb{H}^n$ and $\gamma \in \Gamma$,

$$\gamma_*\mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{\delta_\mu \beta_\xi(y,x)}, \tag{2.11}$$

where $\gamma_*\mu_x(F) := \mu_x(\gamma^{-1}(F))$ for any Borel subset F of $\partial\mathbb{H}^n$.

Fix $o \in \mathbb{H}^n$. We consider the measure on $\tilde{E}$ given by

$$d\mu_{\tilde{E}}(v) = e^{\delta_\mu \beta_{v^+}(o,\pi(v))}\, d\mu_o(v^+). \tag{2.12}$$

By (2.11), $\mu_{\tilde{E}}$ is independent of the choice of $o \in \mathbb{H}^n$ and $\gamma_*\mu_{\tilde{E}} = \mu_{\gamma\tilde{E}}$ for any $\gamma \in \Gamma$. Let $\mu_{\Gamma_{\tilde{E}}\backslash\tilde{E}}$ be the locally finite Borel measure on $\Gamma_{\tilde{E}}\backslash\tilde{E}$ induced by $\mu_{\tilde{E}}$ as follows: For any $f \in C_c(\tilde{E})$, let $\bar{f}(\Gamma_{\tilde{E}}v) = \sum_{\gamma\in\Gamma} f(\gamma v)$, for all $v \in \tilde{E}$. Then $f \mapsto \bar{f}$ is a surjective map from $C_c(\tilde{E})$ from to $C_c(\Gamma_{\tilde{E}}\backslash\tilde{E})$, and

$$\int_{\Gamma_{\tilde{E}}\backslash\tilde{E}} \bar{f}\, d\mu_{\Gamma_{\tilde{E}}\backslash\tilde{E}} := \int_{\tilde{E}} f\, d\mu_{\tilde{E}} \tag{2.13}$$

is well defined; see [29, Chapter 1] for a similar construction.

Now let μ_E be the measure on $E = \mathbf{p}(\tilde{E})$ defined as the pushforward of $\mu_{\Gamma_{\tilde{E}}\backslash\tilde{E}}$ from $\Gamma_{\tilde{E}}\backslash\tilde{E}$ to $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ under the map $\Gamma_{\tilde{E}}v \mapsto \Gamma v$. Thus for any set $B \subset \mathrm{T}^1(\mathbb{H}^n)$ such that $\mathbf{p}$ is injective on B, and any measurable nonnegative function f on $E \cap \mathbf{p}(B)$,

$$\begin{aligned}\int_{E\cap\mathbf{p}(B)} f\, d\mu_E &= \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \int_{u\in\gamma\tilde{E}\cap B} f(\mathbf{p}(u))\, d\mu_{\gamma\tilde{E}}(u)\\ &= \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \int_{u\in\tilde{E}\cap\gamma^{-1}B} f(\mathbf{p}(u))\, d\mu_{\tilde{E}}(u),\end{aligned} \tag{2.14}$$

where the integration over an empty set is defined to be 0. Therefore by Proposition 2.9 we obtain the following:

Proposition 2.10. *Let $0 < \epsilon \le \epsilon_0$ and $r > r_\epsilon$. Then for all borel measurable functions $\Psi \ge 0$ on $\mathrm{T}^1(X)$ with $\mathrm{supp}(\Psi) \subset \mathbf{p}(B_{\epsilon-})$ and $f \ge 0$ on E, we have*

$$\begin{aligned}\int_{u\in E} \Psi(\mathcal{G}^r(u))f(u)\, d\mu_E(u) &= \int_{E\cap\mathbf{p}(\mathcal{G}^{-r}(B_{\epsilon\pm}))} \Psi(\mathcal{G}^r(u))f(u)\, d\mu_E(u)\\ &= \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \int_{\mathcal{G}^{-r}(\gamma^{-1}B_{\epsilon\pm})\cap\tilde{E}} \Psi(\mathcal{G}^r(u))f(u)\, d\mu_{\tilde{E}}(u)\\ &= \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \int_{\tilde{E}_{r,\gamma}} \Psi(\mathcal{G}^r(\mathbf{p}(u)))f(\mathbf{p}(u))\, d\mu_{\tilde{E}}(u).\end{aligned}$$

Remark 2.11. (1) For the counting application in section 7, we will use the results of this section only for the case when the map $\Gamma_{\tilde{E}}\backslash\tilde{E} \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is proper, in which case μ_E is a locally finite Borel measure.

(2) In the general case, μ_E may not be σ-finite, but it is an *s-finite measure*; namely, a countable sum of finite measures (with possibly non-disjoint supports).

(3) If the dimension of $\tilde{S} = \pi(\tilde{E})$ in $\mathbb{H}^n$ is 0 or $n-1$, the map $\Gamma_{\tilde{E}}\backslash\tilde{E}$ to $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is *injective*, and hence μ_E is σ-finite on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

2.6.1. *Measures on horospherical foliation and semi-invariance under geodesic flow.* The conformal density $\{\mu_x\}$ induces a Γ-equivariant family of measures $\{\mu_{\mathcal{H}^+_u} : u \in \mathrm{T}^1(\mathbb{H}^n)\}$ on the unstable horospherical foliation on $\mathrm{T}^1(\mathbb{H}^n)$:

$$d\mu_{\mathcal{H}^+_u}(v) = e^{\delta_\mu \beta_{v^+}(o,\pi(v))}\, d\mu_o(v^+). \tag{2.15}$$

For any $r \in \mathbb{R}$, since $\mathcal{G}^r(v)^+ = v^+$ and

$$\beta_{v^+}(o, \pi(\mathcal{G}^r(v))) - \beta_{v^+}(o,\pi(v)) = \beta_{v^+}(\pi(v), \pi(\mathcal{G}^r(v))) = r,$$

by (2.11), we get for all $\gamma \in \Gamma$ and $r \in \mathbb{R}$,

$$\gamma_* \mu_{\mathcal{H}^+_u} = \mu_{\mathcal{H}^+_{\gamma u}} \quad \text{and} \quad \mathcal{G}^r_* \mu_{\mathcal{H}^+_u} = e^{-\delta_\mu r} \mu_{\mathcal{H}^+_{\mathcal{G}^r(u)}}. \tag{2.16}$$

2.7. **On transversal intersections of $\mathcal{G}^r(\Gamma\tilde{E})$ with B.** Let a box B, $\epsilon_0 > 0$, $C_1 > 0$ and $C_2 > 0$ be as described in the beginning of §2.5. For any $0 < \epsilon \le \epsilon_0$, we put

$$r_\epsilon = \log((C_1+1)C_1C_2/\epsilon). \tag{2.17}$$

Proposition 2.12. *Let $0 < \epsilon \le \epsilon_0$, $r > r_\epsilon$, and $\{t\} = T \cap \mathcal{G}^r(\gamma\tilde{E})$ for some $\gamma \in \Gamma$. Then for all measurable functions $\Psi \ge 0$ on B_{ϵ_0+} and $f \ge 0$ on $\tilde{E}$,*

$$\begin{aligned} &(e^{-\delta_\mu \epsilon}) f^-_\epsilon(\mathcal{G}^{-r}(\gamma^{-1}t)) \int_{tP} \Psi^-_\epsilon \, d\mu_{\mathcal{H}^+_t} \\ \le & e^{\delta_\mu r} \int_{w \in \tilde{E}_{r,\gamma}} \Psi(\mathcal{G}^r(\gamma w)) f(w) d\mu_{\tilde{E}}(w) \\ \le & (e^{\delta_\mu \epsilon}) f^+_\epsilon(\mathcal{G}^{-r}(\gamma^{-1}t)) \int_{tP} \Psi^+_\epsilon \, d\mu_{\mathcal{H}^+_t}, \end{aligned}$$

where $f^\pm_\epsilon$ on $\tilde{E}$ and $\Psi^\pm_\epsilon$ on $B_{\epsilon+}$ are defined as

$$\begin{array}{rcl} f^+_\epsilon(u) &=& \sup_{\{u_1 \in \tilde{E} : d(u_1,u) \le \epsilon\}} f(u_1), \\ f^-_\epsilon(u) &=& \inf_{\{u_1 \in \tilde{E} : d(u_1,u) \le \epsilon\}} f(u_1), \\ \Psi^+_\epsilon(tp) &=& \sup_{\{t_1 \in T_{\epsilon+} : d(t_1p, tp) \le \epsilon\}} \Psi(t_1 p), \\ \Psi^-_\epsilon(tp) &=& \inf_{\{t_1 \in T_{\epsilon+} : d(t_1p, tp) \le \epsilon\}} \Psi(t_1 p). \end{array} \tag{2.18}$$

Proof. Let $v = \mathcal{G}^{-r}(\gamma^{-1}t) \in \tilde{E}$. Let $\phi : tP \subset \mathcal{H}_t^+ \to \tilde{E}_{r,\gamma} \subset \tilde{E}$ be the map given by $\phi(tp) = w := \xi_v(y)$, where $p \in P$ and $y = \mathcal{G}^{-r}(\gamma^{-1}tp)$. By Lemma 2.6,

$$d(y,w) < C_1C_2e^{-r} < \epsilon, \quad d(v,w) < (C_1+1)C_2e^{-r} < \epsilon \tag{2.19}$$

and since $w^+ = y^+$,

$$d(\mathcal{G}^r(\gamma y), \mathcal{G}^r(\gamma w)) = d(\mathcal{G}^r(y), \mathcal{G}^r(w)) \le d(y,w) < \epsilon,$$

and by Proposition 2.9, $\mathcal{G}^r(\gamma w) \in T_{\epsilon+}p$. Therefore

$$f_\epsilon^-(v) \le f(w) \le f_\epsilon^+(v); \tag{2.20}$$

$$\Psi_\epsilon^-(\mathcal{G}^r(\gamma y)) \le \Psi(\mathcal{G}^r(\gamma w)) \le \Psi_\epsilon^+(\mathcal{G}^r(\gamma y)). \tag{2.21}$$

For the map $tp \mapsto y := \mathcal{G}^{-r}(\gamma^{-1}tp)$, by (2.16),

$$e^{\delta_\mu r} d\mu_{\mathcal{H}_v^+}(y) = d\mu_{\mathcal{H}_t^+}(tp). \tag{2.22}$$

For the map $y \mapsto w = \xi_v(y)$, by (2.12) and (2.15), since $w^+ = y^+$,

$$d\mu_{\tilde{E}}(w) = \frac{e^{\delta_\mu \beta_{w^+}(o,\pi(w))}}{e^{\delta_\mu \beta_{y^+}(o,\pi(y))}} d\mu_{\mathcal{H}_v^+}(y) = e^{\delta_\mu \beta_{w^+}(\pi(y),\pi(w))} d\mu_{\mathcal{H}_v^+}(y). \tag{2.23}$$

By (2.19), $|\beta_{w^+}(\pi(y),\pi(w))| \le d(\pi(y),\pi(w)) \le \epsilon$. Therefore

$$e^{-\delta_\mu \epsilon} < d\mu_{\tilde{E}}(w)/d\mu_{\mathcal{H}_v^+}(y) < e^{\delta_\mu \epsilon}. \tag{2.24}$$

Combining (2.22) and (2.24), for the map $w = \phi(tp)$ we get

$$e^{-\delta_\mu \epsilon} \le e^{\delta_\mu r} \frac{d\mu_{\tilde{E}}(w)}{d\mu_{\mathcal{H}_t^+}(tp)} \le e^{\delta_\mu \epsilon}. \tag{2.25}$$

By noting that $\mathcal{G}^{-r}(\gamma^{-1}t) = v$ and $tp = \mathcal{G}^r(\gamma y)$, the conclusion of the proposition follows from (2.20), (2.21) and (2.25). □

Notation 2.13. For $r \ge 0$ and $t \in T \cap \mathcal{G}^r(\Gamma \tilde{E})$, in view of Lemma 2.7 let

$$\bar{\Gamma}_{r,t} = \{[\gamma] \in \Gamma/\Gamma_{\tilde{E}} : \{t\} = T \cap \mathcal{G}^r(\gamma \tilde{E})\}. \tag{2.26}$$

Since $\mathbf{p}$ is injective on B_{ϵ_0+}, for notational convenience, we identify $t \in T_{\epsilon_0+}$ with its image $\mathbf{p}(t) \in \mathbf{p}(T) \subset X$. Therefore we have

$$\{[\gamma] \in \Gamma/\Gamma_{\tilde{E}} : \tilde{E}_{r,\gamma} \neq \emptyset\} = \bigcup_{t \in T \cap \mathcal{G}^r(\Gamma \tilde{E})} \bar{\Gamma}_{r,t} = \bigcup_{t \in T \cap \mathcal{G}^r(E)} \bar{\Gamma}_{r,t}. \tag{2.27}$$

Combining Proposition 2.10 and Proposition 2.12, in view of (2.27) we deduce the following:

Corollary 2.14. *Let $0 < \epsilon \le \epsilon_0$ and $r > r_\epsilon$. For all measurable functions $\Psi \ge 0$ on B_{ϵ_0+} with* $\operatorname{supp}(\Psi) \subset B_{\epsilon-}$ *and $f \ge 0$ on E, we have*

$$\begin{aligned}
&(e^{-\delta_\mu \epsilon}) \sum_{t \in T \cap \mathcal{G}^r(E)} \#(\bar{\Gamma}_{r,t}) f_\epsilon^-(\mathcal{G}^{-r}(t)) \cdot \int_{tP} \Psi_\epsilon^- \, d\mu_{\mathcal{H}_t^+} \\
\le & e^{\delta_\mu r} \int_E \Psi(\mathcal{G}^r(u)) f(u) \, d\mu_E(u) \\
\le & (e^{\delta_\mu \epsilon}) \sum_{t \in T \cap \mathcal{G}^r(E)} \#(\bar{\Gamma}_{r,t}) \cdot f_\epsilon^+(\mathcal{G}^{-r}(t)) \cdot \int_{tP} \Psi_\epsilon^+ \, d\mu_{\mathcal{H}_t^+},
\end{aligned}$$

where $f_\epsilon^\pm$ on $B_{\epsilon+}$ and $\Psi_\epsilon^\pm$ on E are defined as in (2.18).

2.8. **Haar system and admissible boxes.**

Lemma 2.15 ([31])**.** *For a uniformly continuous $\Psi \in C(B)$, the map*

$$t \in T \mapsto \int_{tP} \Psi \, d\mu_{\mathcal{H}_t^+}$$

is uniformly continuous. In particular the map $t \mapsto \mu_{\mathcal{H}_t^+}(tP)$ is uniformly continuous.

Proof. Note that $(tp)^+ = p^+$. Therefore by (2.15)

$$\int_{tP} \Psi \, d\mu_{\mathcal{H}_t^+} = \int_P \Psi(tp) e^{\delta_\mu \beta_{p^+}(o, \pi(tp))} \, d\mu_o(p^+).$$

Put $\phi(tp) = \Psi(tp) e^{\delta_\mu \beta_{p^+}(o,\pi(tp))}$. Since ϕ is uniformly continuous on B,

$$|\int_{t_1 P} \Psi \, d\mu_{\mathcal{H}_{t_1}^+} - \int_{t_2 P} \Psi \, d\mu_{\mathcal{H}_{t_2}^+}| \le \mu_o(\operatorname{Viz}(P)) \cdot \sup_{p \in P} |\phi(t_1 p) - \phi(t_2 p)| \to 0$$

as $d(t_1, t_2) \to 0$. □

Definition 2.16. A box $B = TP$ as defined in subsection 2.4 is called *admissible with respect to the conformal density* $\{\mu_x\}$, if every plaque of B has a positive measure with respect to $\{\mu_{\mathcal{H}^+}\}$; that is, $\mu_{\mathcal{H}_t^+}(tP) > 0$ for all $t \in T$, or equivalently,

$$\mu_x(\operatorname{Viz}(tP)) = \mu_x(P^+) > 0 \text{ for some (and hence all) } x \in \mathbb{H}^n.$$

Lemma 2.17. *Fix a conformal density $\{\mu_x\}_{x \in \mathbb{H}^n}$ on $\partial \mathbb{H}^n$. Then for any $u \in \mathrm{T}^1(\mathbb{H}^n)$, there exists an admissible box around u with respect to $\{\mu_x\}$.*

Proof. Fix any $x \in \mathbb{H}^n$. Since Γ_{u^-} is virtually abelian, and since we assume that Γ is non-elementary, Γ does not fix u^-. Therefore by the Γ-invariance and the conformality of the density $\{\mu_x\}$, we have

$\mathrm{supp}(\mu_x) \neq \{u^-\}$. Since $\mathrm{Viz} : \mathcal{H}_u^+ \to \partial \mathbb{H}^n \smallsetminus \{u^-\}$ is a diffeomorphism, there exists $u_1 \in \mathcal{H}_u^+$ such that $u_1^+ = \mathrm{Viz}(u_1) \in \mathrm{supp}(\mu_x)$. If $\gamma u = u_1$ for any $\gamma \in \Gamma$, then by the conformality, $u \in \mathrm{supp}(\mu_{\mathcal{H}_u^+})$ and we replace u_1 by u. Since $\mathbf{p}$ is injective on $\{u, u_1\}$, there exists a relatively compact open subset P of $\mathcal{H}_u^+$ containing $\{u, u_1\}$ such that $\mathbf{p}$ is injective on an open set Ω of $\mathrm{T}^1(\mathbb{H}^n)$ containing $\overline{P}$. Then $\mu_x(\mathrm{Viz}(P)) > 0$. By Lemma 2.15, we can choose T a enough ball in $\mathrm{Viz}^{-1}(u^+)$ so that some neighborhood of the closure of $B = TP$ is contained in Ω. Now $B = TP$ is an admissible box. ☐

2.8.1. Let $B = TP$ be an admissible box with respect to a conformal density $\{\mu_x\}$ such that $\mathbf{p}$ is injective on a neighborhood of the closure of B_{ϵ_0+} for some $\epsilon_0 > 0$. Let C_1, C_2 be as described in the beginning of §2.5. For notational convenience, we will *identify* T_{ϵ_0+} and B_{ϵ_0+} with their respective images in $\mathrm{T}^1(X)$ under $\mathbf{p}$.

Proposition 2.18. *Let $0 < \epsilon \leq \epsilon_0$ and $r > r_\epsilon$ (see (2.17)). Then for all measurable functions $\psi \geq 0$ on T_{ϵ_0+} with $\mathrm{supp}(\psi) \subset T_{\epsilon-}$ and $f \geq 0$ on E, we have*

$$\begin{aligned}&(e^{-\delta_\mu \epsilon}) \int_E \Psi_\epsilon^-(\mathcal{G}^r(w)) f_\epsilon^-(w)\, d\mu_E(w)\\ \leq& e^{-\delta_\mu r} \sum_{t \in T \cap \mathcal{G}^r(E)} \#(\bar{\Gamma}_{r,t}) \cdot \psi(t) f(\mathcal{G}^{-r}(t))\\ \leq& (e^{-\delta_\mu \epsilon}) \int_E \Psi_\epsilon^+(\mathcal{G}^r(w)) f_\epsilon^+(w) d\mu_E(w),\end{aligned}$$

where the function Ψ on B_{ϵ_0+} is defined by

$$\Psi(\mathbf{p}(tp)) := \psi(t)/\mu_{\mathcal{H}_t^+}(tP), \text{ for all } (t,p) \in T_{\epsilon_0+} \times P,$$

and $\Psi_\epsilon^\pm$ on $B_{\epsilon+}$ and $f_\epsilon^\pm$ on E are defined as in (2.18).

Proof. Since $\int_{tP} \Psi\, d\mu_{\mathcal{H}_t^+} = \psi(t)$, the result is straightforward to deduce from Corollary 2.14. ☐

In the section 3, Proposition 2.18 will enable us to describe the limiting distribution of the transversal intersections $T \cap \mathcal{G}^r(E)$ using the mixing of the geodesic flow with respect to m^{BMS} (cf. Theorem 3.5).

2.9. **Some direct consequences.** The results proved in this subsection are also of independent interest. Let the notation be as in §2.8.1.

Corollary 2.19. *Let $0 < \epsilon \leq \epsilon_0$ and f be a measurable function on E such that $f_\epsilon^+ \in L^1(E, \mu_E)$. Then for any $r > r_\epsilon$ and any measurable* *function ψ on T:*

$$\sum_{t\in T\cap\mathcal{G}^r(E)} \#(\bar\Gamma_{r,t})\cdot|\psi(t)f(\mathcal{G}^{-r}(t))|<\infty.$$

In particular, if there exists a Γ-invariant conformal density $\{\mu_x\}$ and $|\mu_E|<\infty$, then

$$\sum_{t\in T\cap\mathcal{G}^r E} \#(\bar\Gamma_{r,t})<\infty.$$

Proof. By Proposition 2.18 with T_ϵ in place of T and declaring ψ to be zero outside T, we obtain the first claim, because

$$\sum_{t\in T\cap\mathcal{G}^r(E)} \#(\bar\Gamma_{r,t})\cdot|\psi(t)f(\mathcal{G}^{-r}(t))|\le(1+\epsilon)e^{\delta_\mu r}\|\Psi_\epsilon^+\|_\infty\cdot\mu_E(|f_\epsilon^+|).$$

To deduce the second claim from the first one, we choose $f=1$ on E and $\psi=1$ on T. □

Definition 2.20 (Radial Limit points)**.** The limit set $\Lambda(\Gamma)$ of Γ is the set of all accumulation points of an orbit $\Gamma(z)$ in $\overline{\mathbb{H}}^n$ for $z\in\mathbb{H}^n$. As Γ acts properly discontinuously on $\mathbb{H}^n$, $\Lambda(\Gamma)$ is contained in $\partial\mathbb{H}^n$.

A point $\xi\in\Lambda(\Gamma)$ is called a *radial limit point* if for some (and hence every) geodesic ray β tending to ξ and some (and hence every) point $x\in\mathbb{H}^n$, there is a sequence $\gamma_i\in\Gamma$ with $\gamma_i x\to\xi$ and $d(\gamma_i x,\beta)$ is bounded.

We denote by $\Lambda_{\mathrm{r}}(\Gamma)$ the set of radial limit points for Γ.

If Γ is non-elementary, $\Lambda_{\mathrm{r}}(\Gamma)$ is a nonempty Γ-invariant subset of $\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is a Γ-minimal closed subset of $\partial\mathbb{H}^n$, we have that $\overline{\Lambda_{\mathrm{r}}(\Gamma)}=\Lambda(\Gamma)$.

Theorem 2.21. *Let C denote the smallest subsphere of $\mathbb{H}^n$ containing $\Lambda(\Gamma)$. Suppose that $C=\partial\tilde S$ or $\dim(C)>\dim(\partial\tilde S)$. If there exists a Γ-invariant conformal density $\{\mu_x : x\in\mathbb{H}^n\}$ such that $|\mu_E|<\infty$, then the natural map $\bar{\mathbf{p}}:\Gamma_{\tilde E}\backslash\tilde E\to\Gamma\backslash\mathrm{T}^1(\mathbb{H}^n)$ is proper.*

Proof. Note that $\Gamma\subset G_C=\{g\in G : gC=C\}$, because if $\gamma\in\Gamma$, then $\gamma C\cap C\supset\Lambda(\Gamma)$, hence by minimality $\gamma C=C$.

Suppose $C=\partial\tilde S$ Then, since $G_{\tilde S}=G_{\partial\tilde S}$, $\Gamma=\Gamma\cap G_C=\Gamma_{\tilde S}=\Gamma_{\tilde E}$ and hence the properness of $\bar{\mathbf{p}}$ is obvious.

Now suppose that $\dim(C)>\dim(\tilde S)$ and that that $\bar{\mathbf{p}}$ is not proper. Then there exist sequences $\gamma_i\in\Gamma$ and $e_i\in\tilde E$ such that $\gamma_i e_i$ converges to a vector $v\in\mathrm{T}^1(\mathbb{H}^n)$ as $i\to\infty$, and

$$\gamma_i\Gamma_{\tilde E}\ne\gamma_j\Gamma_{\tilde E},\ \text{for all } i\ne j. \tag{2.28}$$

Fix $e_0 \in \tilde{E}$. Since $G_{\tilde{E}}$ acts transitively on $\tilde{E}$, there exists $h_i \in G_{\tilde{E}}$ such that $e_i = h_i e_0$. Then $\gamma_i h_i e_0$ converges to v. Therefore there exists $g \in G$ such that $\gamma_i h_i \to g$ and $v = g e_0$.

Now $\mathrm{Viz}(g\tilde{E}) = \partial\mathbb{H}^n - \partial(g\tilde{S})$. Since $\dim(\partial(g\tilde{S})) = \dim(\tilde{S}) < \dim(C)$, we have that $\Lambda(\Gamma) \smallsetminus \partial(g\tilde{S})$ is a nonempty open subset of $\Lambda(\Gamma)$. Since $\Lambda_{\mathrm{r}}(\Gamma)$ is dense in $\Lambda(\Gamma)$, it follows that

$$\Lambda_{\mathrm{r}}(\Gamma) \cap \mathrm{Viz}(g\tilde{E}) \neq \emptyset.$$

Therefore there exists $h_0 \in G_{\tilde{E}}$ such that $\mathrm{Viz}(gh_0e_0) = (gh_0e_0)^+ \in \Lambda_{\mathrm{r}}(\Gamma)$. Hence there exist $r_i \to \infty$ such that $\bar{\mathbf{p}}(\mathcal{G}^{r_i}(gh_0e_0))$ converges to a point in $\mathrm{T}^1(X)$. Then there exists a sequence $\{\gamma_i'\} \subset \Gamma$ such that $\mathcal{G}^{r_i}(\gamma_i' g h_0 e_0) \to u$ for some $u \in \mathrm{T}^1(\mathbb{H}^n)$.

Let $B = TP$ be an admissible box centered at u. Let $\epsilon > 0$ be such that $u \in B_{3\epsilon-}$. Fix $k \in \mathbb{N}$ such that $r_k > r_\epsilon$ (see 2.17) such that for $\gamma' = \gamma_k'$, we have $\mathcal{G}^r(\gamma' g h_0 e_0) \in B_{2\epsilon-}$.

Since $\gamma_i h_i \to g$, $\mathcal{G}^r(\gamma'\gamma_i h_i h_0 e_0) \in B_{\epsilon-}$ for all $i \geq i_0$ for some i_0. Since $h_i h_0 e_0 \in \tilde{E}$, by (2.10) $t_i \in T \cap \mathcal{G}^r(\gamma'\gamma_i\tilde{E})$ for all $i \geq i_0$. Therefore

$$(\Gamma T \cap \mathcal{G}^r\tilde{E}) \supset \{(\gamma'\gamma_i)^{-1}t_i : i \geq i_0\}. \tag{2.29}$$

We claim that for any $i \in \mathbb{N}$,

$$\Gamma_{\tilde{E}}\gamma_i^{-1}(\gamma')^{-1}t_i \neq \Gamma_{\tilde{E}}\gamma_j^{-1}(\gamma')^{-1}t_j, \text{ for all but finitely many } j. \tag{2.30}$$

To see this, since $\mathbf{p}$ is injective on T, if $t_i \neq t_j$, then $\Gamma t_i \neq \Gamma t_j$ and hence (2.30) holds. If $t_i = t_j$, then it follows from (2.28) as $\Gamma \cap G_{(\gamma')^{-1}t_i}$ is finite. Combining (2.29) and (2.30), we get that

$$\#(\Gamma_{\tilde{E}}\backslash(\Gamma T \cap \mathcal{G}^r\tilde{E})) = \infty.$$

We observe that if $t \in T \cap \mathcal{G}^r(E)$, then $\Gamma_{\tilde{E}}\backslash(\Gamma t \cap \mathcal{G}^r\tilde{E}) = \bar{\Gamma}_{r,t}^{-1}t$. If $|\mu_E^{\mathrm{PS}}| < \infty$, then by (2.19) of Corollary 2.19

$$\#(\Gamma_{\tilde{E}}\backslash(\Gamma T \cap \mathcal{G}^r\tilde{E})) \leq \sum_{t \in T \cap \mathcal{G}^r(E)} \#(\bar{\Gamma}_{r,t}) < \infty,$$

which is a contradiction. $\square$

Remark 2.22. (1) Theorem 2.21 holds for Γ Zariski dense: since $\Gamma \subset G_C$ and G_C is Zariski closed, we have $C = \partial\mathbb{H}^n$ for Γ Zariski dense.

(2) Theorem 2.21 holds in the case $\Lambda(\Gamma_{\tilde{S}}) = \partial\tilde{S}$; since $\tilde{S} \subset C$ in this case, and hence we have either $\tilde{S} = C$ or $\dim(C) > \dim(\tilde{S})$.

3. Equidistribution of $\mathcal{G}^r_*\mu_E^{\text{Leb}}$

3.1. **BMS-measure and BR-measure on** $\mathrm{T}^1(X)$**.** As before, let Γ be a non-elementary torsion-free discrete subgroup of G and set $X := \Gamma\backslash\mathbb{H}^n$. Let $\{\mu_x\}$ and $\{\mu'_x\}$ be Γ-invariant conformal densities on $\partial\mathbb{H}^n$ of dimension δ_μ and $\delta_{\mu'}$ respectively. After Roblin [31], we define a measure $m^{\mu,\mu'}$ on $\mathrm{T}^1(X)$ associated to $\{\mu_x\}$ and $\{\mu'_x\}$ as follows. Fix $o \in \mathbb{H}^n$. The the map

$$u \mapsto (u^+, u^-, \beta_{u^-}(o, \pi(u)))$$

is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^n)$ with

$$(\partial\mathbb{H}^n \times \partial\mathbb{H}^n \smallsetminus \{(\xi,\xi) : \xi \in \partial\mathbb{H}^n\}) \times \mathbb{R}.$$

Hence we can define a measure $\tilde{m}^{\mu,\mu'}$ on $\mathrm{T}^1(\mathbb{H}^n)$ by

$$d\tilde{m}^{\mu,\mu'}(u) = e^{\delta_\mu \beta_{u^+}(o,\pi(u))}\, e^{\delta_{\mu'}\beta_{u^-}(o,\pi(u))}\, d\mu_o(u^+) d\mu'_o(u^-) ds, \qquad (3.1)$$

where $s = \beta_{u^-}(o, \pi(u))$. Note that $\tilde{m}^{\mu,\mu'}$ is Γ-invariant. Hence it induces a locally finite measure $m^{\mu,\mu'}$ on $\mathrm{T}^1(X)$ such that if $\mathbf{p}$ is injective on $\Omega \subset \mathrm{T}^1(\mathbb{H}^n)$, then

$$m^{\mu,\mu'}(\mathbf{p}(\Omega)) = \tilde{m}^{\mu,\mu'}(\Omega).$$

This definition is independent of the choice of $o \in \mathbb{H}^n$.

Two important conformal densities on $\mathbb{H}^n$ we will consider are the Patterson-Sullivan density and the G-invariant (Lebesgue) density.

3.1.1. *Critical exponent* δ_Γ. We denote by δ_Γ the critical exponent of Γ which is defined as the abscissa of convergence of a Poincare series $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma(o))}$ for some $o \in \mathbb{H}^n$; that is, the series converges for $s > \delta_\Gamma$ and diverges for $s < \delta_\Gamma$ and the convergence property is independent of the choice of $o \in \mathbb{H}^n$.

As Γ is non-elementary, we have $\delta_\Gamma > 0$. Generalizing the work of Patterson [26] for $n = 2$, Sullivan [36] constructed a Γ-invariant conformal density $\{\nu_x : x \in \mathbb{H}^n\}$ of dimension δ_Γ supported on $\Lambda(\Gamma)$, which is unique up to homothety, and called the *Patterson-Sullivan density*. From now on, we will simply write δ instead of δ_Γ.

We denote by $\{m_x : x \in \mathbb{H}^n\}$ a G-invariant conformal density on the boundary $\partial\mathbb{H}^n$ of dimension $(n-1)$, which is unique up to homothety, and each m_x is invariant under the maximal compact subgroup G_x. It will be called the *Lebesgue density*.

The measure $m^{\nu,\nu}$ on $\mathrm{T}^1(X)$ is called the *Bowen-Margulis-Sullivan measure* m^{BMS} associated with $\{\nu_x\}$ ([5], [20], [37]):

$$dm^{\text{BMS}}(u) = e^{\delta\beta_{u^+}(o,\pi(u))}\, e^{\delta\beta_{u^-}(o,\pi(u))}\, d\nu_o(u^+) d\nu_o(u^-) ds. \qquad (3.2)$$

The measure $m^{\nu,m}$ is called the *Burger-Roblin measure* m^{BR} associated with $\{\nu_x\}$ and $\{m_x\}$ ([6], [31]):

$$dm^{\mathrm{BR}}(u) = e^{(n-1)\beta_{u^+}(o,\pi(u))}\, e^{\delta\beta_{u^-}(o,\pi(u))}\, dm_o(u^+)d\nu_o(u^-)ds. \tag{3.3}$$

We note that the support of m^{BMS} and m^{BR} are given respectively by $\{u \in \mathrm{T}^1(X) : u^+, u^- \in \Lambda(\Gamma)\}$ and $\{u \in \mathrm{T}^1(X) : u^- \in \Lambda(\Gamma)\}$.

3.2. **Relation to classification of measures invariant under horocycles.** Burger [6] showed that for a convex cocompact hyperbolic surface $\Gamma\backslash\mathbb{H}^2$ with $\delta > 1/2$, m^{BR} is a unique ergodic horocycle invariant locally finite measure which is not supported on closed horocycles. Roblin extended Burger's result in much greater generality. By identifying the space $\Omega_{\mathcal{H}}$ of all unstable horospheres with $\partial\mathbb{H}^n \times \mathbb{R}$ by $\mathcal{H}^+(u) \mapsto (u^-, \beta_{u^-}(o, \pi(u)))$, one defines the measure $d\hat{\mu}(\mathcal{H}) = d\nu_o(\xi)e^{\delta s}ds$ for $\mathcal{H} = (\xi, s)$. Then Roblin's theorem [31, Thm. 6.6] says that if $|m^{\mathrm{BMS}}| < \infty$, then $\hat{\mu}$ is the unique Radon Γ-invariant measure on $\Lambda_{\mathrm{r}}(\Gamma) \times \mathbb{R} \subset \Omega_{\mathcal{H}}$. This important classification result is not used in this article, but it suggests that the asymptotic distribution of expanding horospheres should be described by m^{BR}.

3.3. **Patterson-Sullivan and Lebesgue measures on $\tilde{E}$, $\mathcal{H}_u^+$ and E.** Let $\tilde{S}$ and $\tilde{E}$ be as in the subsection 2.3. The following measures are special cases of the measures defined in the subsection 2.6.

Fix $o \in \mathbb{H}^n$. Define the Borel measure $\mu_{\tilde{E}}^{\mathrm{Leb}}$ on $\tilde{E}$ such that

$$d\mu_{\tilde{E}}^{\mathrm{Leb}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))}dm_o(v^+). \tag{3.4}$$

Since $\{m_x\}$ is a G-invariant conformal density on $\partial\mathbb{H}^n$, the measure $\mu_{\tilde{E}}^{\mathrm{Leb}}$ is G-invariant; that is, $g_*\mu_{\tilde{E}}^{\mathrm{Leb}} = \mu_{g(\tilde{E})}^{\mathrm{Leb}}$. In particular, it is a $G_{\tilde{E}}$ invariant measure on $\tilde{E}$.

Define the Borel measure $\mu_{\tilde{E}}^{\mathrm{PS}}$ on $\tilde{E}$ such that

$$d\mu_{\tilde{E}}^{\mathrm{PS}}(v) = e^{\delta\beta_{v^+}(o,\pi(v))}d\nu_o(v^+). \tag{3.5}$$

We note that $\mu_{\tilde{E}}^{\mathrm{PS}}$ is a Γ-invariant measure.

As described in the section 2.6, we denote by μ_E^{Leb} and μ_E^{PS} the measures on $E = \mathbf{p}(\tilde{E})$ induced by $\mu_{\tilde{E}}^{\mathrm{Leb}}$ and $\mu_{\tilde{E}}^{\mathrm{PS}}$ respectively. Each of them is a pushforward of the corresponding locally finite measure on $\Gamma_{\tilde{E}}\backslash\tilde{E}$.

As in section 2.6.1, we have families of measures $\mu^{\mathrm{PS}} = \{\mu_{\mathcal{H}^+}^{\mathrm{PS}}\}$ and $\mu^{\mathrm{Leb}} = \{\mu_{\mathcal{H}^+}^{\mathrm{Leb}}\}$ on the unstable horospherical foliation satisfying

$$\mu_{\mathcal{G}^r(\mathcal{H}^+)}^{\mathrm{PS}}(\mathcal{G}^r(F)) = e^{\delta r}\mu_{\mathcal{H}^+}^{\mathrm{PS}}(F) \quad\text{and}\quad \mu_{\mathcal{G}^r(\mathcal{H}^+)}^{\mathrm{Leb}}(\mathcal{G}^r(F)) = e^{(n-1)r}\mu_{\mathcal{H}^+}^{\mathrm{Leb}}(F)$$

for any Borel subset F of $\mathbf{p}(\mathcal{H}^+)$.

3.4. **Transverse measures for $m^{\rm BMS}$.** For each measurable T contained in a weak stable leaf of the geodesic flow on $\mathrm{T}^1(\mathbb{H}^n)$, called a *transversal*, define a measure λ_T on T by

$$d\lambda_T(t) = e^{-\delta s}\, d\nu_o(t^-) ds \tag{3.6}$$

where $s = \beta_{t^-}(o, \pi(t))$. If $B = TP$ is any box and $p \in P$, then $(tp)^- = t^-$ and $\mathcal{H}^+_{tp} = \mathcal{H}^+_t$, and hence $\beta_{(tp)^-}(o, \pi(tp)) = \beta_{t^-}(o, \pi(t))$. Hence

$$d\lambda_{Tp}(tp) = d\lambda_T(t);$$

that is, λ_T is holonomy invariant, where the holonomy is given by $t \mapsto tp$.

Now for any $\Psi \in C(B)$, by (3.2)- (3.6), we have

$$\int_B f\, dm^{\rm BMS} = \int_T \int_P \Psi(tp)\, d\mu^{\rm PS}_{\mathcal{H}^+_t}(tp) d\lambda_T(t) \tag{3.7}$$

$$\int_B f\, dm^{\rm BR} = \int_T \int_P \Psi(tp)\, d\mu^{\rm Leb}_{\mathcal{H}^+_t}(tp) d\lambda_T(t). \tag{3.8}$$

3.4.1. *Backward admissible box.*

Lemma 3.1. *For any $u \in \mathrm{T}^1(\mathbb{H}^n)$ and $\epsilon > 0$, there exists a box $B = TP$ about u such that*

(1) $|\lambda_T| > 0$*; or equivalently* $\nu_o(\{t^- : t \in T\}) > 0$*, and*
(2) $\limsup_{r\to\infty} d(\mathcal{G}^r(tp), \mathcal{G}^r(t'p)) < \epsilon$*, for all* $t, t' \in T$ *and* $p \in P$*.*

Such a box B as above will be called a *backward admissible box with asymptotically ϵ-thin transversals.*

Proof. As in the proof of Lemma 2.17, there exists a relatively compact open neighborhood P^- of u in $\mathcal{H}^-_u$ such that $\nu_o(\{t^- : t \in P^-\}) > 0$, and $\mathbf{p}$ is injective on a neighborhood of the closure of P^-. Let $r_0 = -\log(\epsilon/4\,\mathrm{diam}(P^-))$. Then $\mathrm{diam}(\mathcal{G}^{r_0}(P^-)) = \epsilon/4$ and $\mathbf{p}$ is injective on a neighborhood of the closure of $\mathcal{G}^{r_0}(P^-)$. Let T_1 be an open relatively compact neighborhood of $\mathcal{G}^{r_0}(P^-)$ in $\mathrm{Viz}^{-1}(u^+)$ and P_1 be an open relatively compact neighborhood of $\mathcal{G}^{r_0}(u)$ in $\mathcal{H}^+_{\mathcal{G}^{r_0}(u)}$ such that T_1P_1 is a box about $\mathcal{G}^{r_0}(u)$ contained in a ball of radius $\epsilon/2$ about u. Let $T = \mathcal{G}^{-r_0}(T_1)$ and $P = \mathcal{G}^{-r_0}(P_1)$. Then $B = TP$ has the required properties. The property (1) holds because

$$\{t^- : t \in T\} = \{t^- : t \in T_1\} \supset \{t^- : t \in \mathcal{G}^{r_0}(P^-)\} = \{t^- : t \in P^-\}.$$

For the property (2), let $t_1 = \mathcal{G}^{r_0}(t)$ and $t_1' = \mathcal{G}^{r_0}(t')$ in T_1 and $p_1 = \mathcal{G}^{r_0}(p) \in P_1$. Since $(t_1p_1)^+ = (t_1'p_1)^+$, for any $r > r_0$,

$$d(\mathcal{G}^r(tp), \mathcal{G}^r(t'p)) = d(\mathcal{G}^{r-r_0}(t_1p_1), \mathcal{G}^{r-r_0}(t_1'p_1)) \le d(t_1p_1, t_1'p_1) \le \epsilon.$$

□

3.5. **Mixing of geodesic flow.** We assume that $|m^{\mathrm{BMS}}| < \infty$ for the rest of this section. This implies that Γ is of divergent type, that is, $\sum_{\gamma\in\Gamma} e^{-\delta d(o,\gamma o)} = \infty$ and that the Γ-invariant conformal density of dimension δ is unique up to homothety (see [31, Coro.1.8]).

Hence, up to homothety, ν_x is the weak-limit as $s \to \delta^+$ of the family of measures

$$\nu_{x,o}(s) := \frac{1}{\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)}} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma o)} \delta_{\gamma o},$$

where $\delta_{\gamma o}$ denotes the unit mass at γo for some $o \in \mathbb{H}^n$.

The most crucial ergodic theoretic result involved in this work is the mixing of geodesic flow which was obtained by Rudolph for Γ geometrically finite and by Babillot in general:

Theorem 3.2 (Rudolph [32], Roblin [31], Babillot [1])**.** *For any $\Psi_1 \in L^2(\mathrm{T}^1(X), m^{\mathrm{BMS}})$ and $\Psi_2 \in L^2(\mathrm{T}^1(X), m^{\mathrm{BMS}})$,*

$$\lim_{r\to\infty} \int_{\mathrm{T}^1(X)} \Psi_1(\mathcal{G}^r(x))\Psi_2(x)\, dm^{\mathrm{BMS}}(x) = \frac{1}{|m^{\mathrm{BMS}}|}\, m^{\mathrm{BMS}}(\Psi_1)\cdot m^{\mathrm{BMS}}(\Psi_2).$$

From this theorem, we derive the following result, which generalizes the corresponding result for PS-measures on unstable horospheres due to Roblin [31, Corollary 3.2].

Theorem 3.3. *For any $\Psi \in C_c(\mathrm{T}^1(X))$ and $f \in L^1(E, \mu_E^{\mathrm{PS}})$,*

$$\lim_{r\to\infty} \int_{x\in E} \Psi(\mathcal{G}^r(x)) f(x)\, d\mu_E^{\mathrm{PS}}(x) = \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi). \tag{3.9}$$

We will deduce the above statement from its following version.

Proposition 3.4. *Let $\Psi \in L^1(\mathrm{T}^1(X), m^{\mathrm{BMS}})$ and $f \in L^1(E, \mu_E^{\mathrm{PS}})$, both nonnegative, bounded and vanish outside compact sets. Then for any $\epsilon > 0$,*

$$\limsup_{r\to\infty} \int_{x\in E} \Psi(\mathcal{G}^r(x)) f(x)\, d\mu_E^{\mathrm{PS}}(x) \le \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi_\epsilon^+) \tag{3.10}$$

$$\liminf_{r\to\infty} \int_{x\in E} \Psi(\mathcal{G}^r(x)) f(x)\, d\mu_E^{\mathrm{PS}}(x) \ge \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BMS}}(\Psi_\epsilon^-), \tag{3.11}$$

where, for any $u \in \mathrm{T}^1(\mathbb{H}^n)$,

$$\begin{aligned} \Psi_\epsilon^+(\mathbf{p}(u)) &:= \sup\{\Psi(\mathbf{p}(v)) : d(v,u) < \epsilon,\ v \in \mathrm{Viz}^{-1}(u^+)\}, \\ \Psi_\epsilon^-(\mathbf{p}(u)) &:= \inf\{\Psi(\mathbf{p}(v)) : d(v,u) < \epsilon,\ v \in \mathrm{Viz}^{-1}(u^+)\}. \end{aligned} \tag{3.12}$$

Proof. By Lemma 3.1, there exists a finite open cover $\mathcal{B}$ of $\mathrm{supp}(f) \subset E \subset \mathrm{T}^1(X)$ consisting of backward admissible boxes B with asymptotically ϵ-thin transversals; we identify $B \subset \mathrm{T}^1(\mathbb{H}^n)$ with $\mathbf{p}(B)$. By considering a partition of unity subordinate to this cover, $f = \sum_{B\in\mathcal{B}} \phi_B$, where $\phi_B \in L^1(E, \mu_E^{\mathrm{PS}})$ is a non-negative function whose support is contained in $\mathbf{p}(B)$. Therefore it is enough to prove (3.10) and (3.11) for ϕ_B in place of f for each $B \in \mathcal{B}$.

Fix any $B \in \mathcal{B}$. For each $[\gamma] \in \Gamma/\Gamma_{\tilde{E}}$, let $\phi_\gamma(w) = \phi_B(w)$ for all $w \in \gamma\tilde{E}$. By (2.14),

$$\mu_E^{\mathrm{PS}}(\phi_B) = \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \mu_{\gamma\tilde{E}}^{\mathrm{PS}}(\phi_\gamma), \text{ and}$$

$$\int_{x\in E} \Psi(\mathcal{G}^r(x))\phi_B(x)\, d\mu_E^{\mathrm{PS}}(x) = \sum_{[\gamma]\in\Gamma/\Gamma_{\tilde{E}}} \int_{w\in\gamma\tilde{E}\cap B} \Psi(\mathcal{G}^r(w))\phi_\gamma(w)\, d\mu_{\gamma\tilde{E}}^{\mathrm{PS}}(w).$$

Therefore to prove (3.10) and (3.11) for ϕ_B in place of f, it is enough to prove the following: for any $\gamma \in \Gamma$ and $\phi := \phi_\gamma \in L^1(\gamma\tilde{E}, \mu_{\gamma\tilde{E}}^{\mathrm{PS}})$ vanishing outside $\gamma\tilde{E} \cap B$, we have

$$\limsup_{r\to\infty} \int_{w\in\gamma\tilde{E}\cap B} \Psi(\mathcal{G}^r(w))\phi(w)\, d\mu_{\gamma\tilde{E}}^{\mathrm{PS}}(w) \le \frac{\mu_{\gamma\tilde{E}}^{\mathrm{PS}}(\phi)}{|m^{\mathrm{BMS}}|} m^{\mathrm{BMS}}(\Psi_\epsilon^+); \quad (3.13)$$

$$\liminf_{r\to\infty} \int_{w\in\gamma\tilde{E}\cap B} \Psi(\mathcal{G}^r(w))\phi(w)\, d\mu_{\gamma\tilde{E}}^{\mathrm{PS}}(w) \ge \frac{\mu_{\gamma\tilde{E}}^{\mathrm{PS}}(\phi)}{|m^{\mathrm{BMS}}|} m^{\mathrm{BMS}}(\Psi_\epsilon^-). \quad (3.14)$$

Now we express $B = TP$. If $\gamma\tilde{E}\cap B = \emptyset$, then both sides of (3.13) are zero and hence the claim is true. Otherwise, there exists $(t_1, p_1) \in T\times P$ such that $v := t_1p_1 \in \gamma\tilde{E}$. We recall that as in §2.3.1, $\xi_v : \mathcal{H}_v^+ \smallsetminus (\gamma\cdot \mathrm{Viz}^{-1}(\partial\tilde{S})) \to \gamma\tilde{E} \smallsetminus \{-v\}$ and $q_v : \gamma\tilde{E} \smallsetminus \{-v\} \to \mathcal{H}_v^+ \smallsetminus (\gamma\cdot\mathrm{Viz}^{-1}(\partial\tilde{S}))$ are differentiable inverses of each other.

Letting

$$P_1 = \{p \in P : \xi_v(t_1p) \in Tp\},$$

we claim that

$$\gamma\tilde{E} \cap B = \{\xi_v(t_1p) : p \in P_1\}. \quad (3.15)$$

To see this, if $tp \in \gamma\tilde{E}$ for some $(t,p) \in T \times P$, then

$$q_v(tp) = \mathcal{H}_v^+ \cap \mathrm{Viz}^{-1}((tp)^+) = \mathcal{H}_{t_1}^+ \cap \mathrm{Viz}^{-1}(p^+) = t_1p.$$

Hence $\xi_v(t_1p) = tp$, and so $p \in P_1$. The opposite inclusion is obvious.

We define a map $\rho : TP \to \gamma\tilde{E}$ as follows:

$$\rho(tp) = \xi_v(t_1p), \text{ for all } (t,p) \in T \times P.$$

For any $t \in T$, for the restricted map $\rho : tP \to \gamma\tilde{E}$, by (2.12) and (2.15), since $(tp)^+ = p^+ = \rho(tp)^+$, we have

$$d\mu^{\mathrm{PS}}_{\gamma\tilde{E}}(\rho(tp))/d\mu^{\mathrm{PS}}_{\mathcal{H}^+_t}(tp) = e^{\beta_{p^+}(\pi(tp),\pi(\rho(tp)))}. \tag{3.16}$$

In view of this, we define $\Phi \in L^2(\mathrm{T}^1(X), \mu^{\mathrm{BMS}})$ as follows: $\Phi(x) = 0$ if $x \in X \smallsetminus B$ and

$$\Phi(tp) = \phi(\rho(tp))e^{\beta_{p^+}(\pi(tp),\pi(\rho(tp)))}, \text{ if } x = tp \in B. \tag{3.17}$$

We note that

$$\Phi(tp) \neq 0 \Rightarrow \rho(tp) \in B \Rightarrow p \in P_1. \tag{3.18}$$

And for $t \in T$ and $p \in P_1$, we have $\{\rho(tp), tp\} \subset Tp$. Since $\mathcal{G}^r(B)$ has ϵ-thin transversals as $r \to \infty$ (see Lemma 3.1(2)):

$$\limsup_{r\to\infty} d(\mathcal{G}^r(\rho(tp)), \mathcal{G}^r(tp)) \leq \epsilon \text{ for all } p \in P_1. \tag{3.19}$$

By Theorem 3.2,

$$\frac{1}{|m^{\mathrm{BMS}}|}\, m^{\mathrm{BMS}}(\Psi^+_\epsilon) \cdot m^{\mathrm{BMS}}(\Phi) \tag{3.20}$$

$$= \lim_{r\to\infty} \int_B \Psi^+_\epsilon(\mathcal{G}^r(x))\Phi(x)\, dm^{\mathrm{BMS}}(x) \tag{3.21}$$

$$= \lim_{r\to\infty} \int_{t\in T} \left(\int_{p\in P_1} \Psi^+_\epsilon(\mathcal{G}^r(tp))\Phi(tp)\, d\mu^{\mathrm{PS}}_{\mathcal{H}^+_t}(tp) \right) d\lambda_T(t) \tag{3.22}$$

$$= \lim_{r\to\infty} \int_{t\in T} \left(\int_{p\in P_1} \Psi^+_\epsilon(\mathcal{G}^r(tp))\phi(\rho(tp))\, d\mu^{\mathrm{PS}}_{\gamma\tilde{E}}(\rho(tp)) \right) d\lambda_T(t) \tag{3.23}$$

$$\geq |\lambda_T| \cdot \limsup_{r\to\infty} \int_{w\in\gamma\tilde{E}\cap B} \Psi(\mathcal{G}^r(w))\phi(w)\, d\mu^{\mathrm{PS}}_{\gamma\tilde{E}}(w) \tag{3.24}$$

where (3.22) follows from (3.7) and (3.18), (3.23) follows from (3.15), (3.16) and (3.17), and to justify (3.24) we put $w = \rho(tp)$ and use (3.12) and (3.19).

By putting $\Psi(x) = 1 = \Psi^+_\epsilon(x)$ in (3.21)–(3.24) with equality in (3.24), we get

$$m^{\mathrm{BMS}}(\Phi) = |\lambda_T| \cdot \mu^{\mathrm{PS}}_{\gamma\tilde{E}}(\phi) < \infty. \tag{3.25}$$

Now (3.13) is deduced by comparing (3.20), (3.24) and (3.25), and noting that $|\lambda_T| \neq 0$ by the backward admissibility of B. Similarly we can deduce (3.14). □

Proof of Theorem 3.3. Since both the sides of (3.9) are linear in Ψ, it is enough to prove it for $\Psi \geq 0$. Since Ψ is uniformly continuous and $|m^{\mathrm{BMS}}| < \infty$,

$$\lim_{\epsilon\to 0} m^{\mathrm{BMS}}(\Psi^+_\epsilon - \Psi^-_\epsilon) = 0.$$

Therefore by Proposition 3.4, we have that (3.9) holds for all non-negative bounded measurable f with compact support on E. Since the set of such f's is dense in $L^1(E, \mu_E^{\mathrm{PS}})$ and both sides of (3.9) are linear and continuous in $f \in L^1(E, \mu_E^{\mathrm{PS}})$, and (3.9) holds for all $f \in L^1(E, \mu_E^{\mathrm{PS}})$. $\square$

The following result is one of the basic tools developed in this article.

Theorem 3.5 (Transversal equidistribution)**.** *Let $f \in L^1(E, \mu_E^{\mathrm{PS}})$ such that $\mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-) \to 0$ as $\epsilon \to 0$. Let $\psi \in C_c(T)$ for a transversal T of a box B (§2.4). Then*

$$\lim_{r\to\infty} e^{-\delta r} \sum_{t \in T \cap \mathcal{G}^r(E)} \#(\bar{\Gamma}_{r,t}) \cdot \psi(t) \cdot f(\mathcal{G}^{-r}(t)) = \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|} \cdot \lambda_T(\psi), \quad (3.26)$$

where

$$f_\epsilon^+(x) = \sup_{\{y \in E : d(y,x) < \epsilon\}} f(y) \text{ and } f_\epsilon^-(x) = \inf_{\{y \in E : d(y,x) < \epsilon\}} f(y),$$

the transverse measure λ_T is defined by (3.6) *and $\bar{\Gamma}_{r,t}$ is defined by* (2.27)*.*

Proof. Since both sides of (3.26) are linear in f and in ψ, without loss of generality we may assume that $f \geq 0$ and $\psi \geq 0$. By Lemma 2.17, $\operatorname{supp}(\psi)$ can be covered by finitely many admissible boxes. By a partition of unity argument, in view of Remark 2.5, we may assume without loss of generality that T is a transversal of an admissible box B.

Let $\epsilon_0 > 0$ be such that $\mathbf{p}$ is injective on B_{ϵ_0+} and that ψ vanishes outside T_{ϵ_0-}. We extend ψ to a continuous function on T_{ϵ_0+} by putting $\psi = 0$ on $T_{\epsilon_0+} \smallsetminus T$. Since B is admissible, due to Lemma 2.15, if we define

$$\Psi(tp) = \psi(t)/\mu_{\mathcal{H}_t^+}^{\mathrm{PS}}(tP), \text{ for all } (t,p) \in T_{\epsilon_0+} \times P,$$

then Ψ is a bounded continuous function on B_{ϵ_0+} vanishing outside B_{ϵ_0-}. If $\Psi_\epsilon^\pm \in C(B_{\epsilon+})$ are defined as in (3.12) for $0 < \epsilon \leq \epsilon_0$, then

$$\lim_{\epsilon\to 0} \|\Psi_\epsilon^+ - \Psi_\epsilon^-\|_\infty = 0. \quad (3.27)$$

By Proposition 3.4,

$$\begin{aligned} \limsup_{r\to\infty} \int_E \Psi_\epsilon^+(\mathcal{G}^r(v)) f_\epsilon^+(v)\, d\mu_E^{\mathrm{PS}}(v) &\leq \tfrac{\mu_E^{\mathrm{PS}}(f_\epsilon^+) m^{\mathrm{BMS}}(\Psi_\epsilon^+)}{|m^{\mathrm{BMS}}|}, \\ \liminf_{r\to\infty} \int_E \Psi_\epsilon^-(\mathcal{G}^r(v)) f_\epsilon^-(v)\, d\mu_E^{\mathrm{PS}}(v) &\geq \tfrac{\mu_E^{\mathrm{PS}}(f_\epsilon^-) m^{\mathrm{BMS}}(\Psi_\epsilon^-)}{|m^{\mathrm{BMS}}|}. \end{aligned} \quad (3.28)$$

Since $m^{\mathrm{BMS}}(B_{\epsilon_0+}) < \infty$, by (3.27), we have that $m^{\mathrm{BMS}}(\Psi_\epsilon^+ - \Psi_\epsilon^-) \to 0$ as $\epsilon \to 0$. By our assumption, $\mu_E^{\mathrm{PS}}(|f_\epsilon^\pm - f|) \to 0$ as $\epsilon \to 0$. Therefore by Proposition 2.18 and (3.28),

$$\lim_{r\to\infty} e^{-\delta r} \sum_{t\in T\cap\mathcal{G}^r(E)} \#(\bar\Gamma_{r,t})\cdot\psi(t)\cdot f(\mathcal{G}^{-r}(t)) = \frac{\mu_E^{\mathrm{PS}}(f)m^{\mathrm{BMS}}(\Psi)}{|m^{\mathrm{BMS}}|}.$$

And

$$m^{\mathrm{BMS}}(\Psi) = \int_T d\mu_T(t)\left(\int_{tP}\Psi(tp)d\mu^{\mathrm{PS}}_{\mathcal{H}_t^+}\right) = \lambda_T(\psi).$$

□

Now we state and prove the main equidistribution result of this article which is more general than Theorem 1.8.

Theorem 3.6. *Let $f\in L^1(E,\mu_E^{\mathrm{PS}})$ such that $\mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-)\to 0$ as $\epsilon\to 0$. Let $\Psi\in C_c(\mathrm{T}^1(X))$. Then*

$$\lim_{r\to\infty} e^{(n-1-\delta)r}\int_{u\in E}\Psi(\mathcal{G}^r(u))f(u)\,d\mu_E^{\mathrm{Leb}}(u) = \frac{\mu_E^{\mathrm{PS}}(f)}{|m^{\mathrm{BMS}}|}m^{\mathrm{BR}}(\Psi).$$

In particular, the result applies to $f=\chi_F$ for a Borel measurable $F\subset E$ such that $\mu_E^{\mathrm{PS}}(F_{\epsilon_1})<\infty$ for some $\epsilon_1>0$ and $\mu_E^{\mathrm{PS}}(\partial F)=0$.

Proof. By Lemma 2.17, the boxes admissible with respect to $\{\mu^{\mathrm{PS}}_{\mathcal{H}^+}\}$ form a basis of open sets in $\mathrm{T}^1(X)$. By a partition of unity argument, without loss of generality we may assume that $\mathrm{supp}(\Psi)\subset B$ for an admissible box $B=TP$. Let $\epsilon_0>0$ be such that $\Psi=0$ outside $B_{\epsilon_0 -}$. For $0<\epsilon\le\epsilon_0$, let $\Psi_\epsilon^\pm$ be defined as in (2.18). Then

$$\lim_{\epsilon\to 0}\|\Psi_\epsilon^+ - \Psi_\epsilon^-\|_\infty = 0. \tag{3.29}$$

For $t\in T_{\epsilon_0}$, and $\epsilon>0$, define $\psi_\epsilon^\pm(t) = \int_{tP}\Psi_\epsilon^\pm\,d\mu^{\mathrm{Leb}}_{\mathcal{H}_t^+}$. By Lemma 2.15, $\psi_\epsilon^\pm\in C_c(T)$ for any $0<\epsilon<\epsilon_0/2$.

For the conformal density, $\{\mu_x\}=\{m_x\}$, we have $\delta_\mu = n-1$, and by multiplying all the terms in the conclusion of Corollary 2.14 by $e^{-\delta r}$, for $r>r_\epsilon$ (see (2.17)), we get

$$\begin{aligned}
&(e^{-(n-1)\epsilon})e^{-\delta r}\sum_{t\in T\cap\mathcal{G}^r(E)}\#(\bar\Gamma_{r,t})\cdot\psi_\epsilon^-(t)\cdot f_\epsilon^-(\mathcal{G}^{-r}(t))\\
\le& e^{(n-1-\delta)r}\int_E\Psi(\mathcal{G}^r(u))f(u)\,d\mu_E^{\mathrm{Leb}}(u)\\
\le&(e^{(n-1)\epsilon})e^{-\delta r}\sum_{t\in T\cap\mathcal{G}^r(E)}\#(\bar\Gamma_{r,t})\cdot\psi_\epsilon^+(t)\cdot f_\epsilon^+(\mathcal{G}^{-r}(t)).
\end{aligned}$$

Define $\psi(t) := \int_{tP}\Psi(tp)\,d\mu^{\mathrm{Leb}}_{\mathcal{H}_t^+}$ for all $t\in T$. Then $\lambda_T(\psi)=m^{\mathrm{BR}}(\Psi)$ and $\lambda_T(\psi_\epsilon^\pm)=m^{\mathrm{BR}}(\Psi_\epsilon^\pm)$. Since $m^{\mathrm{BMS}}(B_{\epsilon_0+})<\infty$, by (3.29),

$$\lambda_T(\psi^+)-\lambda_T(\psi^-) = m^{\mathrm{BR}}(\Psi_\epsilon^+ - \Psi_\epsilon^-)\to 0,\ \text{as } \epsilon\to 0.$$

And since $\mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-) \to 0$, by Theorem 3.5

$$\lim_{r\to\infty} e^{(n-1-\delta)r} \int_E \Psi(\mathcal{G}^r(u)) f(u)\, d\mu_E^{\mathrm{Leb}}(u) = \frac{\mu_E^{\mathrm{PS}}(f)\lambda_T(\psi)}{|m^{\mathrm{BMS}}|}.$$

Since $\lambda_T(\psi) = m^{\mathrm{BR}}(\Psi)$, we prove the claim.

In the particular case of $f = \chi_F$, we have

$$\inf_{\epsilon>0} f_\epsilon^+ = \chi_{\overline{F}} \quad \text{and} \quad \sup_{\epsilon>0} f_\epsilon^- = \chi_{\mathrm{int}(F)}, \text{ and}$$

if $\mu_E^{\mathrm{PS}}(f_{\epsilon_1}^+) = \mu_E^{\mathrm{PS}}(F_{\epsilon_1}) < \infty$, then $\lim_{\epsilon\to 0} \mu_E^{\mathrm{PS}}(f_\epsilon^+ - f_\epsilon^-) = \mu_E^{\mathrm{PS}}(\partial F)$. □

The idea of the above proof was influenced by the work of Schapira [34].

Our proof also yields the following variation of Theorem 3.6.

Theorem 3.7. *Let $\tilde F \subset \tilde E$ be a Borel subset such that $\mu_{\tilde E}^{\mathrm{PS}}(\tilde F_\epsilon) < \infty$ for some $\epsilon > 0$ and $\mu_{\tilde E}^{\mathrm{PS}}(\partial \tilde F) = 0$. Then for any $\psi \in C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$,*

$$\lim_{t\to+\infty} e^{(n-1-\delta)t} \cdot \int_{\tilde F} \psi(\mathcal{G}^t(v))\, d\mu_{\tilde E}^{\mathrm{Leb}}(v) = \frac{\mu_{\tilde E}^{\mathrm{PS}}(\tilde F)}{|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BR}}(\psi).$$

3.6. **Integrability of the base eigenfunction ϕ_0.**

Proof of theorem 1.17. We want to prove equivalence of the following:

(1) $\phi_0 \in L^1(\Gamma\backslash\mathbb{H}^n, d\,\mathrm{Vol}_{\mathrm{Riem}})$;
(2) $|m^{\mathrm{BR}}| < \infty$;
(3) Γ is a lattice in G.

The pushforward of m^{BR} from $\Gamma\backslash\,\mathrm{T}^1(\mathbb{H}^1)$ to $\Gamma\backslash\mathbb{H}^n$ is the measure corresponding to $\phi_0\, d\,\mathrm{Vol}_{\mathrm{Riem}}$ (see [17, Lemma 6.7]). Therefore (1) and (2) are equivalent.

To prove that (2) implies (3), suppose that $|m^{\mathrm{BR}}| < \infty$. Since the left G-action on $\mathrm{T}^1(\mathbb{H}^n)$ is transitive, we may identify $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ with $\Gamma\backslash G/M$ for a compact subgroup M. We lift the measure m^{BR} to a measure m on $\Gamma\backslash G$ as follows: for any $f \in C_c(\Gamma\backslash G)$, we define $m(f) = m^{\mathrm{BR}}(\bar f)$, where $\bar f(\Gamma g M) = \int_{x\in M} f(\Gamma g x)\, dx$, where dx is the probability Haar measure on M. Denote by U the horospherical subgroup of G whose orbits in G projects to the unstable horospheres in $\mathrm{T}^1(\mathbb{H}^n)$. Then M normalizes U and any unimodular proper closed subgroup of G containing U is contained in the subgroup MU. As m is invariant under $G_{\mathcal{H}^+}$ for any unstable horosphere $\mathcal{H}^+$, it follows that m is a U-invariant *finite* measure on $\Gamma\backslash G$. By Ratner's theorem [30], any ergodic component, say, λ, of m is a homogeneous measure in the sense that λ is a H-invariant finite measure supported on a closed orbit $x_0 H$ for some $x_0 \in \Gamma\backslash G$ and a unimodular closed subgroup H of G containing U. If $H \neq G$, then $H \subset MU$ and $\Gamma \cap H$ is co-compact in H. It follows by a theorem of Bieberbach ([4, Theorem 2.25]) that $\Gamma \cap U$ is co-compact in U. Hence $H = U$. Thus we can write $m = m_1 + m_2$, where m_1 is G-invariant and m_2 is supported on a union of compact U-orbits.

If $m_1 = 0$, then $m = m_2$, and hence the projection of the support of m^{BR} in $\mathrm{T}^1(\mathbb{H}^n)$ is a union of compact unstable horospheres. It follows that the Patterson-Sullivan density is concentrated on the set of parabolic fixed points of Γ, which is a contradiction.

If $m_1 \neq 0$, then m_1 is a finite G-invariant measure on $\Gamma\backslash G$; that is, Γ is a lattice in G. Hence (2) implies (3).

If Γ is a lattice, then $\{\nu_x\} = \{m_x\}$ up to a constant multiple. Hence m^{BR} is the projection of a finite G-invariant measure of $\Gamma\backslash G$ to $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$. Hence (3) implies (2). □

4. Geometric finiteness of closed totally geodesic immersions

for obtaining the criterion for the finiteness of μ_E^{PS} in §6.

4.1. **Parabolic fixed points and minimal subspaces.** Let Γ be a torsion free discrete subgroup of G.

Definition 4.1. An element $g \in G$ is called *parabolic* if $\mathrm{Fix}(g) := \{\xi \in \partial\mathbb{H}^n : g\xi = \xi\}$ is a singleton set. An element $\xi \in \partial\mathbb{H}^n$ is called a *parabolic fixed point of* Γ if there exists a parabolic element $\gamma \in \Gamma$ such that $\mathrm{Fix}(\gamma) = \{\xi\}$. Note that if ξ is a parabolic fixed point for Γ, then $\xi \in \Lambda(\Gamma)$. Let $\Lambda_{\mathrm{p}}(\Gamma)$ denote the set of parabolic fixed points of Γ.

Let $\xi \in \Lambda_{\mathrm{p}}(\Gamma)$. In order to analyze the action of Γ_ξ on $\partial\mathbb{H}^n \smallsetminus \{\xi\}$, it is convenient to use the upper half space model $\mathbb{R}^n_+ = \{(x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$ for $\mathbb{H}^n$, where ξ corresponds to ∞ and $\partial\mathbb{H}^n \smallsetminus \{\xi\}$ corresponds to $\partial\mathbb{R}^n_+ = \{(x,0) : x \in \mathbb{R}^{n-1}\}$. The subgroup Γ_∞ acts properly discontinuously via affine isometries on $\partial\mathbb{H}^n \smallsetminus \{\infty\} \cong \mathbb{R}^{n-1}$; at this stage we will treat $\mathbb{R}^{n-1}$ only as an affine space, and we will choose its origin 0 later. Moreover the action of Γ_∞ preserves every horosphere $\mathbb{R}^{n-1} \times \{y\}$, where $y > 0$, based at ∞.

By a theorem of Bieberbach ([4, 2.2.5]), Γ_∞ contains a normal abelian subgroup of finite index, say Γ'_∞. By [4, 2.1.5], any (nonempty) Γ'_∞-invariant affine subspace of $\mathbb{R}^{n-1}$ contains a (nonempty) minimal Γ'_∞-invariant affine subspace, we call such an affine subspace a Γ'_∞*-minimal subspace*. By [4, 2.2.6], Γ'_∞ acts cocompactly via translations on any Γ'_∞-minimal subspace. Moreover, any two Γ'_∞-minimal subspaces are parallel, and if v_1 and v_2 belong to any two Γ'_∞-minimal subspaces, then $\gamma v_1 - \gamma v_2 = v_1 - v_2$ for all $\gamma \in \Gamma'_\infty$. Let $\mathrm{rank}(\Gamma_\infty)$ denote the rank of the (torsion free) $\mathbb{Z}$-module Γ'_∞; it is independent of the choice of Γ'_∞, and it equals the dimension of a Γ'_∞-minimal subspace.

Definition 4.2. A parabolic fixed point $\xi \in \Lambda_{\rm p}(\Gamma)$ is said to be *bounded* if $\Gamma_\xi \backslash (\Lambda(\Gamma) \smallsetminus \{\xi\})$ is compact. Denote by $\Lambda_{\rm bp}(\Gamma)$ the set of all bounded parabolic fixed points for Γ. Therefore $\infty \in \Lambda_{\rm bp}(\Gamma)$ *if and only if* $\infty \in \Lambda_{\rm p}(\Gamma)$ and

$$\Lambda(\Gamma) \smallsetminus \{\infty\} \subset \{x \in \mathbb{R}^{n-1} : d_{\rm Euc}(x, L) \leq r_0\}, \tag{4.1}$$

for some $r_0 > 0$, where L is a Γ'_∞-minimal subspace.

4.2. **On geometric finiteness of $\Gamma_{\tilde{S}}$.** For the rest of this section, let $\tilde{S}$ be a proper connected totally geodesic subspace of $\mathbb{H}^n$ such that the natural projection map $\Gamma_{\tilde{S}} \backslash \tilde{S} \to X = \Gamma \backslash \mathbb{H}^n$ is proper, or equivalently, the map $\Gamma_{\tilde{S}} \backslash G_{\tilde{S}} \to \Gamma \backslash G$ is proper, or equivalently $\Gamma G_{\tilde{S}}$ is closed in G. Since $\tilde{S}$ is totally geodesic, the geometric boundary $\partial \tilde{S}$ is the intersection of $\partial \mathbb{H}^n$ with the closure of $\tilde{S}$ in $\overline{\mathbb{H}^n}$.

Proposition 4.3. *Let $\infty \in \Lambda_{\rm p}(\Gamma) \cap \partial \tilde{S}$. Let L be a Γ'_∞-minimal subspace of $\partial \mathbb{H}^n \smallsetminus \{\infty\} \cong \mathbb{R}^{n-1}$ and choose the origin $0 \in L$. Then the intersection of L with the (parallel) translate of the affine subspace $\partial \tilde{S} \smallsetminus \{\infty\}$ through 0 is a $(\Gamma'_\infty \cap G_{\tilde{S}})$-minimal subspace.*

Proof. Let $\Gamma' = \Gamma'_\infty$, $\Delta = \Gamma' \cap G_{\tilde{S}}$, and the affine subspace $F = \partial \tilde{S} \smallsetminus \{\infty\}$. Since $\Delta F = F$, let v belong to a Δ-minimal subspace of F. Since v and 0 belong to two Δ-minimal subspaces, $\gamma v - \gamma 0 = v - 0 = v$. Since $\gamma v \in F$, we have $\gamma 0 = \gamma v - v \in F - v$. Since $0 \in F - v$, we have $\gamma 0 \in \gamma(F - v) \cap (F - v)$. Now $\gamma(F - v)$ and $\gamma F = F$ are parallel. Therefore $F - v$ and $\gamma(F - v)$ are parallel, and since they intersect, $\gamma(F - v) = F - v$. Thus $\Delta(F - v) = F - v$. Therefore Δ-action preserves $L_0 := L \cap (F - v)$. We want to prove that $\Gamma' \cap G_{\tilde{S}}$ acts cocompactly on L_0.

Since $\infty \in \Lambda_{\rm p}(\Gamma)$, by [4, Lemma 3.2.1] Γ_∞ consists of parabolic elements of G_∞; that is $\Gamma_\infty \subset MN$, where N is the maximal unipotent subgroup of G which acts transitively on $\mathbb{R}^{n-1} = \partial \mathbb{H}^n \smallsetminus \{\infty\}$ via translations and M is a compact subgroup of G normalizing N and acts on $\mathbb{R}^{n-1}$ by Euclidean isometries fixing 0. Let $U = \{g \in N : gL = L\}$. Then U acts transitively on L by translations. Since $0 \in L$ and Γ' acts cocompactly on L via translations, the connected component of the Zariski closure of Γ' in G is a connected abelian subgroup of the form $M_L U$, where $M_L \subset M$ and M_L acts trivially on L.

Since $\Gamma' \backslash L$ is a compact Euclidean torus, the closure of the image of L_0 in $\Gamma' \backslash L$ equals the image of an affine subspace, say L_1, of L. Thus $\overline{\Gamma' L_0} = \Gamma' L_1$. For $i = 0, 1$, let $U_i = \{u \in U : uL_i = L_i\}$.

Then U_i acts transitively on L_i, and $\overline{\Gamma' M_L U_0} = \Gamma' M_L U_1$. Therefore the identity component of $\overline{\Gamma' U_0}$ is of the form $M_1 U_1$, where $M_1 \subset M_L$ and $(\Gamma' \cap M_1 U_1)\backslash M_1 U_1$ is compact. In particular, $\Gamma' \cap M_1 U_1$ acts cocompactly on L_1.

By our assumption $\Gamma G_{\tilde{S}}$ is a closed subset of G. Therefore $\overline{\Gamma' U_0} \subset \Gamma G_{\tilde{S}}$. Since $G_{\tilde{S}}$ is the identity component in $\Gamma G_{\tilde{S}}$, we have $M_1 U_1 \subset G_{\tilde{S}}$. It follows that U_1 preserves L_0. Since U_1 acts transitively on L_1, $L_1 \subset L_0$; hence $L_1 = L_0$. In particular, $\Gamma' \cap M_1 U_1$ acts cocompactly on L_0. Therefore $\Delta = \Gamma' \cap G_{\tilde{S}}$ acts cocompactly on L_0. □

Proposition 4.4. *Let $\infty \in \Lambda_{\mathrm{bp}}(\Gamma) \cap \partial\tilde{S}$ and $\Gamma_{\tilde{S}} := \Gamma \cap G_{\tilde{S}}$. Then*

$$\begin{cases} \infty \in \Lambda_{\mathrm{bp}}(\Gamma_{\tilde{S}}) & \text{if } \Gamma_\infty \cap \Gamma_{\tilde{S}} \text{ is infinite;} \\ \infty \notin \Lambda(\Gamma_{\tilde{S}}) & \text{if } \Gamma_\infty \cap \Gamma_{\tilde{S}} \text{ is finite, hence trivial.} \end{cases}$$

Proof. Let the notation be as in Proposition 4.3. Since $\infty \in \Lambda_{\mathrm{bp}}(\Gamma)$, by (4.1), $\Lambda(\Gamma) \smallsetminus \{\infty\}$ is contained in a bounded neighborhood of L, and hence in a bounded neighborhood of $L+v$. Therefore $(\Lambda(\Gamma)\smallsetminus\{\infty\})\cap\partial\tilde{S}$ is contained in a bounded neighborhood of $L+v$ intersected with $F = \partial\tilde{S} \smallsetminus \{\infty\}$, and hence in a bounded neighborhood of $L_0 = L \cap (F - v)$ as well. By Proposition 4.3, L_0 is a $(\Gamma_{\tilde{S}} \cap \Gamma')$-minimal subspace. Now if $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ is infinite, or equivalently $\infty \in \Lambda_{\mathrm{p}}(\Gamma_{\tilde{S}})$, then $\infty \in \Lambda_{\mathrm{bp}}(\Gamma_{\tilde{S}})$.

Suppose that $\Gamma_\infty \cap \Gamma_{\tilde{S}}$ is finite. Then L_0 is a singleton set. Therefore $\Lambda(\Gamma_{\tilde{S}}) \smallsetminus \{\infty\}$ is contained in a bounded subset of $\partial\tilde{S} \smallsetminus \{\infty\}$. Then $\infty \in \partial\tilde{S}$ is isolated from $\Lambda(\Gamma_{\tilde{S}})$. Since the limit set of a non-elementary hyperbolic group is perfect, it follows that $\Gamma_{\tilde{S}}$ is elementary, and hence $\Gamma_{\tilde{S}}$ is either parabolic or loxodromic. Now suppose that $\infty \in \Lambda(\Gamma_{\tilde{S}})$. In the parabolic case $\Lambda(\Gamma_{\tilde{S}}) = \{\infty\} = \Lambda_{\mathrm{p}}(\Gamma_{\tilde{S}})$, contradicting the assumption that $\Gamma_\infty \cap \Gamma_{\tilde{S}} = \{e\}$. In the loxodromic case, $\infty \in \Lambda_{\mathrm{r}}(\Gamma_{\tilde{S}}) \subset \Lambda_{\mathrm{r}}(\Gamma)$, contradicting the assumption that $\infty \in \Lambda_{\mathrm{p}}(\Gamma)$. □

Lemma 4.5. *We have*

$$\Lambda_{\mathrm{r}}(\Gamma) \cap \partial\tilde{S} = \Lambda_{\mathrm{r}}(\Gamma_{\tilde{S}}).$$

Proof. Let $\xi \in \Lambda_{\mathrm{r}}(\Gamma) \cap \partial\tilde{S}$. As $\tilde{S}$ is totally geodesic, there exists a geodesic ray, say, β, lying in $\tilde{S}$ pointing toward ξ. Since ξ is a radial limit point, $\Gamma\beta$ accumulates on a compact subset of $\mathbb{H}^n$. By the assumption that the natural projection map $\Gamma_{\tilde{S}}\backslash\tilde{S} \to X$ is proper, $\Gamma_{\tilde{S}}\beta$ accumulates on a compact subset of $\tilde{S}$. This implies $\xi \in \Lambda_{\mathrm{r}}(\Gamma_{\tilde{S}})$. The other direction for the inclusion is clear. □

In [4], Bowditch proved the equivalence of several definitions of geometrically finite hyperbolic groups. In particular, we have:

Theorem 4.6 ([2], [4], [21])**.** *Γ is geometrically finite if and only if $\Lambda(\Gamma) = \Lambda_{\rm r}(\Gamma) \cup \Lambda_{\rm bp}(\Gamma)$.*

Hence, for geometrically finite Γ, we have $\Lambda_{\rm p}(\Gamma) = \Lambda_{\rm bp}(\Gamma)$.

Theorem 4.7. *If Γ is geometrically finite, then $\Gamma_{\tilde S}$ is geometrically finite.*

Proof. Since $\Lambda(\Gamma) = \Lambda_{\rm r}(\Gamma) \cup \Lambda_{\rm bp}(\Gamma)$, it follows from Proposition 4.4 and Lemma 4.5 that $\Lambda(\Gamma_{\tilde S}) = \Lambda_{\rm bp}(\Gamma_{\tilde S}) \cup \Lambda_{\rm r}(\Gamma_{\tilde S})$, proving the claim by Theorem 4.6. □

4.3. **Compactness of $\operatorname{supp}(\mu_E^{\rm PS})$ for Horospherical E.**

Theorem 4.8 (Dal'bo [7])**.** *Let Γ be geometrically finite. For a horosphere $\mathcal{H}$ in $\mathrm{T}^1(\mathbb{H}^n)$ based at $\xi \in \partial\mathbb{H}^n$, $E := \mathbf{p}(\mathcal{H})$ is closed in $\mathrm{T}^1(X)$ if and only if either $\xi \notin \Lambda(\Gamma)$ or $\xi \in \Lambda_{\rm p}(\Gamma)$.*

Theorem 4.9. *Let Γ be geometrically finite. If $E := \mathbf{p}(\mathcal{H})$ is a closed horosphere in $\mathrm{T}^1(X)$, then $\operatorname{supp}(\mu_E^{\rm PS})$ is compact.*

Proof. Let $\xi \in \partial\mathbb{H}^n$ be the base point for $\mathcal{H}$. The restriction of the visual map $\mathrm{Vis} : v \mapsto v^+$ induces a homeomorphism $\psi : \mathcal{H} \to \partial\mathbb{H}^n \smallsetminus \{\xi\}$. As E is closed, by Theorem 4.8, either $\xi \notin \Lambda(\Gamma)$ or ξ is a bounded parabolic fixed point. If $\xi \notin \Lambda(\Gamma)$, then $\Lambda(\Gamma)$ is a compact subset of $\partial\mathbb{H}^n \smallsetminus \{\xi\}$. Since $\operatorname{supp}(\mu_E^{\rm PS}) = \mathbf{p}(\psi^{-1}(\Lambda(\Gamma)))$, it follows that $\operatorname{supp}(\mu_E^{\rm PS})$ is compact.

Suppose now that ξ is a bounded parabolic fixed point. By Definition 4.2, $\Gamma_\xi\backslash(\Lambda(\Gamma) \smallsetminus \{\xi\})$ is compact. Since Γ_ξ is discrete, it preserves the horosphere $\mathcal{H}$ based at ξ, and $\Gamma_\xi = \Gamma_{\mathcal{H}}$. Therefore ψ induces a homeomorphism between $\Gamma_{\mathcal{H}}\backslash\mathcal{H}$ and $\Gamma_{\mathcal{H}}\backslash(\partial\mathbb{H}^n \smallsetminus \{\xi\})$. It follows that $\Gamma_{\mathcal{H}}\backslash\psi^{-1}(\Lambda(\Gamma) \smallsetminus \{\xi\})$ is compact and is equal to $\operatorname{supp}(\mu_E^{\rm PS})$. □

5. On the cuspidal neighborhoods of $\Lambda_{\rm bp}(\Gamma) \cap \partial\tilde S$

5.1. Throughout this section, let Γ be a torsion-free discrete subgroup of G and $\tilde S$ a connected complete totally geodesic subspace of $\mathbb{H}^n$ such that the natural projection $\Gamma_{\tilde S}\backslash\tilde S \to \Gamma\backslash\mathbb{H}^n$ is a proper map.

The *Dirichlet domain* for $\Gamma_{\tilde S}$ attached to some $a \in \tilde S$ is defined by

$$\mathcal{D}(a, \Gamma_{\tilde S}) := \{s \in \tilde S : d(s, a) \le d(s, \gamma a) \text{ for all } \gamma \in \Gamma_{\tilde S}\}. \tag{5.1}$$

Proposition 5.1. *$\Lambda_{\rm r}(\Gamma) \cap \partial\mathcal{D}(a, \Gamma_{\tilde S}) = \emptyset$.*

Proof. Let $\xi \in \Lambda_r(\Gamma) \cap \partial\mathcal{D}(a, \Gamma_{\tilde S})$. As

$$\overline{\mathcal{D}(a, \Gamma_{\tilde S})} = \mathcal{D}(a, \Gamma_{\tilde S}) \cup (\partial\mathcal{D}(a, \Gamma_{\tilde S}) \cap \partial\mathbb{H}^n)$$

is convex in $\mathbb{H}^n$, there exists a geodesic $\{\xi_t\} \subset \mathcal{D}(a, \Gamma_{\tilde{S}})$ such that $\xi_0 = a$ and $\xi_\infty = \xi$. As $\xi \in \Lambda_r(\Gamma_{\tilde{S}})$ by Lemma 4.5, there exist sequences $t_i \to \infty$ and $\gamma_i \in \Gamma_{\tilde{S}}$ such that $d(\gamma_i \xi_{t_i}, a)$ is uniformly bounded for all i. Since $d(\xi_{t_i}, a) \to \infty$, it follows that for all large i, $d(\xi_{t_i}, \gamma_i^{-1} a) < d(\xi_{t_i}, a)$, yielding that $\xi_{t_i} \notin \mathcal{D}(a, \Gamma_{\tilde{S}})$, a contradiction. □

Let $\tilde{E} \subset \mathrm{T}^1(\mathbb{H}^n)$ denote the set of all normal vectors to $\tilde{S}$. Given $U \subset \partial\mathbb{H}^n$, we define

$$\mathcal{E}_U = \{v \in \tilde{E} : \pi(v) \in \mathcal{D}(a, \Gamma_{\tilde{S}}),\ v^+ \in U \cap \Lambda(\Gamma)\}. \tag{5.2}$$

Remark 5.2. If $\xi \in \Lambda_r(\Gamma) \cap \partial\tilde{S}$, then there exists a neighborhood U of ξ in $\partial\mathbb{H}^n$ such that $\mathcal{E}_U = \emptyset$; to see this, note that if there exists a sequence $\{v_i\} \subset \tilde{E}$ such that $v_i^+ \to \xi$, then $\pi(v_i) \to \xi$, and hence by Proposition 5.1 $\pi(v_i) \notin \mathcal{D}(a, \Gamma_{\tilde{S}})$ for all large i.

In view of Theorem 4.6 and Remark 5.2, the main goal of this section is to describe the structure of $\mathcal{E}_U$ for a neighborhood U of a point in $\Lambda_{\mathrm{bp}}(\Gamma) \cap \partial\tilde{S}$ and to compute the measure $\mu^{\mathrm{PS}}_{\tilde{E}}(\mathcal{E}_U)$.

In this section, we will use the upper half space model $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ and first we assume that

$$\infty \in \partial\tilde{S} \cap \Lambda_{\mathrm{p}}(\Gamma).$$

Here $\mathbb{R}^{n-1}$ is to be treated as an affine space till we make a choice of the origin. Hence $\tilde{S}$ is a vertical plane over the affine subspace $\partial\tilde{S} \smallsetminus \{\infty\}$ of $\mathbb{R}^{n-1}$. For any affine subspace F of $\mathbb{R}^{n-1}$, let $P_F : \mathbb{R}^{n-1} \to F$ denote the orthogonal projection. Let

$$\mathrm{b} : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \to \mathbb{R}^{n-1} \text{ and } \mathrm{h} : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0} \tag{5.3}$$

denote the natural projections.

Let $\Gamma' = \Gamma'_\infty$ be a normal abelian subgroup of Γ_∞ with finite index, as in section 4.1 and fix a Γ'-minimal subspace L of $\mathbb{R}^{n-1}$. Noting that $\mathrm{b}(a) \in \partial\tilde{S} \smallsetminus \{\infty\}$, we choose $0 := P_L(b(a))$, the origin of $\mathbb{R}^{n-1}$. This choice of 0 makes L a linear subspace. Set $W := \{v - \mathrm{b}(a) : v \in \partial\tilde{S} \smallsetminus \{\infty\}\}$, a linear subspace of $\mathbb{R}^{n-1}$, and $\Delta := \Gamma_{\tilde{S}} \cap \Gamma'$. By Proposition 4.3, $L_0 := L \cap W$ is a Δ-minimal (linear) subspace.

Let V be the largest affine subspace of $\mathbb{R}^{n-1}$ such that Δ acts by translations on V. Then $0 \in L \subset V$ and V is the union of all (parallel) Δ-minimal subspaces of $\mathbb{R}^{n-1}$. There exist group homomorphisms $\tau : \Delta \to L_0 \subset \mathbb{R}^{n-1}$ and $\theta : \Delta \to O(n-1)$ such that for any $\gamma \in \Delta$,

$$\gamma(x) = \theta(\gamma)(x) + \tau(\gamma), \text{ for all } x \in \mathbb{R}^{n-1}. \tag{5.4}$$

We note that $V = \{x \in \mathbb{R}^{n-1} : \theta(\Delta)x = x\}$, and $V^\perp$ is the sum of all nontrivial (two-dimensional) $\theta(\Delta)$-irreducible subspaces of $\mathbb{R}^{n-1}$.

Lemma 5.3. *(1)* $W = (W \cap V) + (W \cap V^{\perp})$*;*
(2) $W^{\perp} = (W^{\perp} \cap V) + (W^{\perp} \cap V^{\perp})$.

Proof. Put $F = \partial\tilde{S} \setminus \{\infty\}$. Then $\Delta F = F$, and there exists a Δ-minimal affine subspace $L_{\tilde{S}} \subset F$. Choose $0' \in L_{\tilde{S}} \subset F \cap V$. Since W is a parallel translate of F through 0, we have $W = F - 0'$. As in the proof of Proposition 4.3, $\Delta(W) = W$. Since $0 \in W$, we have $\theta(\Delta)(W) = W$, and hence $\theta(\Delta)(W^{\perp}) = W^{\perp}$. Thus $W \cap V$ is the set of fixed points of $\theta(\Delta)$ in W, and its orthocomplement in W is the sum of all nontrivial $\theta(\Delta)$-irreducible subspaces of W which is same as $W \cap V^{\perp}$. Therefore (1) follows. And (2) is proved similarly. $\square$

For any $v \in \tilde{E}$, $\pi(v) \in \tilde{S}$. By abuse of notation, we write $\mathrm{b}(v) := \mathrm{b}(\pi(v)) \in \partial\tilde{S} \setminus \{\infty\}$ and $\mathrm{h}(v) := \mathrm{h}(\pi(v)) \in \mathbb{R}_{>0}$. We denote by $\sigma(v) \in W^{\perp}$ the unique element in $W^{\perp}$ of norm one satisfying

$$v^{+} := \mathrm{Viz}(v) = \mathrm{b}(v) + \mathrm{h}(v)\sigma(v). \tag{5.5}$$

Bounded parabolic assumption. For the rest of this section we will further assume that $\infty \in \partial(\tilde{S}) \cap \Lambda_{\mathrm{bp}}(\Gamma)$. Hence there exists $R_0 > 0$ such that for all $x \in \Lambda(\Gamma) \cap \mathbb{R}^{n-1}$,

$$\|P_{L^{\perp}}(x)\| \le R_0, \tag{5.6}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Lemma 5.4. *For any* $v \in \tilde{E}$ *with* $v^{+} \in \Lambda(\Gamma)$*,*

$$\|P_{V^{\perp}}(\mathrm{b}(v))\| \le R_0.$$

Proof. Let $0' \in V$ be as in the proof of Lemma 5.3. Since $\mathrm{b}(v) - 0' \in W$ and $0' \in V$, we have $P_{V^{\perp}}(0') = 0$ and by Lemma 5.3,

$$P_{V^{\perp}}(\mathrm{b}(v)) = P_{V^{\perp}}(\mathrm{b}(v) - 0') \in W \text{ and } P_{V^{\perp}}(\sigma(v)) \in W^{\perp}.$$

Therefore by (5.5), $\|P_{V^{\perp}}(\mathrm{b}(v))\| \le \|P_{V^{\perp}}(v^{+})\|$. Since $L \subset V$, we have $V^{\perp} \subset L^{\perp}$, and hence by (5.6), $\|P_{V^{\perp}}(v^{+})\| \le \|P_{L^{\perp}}(v^{+})\| \le R_0$. $\square$

Proposition 5.5. *There exists* $R_1 > 0$ *such that for all* $v \in \mathcal{E}_{\partial\mathbb{H}^n}$*,*

$$\|P_{L_0}(v^{+})\| \le R_1.$$

Proof. Let $v \in \mathcal{E}_{\partial\mathbb{H}^n}$. Then for all $\gamma \in \Delta \subset \Gamma_{\tilde{S}}$,

$$\begin{aligned} d_{\mathrm{hyp}}(\pi(v), a) &\le d_{\mathrm{hyp}}(\gamma\pi(v), a) \\ &\Rightarrow\ d_{\mathrm{eucl}}(\mathrm{b}(v), \mathrm{b}(a)) \le d_{\mathrm{eucl}}(\gamma\,\mathrm{b}(v), \mathrm{b}(a)). \end{aligned} \tag{5.7}$$

Now $\mathrm{b}(v) - \mathrm{b}(a) \in W$, $L_0 \subset W \cap V$, and $P_{L_0}(\mathrm{b}(a)) = 0$. As

$$W = (V^{\perp} \cap W) + L_0 + (W \cap V \cap L_0^{\perp}),$$

which is a sum of $\theta(\Delta)$-invariant orthogonal subspaces of W, we get

$$\gamma\,\mathrm{b}(v) - \mathrm{b}(a) = [\theta(\gamma) P_{V^\perp}(\mathrm{b}(v)) - P_{V^\perp}(\mathrm{b}(a))] + [P_{L_0}(\mathrm{b}(v)) + \tau(\gamma)] + P_{V\cap W\cap L_0^\perp}(\mathrm{b}(v) - \mathrm{b}(a)).$$

Comparing this with (5.7), for any $\gamma \in \Delta$ we get

$$\|P_{L_0}(\mathrm{b}(v))\|^2 \le \|\theta(\gamma) P_{V^\perp}(\mathrm{b}(v)) - P_{V^\perp}(\mathrm{b}(a))\|^2 + \|P_{L_0}(\mathrm{b}(v)) + \tau(\gamma)\|^2. \quad (5.8)$$

Since $\tau(\Delta)$ is a lattice in L_0, the radius of the smallest ball containing a fundamental domain of $\tau(\Delta)$ in L_0 is finite, which we denote by R_2. Then by (5.4) and (5.8), we conclude that

$$\|P_{L_0}(v^+)\|^2 = \|P_{L_0}(\mathrm{b}(v))\|^2 \le (R_0 + \|\mathrm{b}(a)\|)^2 + R_2^2.$$

By setting $R_1 = ((R_0 + \|\mathrm{b}(a)\|)^2 + R_2^2)^{1/2}$, we finish the proof. □

5.2. **Co-rank at ∞ and the structure of $\mathcal{E}_U$.** Set

$$r_\infty := \mathrm{rank}(\Gamma_\infty) - \mathrm{rank}(\Gamma_\infty \cap \Gamma_{\tilde{S}}).$$

More precisely, $r_\infty = \mathrm{rank}(\Gamma') - \mathrm{rank}(\Delta) = \dim(L) - \dim(L_0)$.

Proposition 5.6. *If $r_\infty = 0$, then there exists a neighborhood U of ∞ in $\partial\mathbb{H}^n$ such that $\mathcal{E}_U = \emptyset$, where $\mathcal{E}_U$ is defined in* (5.2).

Proof. As $r_\infty = 0$, we have $L = L_0$. Therefore, for all $x \in \Lambda(\Gamma) \cap \mathbb{R}^{n-1}$,

$$\|P_{L_0^\perp}(x)\| \le R_0.$$

Hence for any $v \in \mathcal{E}_{\partial\mathbb{H}^n}$, by Proposition 5.5,

$$\|v^+\|^2 = \|P_{L_0}(v^+)\|^2 + \|P_{L_0^\perp}(v^+)\|^2 \le R_1^2 + R_0^2.$$

Let $U = \{x \in \mathbb{R}^{n-1} : \|x\|^2 > R_0^2 + R_1^2\} \cup \{\infty\}$. Then $\mathcal{E}_U = \emptyset$. □

In the rest of this section, we now consider the case when

$$r := r_\infty \ge 1.$$

Notation 5.7. For any $\boldsymbol{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ and an ordered r-tuple $(w_1, \ldots, w_r)$ of vectors in $\mathbb{R}^{n-1}$, we set $\boldsymbol{s}\cdot\boldsymbol{w} := s_1 w_1 + \cdots + s_r w_r \in \mathbb{R}^{n-1}$, $\mathbb{R}^r\boldsymbol{w} := \{\boldsymbol{s}\cdot\boldsymbol{w} : \boldsymbol{s} \in \mathbb{R}^r\}$ and $|\boldsymbol{s}| = \max(|s_1|, \ldots, |s_r|)$. For $\boldsymbol{k} \in \mathbb{Z}^r$ and an ordered r-tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of elements of G, we write $\boldsymbol{\gamma}^{\boldsymbol{k}} = \gamma_1^{k_1} \cdots \gamma_r^{k_r} \in G$.

Fix an ordered r-tuple $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_r)$ of elements of $\Gamma' = \Gamma'_\infty$ such that the subgroup generated by $\boldsymbol{\gamma} \cup \Delta$ is of finite index in Γ'. For each γ_i, there exists $w_i \in L$ and $\sigma_i \in O(n-1)$ such that for all $x \in \mathbb{R}^{n-1}$,

$$\gamma_i(x) = \sigma_i(x) + w_i.$$

Moreover σ_i and the translation by w_i commutes, and hence for any $k \in \mathbb{Z}$, $\gamma_i^k(x) = \sigma_i^k(x) + kw_i$.

Setting $\boldsymbol{w} = (w_1, \dots, w_r)$ and $\sigma = (\sigma_1, \cdots, \sigma_r)$, we have that for any $x = y + z \in \mathbb{R}^{n-1}$ with $y \in L^\perp$ and $z \in L$ and $\boldsymbol{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$,

$$\boldsymbol{\gamma}^{\boldsymbol{k}}(x) = \sigma^{\boldsymbol{k}}(y) + z + \boldsymbol{k} \cdot \boldsymbol{w}. \tag{5.9}$$

Let R_0 and R_1 be as in (5.6) and in Proposition 5.5 respectively. Set

$$B_0 := \{x \in L^\perp : \|x\| \le R_0\} \quad \text{and} \quad B_1 := \{x \in L_0 : \|x\| \le R_1\}.$$

Let $M_1 := L \cap L_0^\perp$. Then $\mathbb{Z}^r \cdot P_{M_1}(\boldsymbol{w})$ is a lattice in $M_1 = \mathbb{R}^r P_{M_1}(\boldsymbol{w})$, which admits a relatively compact fundamental domain, say F_1. Let F_2 be a relatively compact fundamental domain for the lattice $\tau(\Delta)$ in L_0. We define the following relatively compact subset of $\mathbb{R}^{n-1}$:

$$\mathcal{F} := B_0 + (B_1 + F_2) + F_1 \subset L^\perp + L_0 + (L \cap L_0^\perp). \tag{5.10}$$

By (5.9),

$$\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F} = \mathcal{F} + \boldsymbol{k} \cdot \boldsymbol{w}.$$

For related variable quantities $x \ge 0$ and $y \ge 0$, the symbol $x \gg y$ means that there exists a constant $C > 0$ such that for all related x and y, $x \ge Cy$, and the symbol $x \asymp y$ means that $x \gg y$ and $y \gg x$.

Proposition 5.8. *There exists $c_0 \ge 1$ such that for all sufficiently large $N \ge 1$,*

$$\mathrm{Viz}(\mathcal{E}_{U_{c_0 N}}) \subset \cup_{|\boldsymbol{k}| \ge N} \Delta \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}) \tag{5.11}$$

where $U_{c_0 N} = \{x \in \mathbb{R}^{n-1} : \|x\| \ge c_0 N\}$.

Proof. Since $\mathbb{R}^{n-1} = L^\perp + L_0 + M_1$ for $M_1 = L_0^\perp \cap L$, we have for any $v \in \mathbb{R}^{n-1}$,

$$v^+ = P_{L^\perp}(v^+) + P_{L_0}(v^+) + P_{M_1}(v^+). \tag{5.12}$$

Let $v \in \mathcal{E}_{\partial \mathbb{H}^n}$. By (5.6) and Proposition 5.5,

$$P_{L^\perp}(v^+) \in B_0 \text{ and } P_{L_0}(v^+) \in B_1. \tag{5.13}$$

In order to control $P_{M_1}(v^+)$, let $\boldsymbol{k} = \boldsymbol{k}(v^+) \in \mathbb{Z}^r$ be such that $P_{M_1}(v^+) \in \boldsymbol{k} \cdot P_{M_1}(\boldsymbol{w}) + F_1$, where $\boldsymbol{k}$ is uniquely determined. Let $\lambda_{\boldsymbol{k}} \in \Delta$ be such that

$$P_{L_0}(\boldsymbol{k} \cdot \boldsymbol{w}) \in \tau(\lambda_{\boldsymbol{k}}) + F_2.$$

Since $\boldsymbol{k} \cdot P_{M_1}(\boldsymbol{w}) - \boldsymbol{k} \cdot \boldsymbol{w} = P_{L_0}(\boldsymbol{k} \cdot \boldsymbol{w})$,

$$P_{M_1}(v^+) \in (\boldsymbol{k} \cdot P_{M_1}(\boldsymbol{w}) - \boldsymbol{k} \cdot \boldsymbol{w}) + (F_1 + \boldsymbol{k} \cdot \boldsymbol{w}) \in \tau(\lambda_{\boldsymbol{k}}) + F_2 + (F_1 + \boldsymbol{k} \cdot \boldsymbol{w}).$$

Therefore by (5.12), for $\boldsymbol{k} = \boldsymbol{k}(v^+)$, we have

$$v^+ \in \mathcal{F} + \boldsymbol{k} \cdot \boldsymbol{w} + \tau(\lambda_{\boldsymbol{k}}) = \lambda_{\boldsymbol{k}} \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}). \tag{5.14}$$

Since $P_{M_1} : \mathbb{R}^r \boldsymbol{w} \to M_1$ is a linear isomorphism, there exists $N_1 \geq 1$ such that for all $\boldsymbol{k} \in \mathbb{Z}^r$ with $|\boldsymbol{k}| > N_1$,

$$\|P_{M_1}(\boldsymbol{k} \cdot \boldsymbol{w})\| \asymp |\boldsymbol{k}|. \tag{5.15}$$

By (5.12) and (5.13), $\|P_{M_1}(v^+) - v^+\| \leq R_0 + R_1$ and $P_{M_1}(v^+) - P_{M_1}(\boldsymbol{k} \cdot \boldsymbol{w}) \in F_1$ for $\boldsymbol{k} = \boldsymbol{k}(v^+)$. It follows that there exists a constant $B > 0$ such that for all $v \in \mathcal{E}_{\partial \mathbb{H}^n}$,

$$\|P_{M_1}(\boldsymbol{k} \cdot \boldsymbol{w})\| - B \leq \|v^+\| \leq \|P_{M_1}(\boldsymbol{k} \cdot \boldsymbol{w})\| + B$$

where $\boldsymbol{k} = \boldsymbol{k}(v^+)$. Hence by (5.15), there exists $N_2 \geq 1$ such that for all $v \in \mathcal{E}_{\partial \mathbb{H}^n}$ with $|\boldsymbol{k}(v^+)| > N_2$,

$$\|v^+\| \asymp |\boldsymbol{k}(v^+)|.$$

In view of (5.14), this finishes the proof. □

Lemma 5.9. *There exists $N_0 \geq 1$ such that for all $\boldsymbol{k} \in \mathbb{Z}^r$ with $|\boldsymbol{k}| > N_0$, the following hold:*

1. *For $\xi \in \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F})$, $\|\xi\| \asymp |\boldsymbol{k}|$.*
2. *For $v \in \tilde{E}$ with $v^+ \in \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F})$, $\mathrm{h}(v) \asymp |\boldsymbol{k}|$.*

Proof. If $\xi \in \boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F}$, then $\|\xi - \boldsymbol{k} \cdot \boldsymbol{w}\| \leq \mathrm{diam}_{\mathrm{eucl}}(\mathcal{F})$. Hence $\|\xi\| \asymp \|\boldsymbol{k} \cdot \boldsymbol{w}\| \asymp |\boldsymbol{k}|$, proving (1).

For $v \in \tilde{E}$ such that $v^+ \in \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F})$, by (5.5),

$$\mathrm{h}(v) \asymp \|P_{W^\perp}(v^+)\| \asymp P_{W^\perp}(\boldsymbol{k} \cdot \boldsymbol{w}).$$

Since $W \cap L = L_0$ and $L = L_0 \oplus \mathbb{R}^r \boldsymbol{w}$, the map $P_{W^\perp} : \mathbb{R}^r \boldsymbol{w} \to W^\perp$ is injective. Therefore

$$\|P_{W^\perp}(\boldsymbol{k} \cdot \boldsymbol{w})\| \asymp \|\boldsymbol{k} \cdot \boldsymbol{w}\| \asymp |\boldsymbol{k}|,$$

from which (2) follows. □

Let $o = (0, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$. For $T \geq 1$, put

$$\mathcal{B}_T = \{v \in \tilde{E} : \beta_\infty(o, \pi(v)) \geq \log T\}, \tag{5.16}$$

that is, $\mathcal{B}_T$ is the intersection with $\tilde{E}$ of a horoball based at ∞. We note that for $v \in \tilde{E}$, $\beta_\infty(o, \pi(v)) = \log h(v)$. Hence in the vertical plane model of $\tilde{S}$, $\mathcal{B}_T$ consists of vectors $v \in \tilde{E}$ whose base points have the Euclidean height at least T.

Proposition 5.10. *Let $\mathcal{F}_0 := B_0 + F_2 + F_1$. Then $\nu_o(\mathcal{F}_0) > 0$, and for all sufficiently large T, there exists $N \asymp T$ such that*

$$\mathrm{Viz}(\mathcal{B}_T) \supset \cup_{|\boldsymbol{k}| \geq N} \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}_0). \tag{5.17}$$

Proof. Since $\Delta F_2 = L_0$ and $\boldsymbol{\gamma}^{\mathbb{Z}^r} F_1 = M_1$,

$$\Delta(\cup_{\boldsymbol{k}\in\mathbb{Z}^r}\boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}_0)) = B_0 + L_0 + M_1 = B_0 + L \supset \Lambda(\Gamma) \setminus \{\infty\}.$$

Therefore if $\nu_o(\mathcal{F}_0) = 0$, then by the conformality, it follows that $\nu_o(\Lambda(\Gamma) \setminus \{\infty\}) = 0$. Since Γ does not fix ∞, by the Γ-invariance of $\{\nu_x\}$, we get $\nu_o(\Lambda(\Gamma)) = 0$, which is a contradiction, proving the first claim.

If $v \in \tilde{E}$ and $v^+ \in \boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}_0)$, then by Lemma 5.9, $h(v) \asymp |\boldsymbol{k}|$. If $h(v) > T$, then $v \in \mathcal{B}_T$. Therefore (5.17) holds for suitable $N \asymp T$. ☐

5.3. **Estimation of $\mu^{\mathrm{PS}}_{\tilde{E}}(\mathcal{E}_U)$.** Let $\mathrm{V}^{-1} : \mathbb{R}^{n-1} \setminus \partial\tilde{S} \to \tilde{E}$ be the inverse of the restriction of the visual map $\mathrm{Viz} : \tilde{E} \to \partial\mathbb{H}^n \setminus \partial\tilde{S} = \mathbb{R}^{n-1} \setminus \partial\tilde{S}$.

Lemma 5.11. *There exists $N_1 \geq 1$ such that for all $\boldsymbol{k} \in \mathbb{Z}^r$ with $|\boldsymbol{k}| > N_1$,*

$$\int_{\xi\in\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F}} e^{\delta\beta_\xi(o,\pi(\mathrm{V}^{-1}(\xi)))}\, d\nu_o(\xi) \asymp |\boldsymbol{k}|^{-\delta}.$$

Proof. We have $\|P_{W^\perp}(\boldsymbol{k}\cdot\boldsymbol{w})\| \asymp |\boldsymbol{k}|$. Hence for sufficiently large $|\boldsymbol{k}|$, we have that $\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F} \cap \partial\tilde{S} = \emptyset$. Note that the Euclidean diameter of the horosphere based at ξ and passing through $o = (0,1) \in \mathbb{R}^{n-1}\times\mathbb{R}_{>0}$ is $1 + \|\xi\|^2$. And the diameter of the horosphere based at ξ and passing through $\pi(\mathrm{V}^{-1}(\xi))$ is $\mathrm{h}(\pi(\mathrm{V}^{-1}(\xi)))$. Therefore the signed hyperbolic distance of the segment cut by these two horospheres on the vertical geodesic ending in ξ is

$$\beta_\xi(o,\pi(\mathrm{V}^{-1}(\xi))) = \log(1+\|\xi\|^2) - \log(\mathrm{h}(\pi(\mathrm{V}^{-1}(\xi)))).$$

Hence by Lemma 5.9,

$$e^{\delta\beta_\xi(o,\pi(\mathrm{V}^{-1}(\xi)))} = \left(\frac{1+\|\xi\|^2}{\mathrm{h}(\pi(\mathrm{V}^{-1}(\xi)))}\right)^\delta \asymp |\boldsymbol{k}|^\delta. \tag{5.18}$$

By conformality and Γ-invariance of Patterson-Sullivan density $\{\nu_x\}$,

$$\begin{aligned}\nu_o(\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F}) = (\boldsymbol{\gamma}^{-\boldsymbol{k}}\nu_o)(\mathcal{F}) = \nu_{\boldsymbol{\gamma}^{-\boldsymbol{k}}\cdot o}(\mathcal{F}) &= \int_{\xi\in\mathcal{F}} \frac{d\nu_{\boldsymbol{\gamma}^{-\boldsymbol{k}}\cdot o}}{d\nu_o}(\xi)\, d\nu_o(\xi)\\ &= \int_{\xi\in\mathcal{F}} e^{-\delta\beta_\xi(\boldsymbol{\gamma}^{-\boldsymbol{k}}o,o)} d\nu_o(\xi). \quad (5.19)\end{aligned}$$

We note that the horosphere based at ξ passing through $\boldsymbol{\gamma}^{-\boldsymbol{k}}o = (-\boldsymbol{k}\cdot\boldsymbol{w}, 1) \in \mathbb{R}^{n-1}\times\mathbb{R}_{>0}$ has diameter $1 + \|\xi + \boldsymbol{k}\cdot\boldsymbol{w}\|^2$. Therefore

$$\beta_\xi(\boldsymbol{\gamma}^{-\boldsymbol{k}}o, o) = \log(1+\|\xi+\boldsymbol{k}\cdot\boldsymbol{w}\|^2) - \log(1+\|\xi\|^2),$$

and hence, since $\|\boldsymbol{k}\cdot\boldsymbol{w}\| \asymp |\boldsymbol{k}|$ for all large $|\boldsymbol{k}|$, we have, for any $\xi \in \mathcal{F}$,

$$e^{-\delta\beta_\xi(\boldsymbol{\gamma}^{\boldsymbol{k}}o,o)} = \left(\frac{1+\|\boldsymbol{k}\cdot\boldsymbol{w}-\xi\|^2}{1+\|\xi\|^2}\right)^{-\delta} \asymp |\boldsymbol{k}|^{-2\delta},$$

Since $\nu_o(\mathcal{F}) > 0$ by Proposition 5.10, we deduce from (5.19) that

$$\nu_o(\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F}) \asymp |\boldsymbol{k}|^{-2\delta}\nu_o(\mathcal{F}) \asymp |\boldsymbol{k}|^{-2\delta}.$$

Together with (5.18), this proves the claim. □

Let $\mathbf{p} : \tilde{E} \to \Gamma_{\tilde{E}}\backslash\tilde{E}$ be the natural quotient map. We note that $\Gamma_{\tilde{E}} = \Gamma_{\tilde{S}}$. From §2.6, we recall that the measure $\mu^{\rm PS}_{\tilde{E}}$, which is $\Gamma_{\tilde{E}}$ invariant, naturally induces a measure on $\Gamma_{\tilde{E}}\backslash\tilde{E}$. The pushforward of this measure from $\Gamma_{\tilde{E}}\backslash\tilde{E}$ to $E = \mathbf{p}(\tilde{E})$ is $\mu^{\rm PS}_E$.

Recall the definition of $c_0 > 0$ and U_{c_0N} from Proposition 5.8.

Proposition 5.12. (1) *For all sufficiently large $N \geq 1$, we have*

$$\mu^{\rm PS}_E(\mathbf{p}(\mathcal{E}_{U_{c_0N}})) \ll \sum_{\boldsymbol{k}\in\mathbb{Z}^r\smallsetminus\{0\}} |\boldsymbol{k}|^{-\delta}.$$

(2) *For all sufficiently large $T \geq 1$, we have*

$$\mu^{\rm PS}_E(\mathbf{p}(\mathcal{B}_T)) \gg \sum_{\boldsymbol{k}\in\mathbb{Z}^r\smallsetminus\{0\}} |\boldsymbol{k}|^{-\delta}$$

Proof. By Proposition 5.8 and by Lemma 5.11, for all large $N \geq 1$,

$$\begin{aligned}\mu^{\rm PS}_E(\mathbf{p}(\mathcal{E}_{U_{c_0N}})) &\leq \sum_{|\boldsymbol{k}|\geq N} \mu^{\rm PS}_{\tilde{E}}(\mathrm{V}^{-1}(\boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}))) \\ &= \sum_{|\boldsymbol{k}|\geq N} \int_{\xi\in\boldsymbol{\gamma}^{\boldsymbol{k}}\mathcal{F}} e^{\delta\beta_\xi(o,\pi(\mathrm{V}^{-1}(\xi)))}\, d\nu_o(\xi) \\ &\ll \sum_{\boldsymbol{k}\geq N} |\boldsymbol{k}|^{-\delta},\end{aligned}$$

proving (1).

Consider the natural quotient map

$$\mathbf{p}_\infty : (\Gamma_{\tilde{E}} \cap \Gamma_\infty)\backslash\tilde{E} \to \Gamma_{\tilde{E}}\backslash\tilde{E}. \tag{5.20}$$

Since $\infty \in \Lambda_{\rm bp}(\Gamma)$, there exists $T_0 > 0$ such that $\mathbf{p}_\infty$ restricted to $(\Gamma_{\tilde{E}} \cap \Gamma_\infty)\backslash\mathcal{B}_T$ is proper and injective for all $T \geq T_0$.

Now since F_2 is a fundamental domain for Δ action on $L_{\tilde{S}}$ and F_1 is a fundamental domain for the action of $\{\boldsymbol{\gamma}^{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}^r\}$ on M_1, the quotient map $\tilde{E} \to \Delta\backslash\tilde{E}$ is injective on $\cup_{|\boldsymbol{k}|\geq N}\mathrm{V}^{-1}(\boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}_0))$. Since

$[\Gamma_{\tilde S}\cap\Gamma_\infty:\Delta]<\infty$ and $\mathbf{p}_\infty$ is injective on $(\Gamma_{\tilde S}\cap\Gamma_\infty)\backslash\mathcal{B}_T$, for all sufficiently large $T\gg 1$,

$$\begin{aligned}\mu_E^{\mathrm{PS}}(\mathbf{p}(\mathcal{B}_T)) &= \mu^{\mathrm{PS}}_{\Gamma_{\tilde E}\backslash\tilde E}(\mathbf{p}_\infty(\mathcal{B}_T)); \quad \text{see (2.13)}\\ &\gg \textstyle\sum_{|\boldsymbol{k}|\ge N}\mu^{\mathrm{PS}}_{\tilde E}(\mathrm{V}^{-1}(\boldsymbol{\gamma}^{\boldsymbol{k}}(\mathcal{F}_0))); \quad \text{by Proposition 5.10}\\ &\gg \textstyle\sum_{\boldsymbol{k}\ge N}|\boldsymbol{k}|^{-\delta}; \quad \text{by Lemma (5.11).}\end{aligned}$$

This proves (2). ☐

6. Parabolic co-rank and Criterion for finiteness of μ_E^{PS}

Let Γ be non-elementary torsion free discrete subgroup of G. Let $\tilde S$, $\tilde E$ and E be as in section 5. In particular, $\tilde S$ is totally geodesic and the map $\Gamma_{\tilde S}\backslash\tilde S\to\Gamma\backslash\mathbb{H}^n$ is proper.

Definition 6.1 (Parabolic corank)**.** Define

$$\text{pb-corank}(\Gamma_{\tilde S})=\max_{\xi\in\Lambda_{\mathrm p}(\Gamma)\cap\partial(\tilde S)}\left(\mathrm{rank}(\Gamma_\xi)-\mathrm{rank}(\Gamma_\xi\cap\Gamma_{\tilde S})\right).$$

When $\Lambda_{\mathrm p}(\Gamma)\cap\partial(\tilde S)=\emptyset$, we set pb-corank$(\Gamma_{\tilde S})=0$.

Lemma 6.2 (Corank Lemma)**.** pb-corank$(\Gamma_{\tilde S})\le\mathrm{codim}(\tilde S)$.

Proof. Suppose $\infty\in\Lambda_{\mathrm p}(\Gamma)\cap\partial\tilde S$. Let L be a Γ'_∞-minimal subspace of $\partial\mathbb{H}^n\smallsetminus\infty$ and let W be the intersection of a translate of $\partial\tilde S\smallsetminus\{\infty\}$ through a point in L . Then by Proposition 4.3, $\mathrm{rank}(\Gamma'_\infty)-\mathrm{rank}(\Gamma_{\tilde S}\cap\Gamma'_\infty)=\dim(L)-\dim(W)\le(n-1)-\dim(\partial\tilde S)=n-\dim\tilde S$. ☐

6.1. Finiteness criterion for geometrically finite Γ.

For the rest of this section we further assume that Γ is geometrically finite.

Theorem 6.3. pb-corank$(\Gamma_{\tilde S})=0\Leftrightarrow\mathrm{supp}(\mu_E^{\mathrm{PS}})$ *is compact.*

Proof. Suppose that $\mathrm{supp}(\mu_E^{\mathrm{PS}})$ is not compact. Fix a Dirichlet domain $\mathcal{D}(a,\Gamma_{\tilde S})$ for the $\Gamma_{\tilde S}$ action on $\tilde S$. Since the projection of $\Gamma_{\tilde E}\backslash\tilde E$ into $\Gamma\backslash\mathrm{T}^1(\mathbb{H}^n)$ is proper, there exists an unbounded sequence $v_m\in\tilde E$ with $\pi(v_m)\in\mathcal{D}(a,\Gamma_{\tilde S})$ and $v_m^+\in\Lambda(\Gamma)$. Since $\Lambda(\Gamma)$ is compact, by passing to a subsequence, we assume that $v_m^+\to\xi$ for some $\xi\in\Lambda(\Gamma)$. Thus for any neighborhood U of ξ in $\partial\mathbb{H}^n$, we have $v_m\in\mathcal{E}_U$ for all large m.

Consider the upper half space model $\mathbb{H}^n=\mathbb{R}^{n-1}\times\mathbb{R}_{>0}$ with ξ identified with ∞ as in §5. As $v_m^+\to\xi=\infty$, by (5.5) we have $\|\mathrm{b}(v_m)\|\to\infty$ or $\mathrm{h}(v_m)\to\infty$ (see (5.3) for notation) and hence $\pi(v_m)\to\infty=\xi$. Therefore $\xi\in\partial(\mathcal{D}(a,\Gamma_{\tilde S}))$. By Proposition 5.1, $\xi\notin\Lambda_{\mathrm r}(\Gamma)$. Since Γ is geometrically finite, by Theorem 4.6, $\xi\in\Lambda_{\mathrm{bp}}(\Gamma)\cap\partial\mathcal{D}(a,\Gamma_{\tilde S})$. Now by Proposition 5.6, pb-corank$(\Gamma_{\tilde S})\ne 0$.

To prove the converse, suppose that there exists $\xi \in \Lambda_{\mathrm{bp}}(\Gamma)\cap\partial\tilde{S}$ such that $r = \operatorname{rank}(\Gamma_\xi) - \operatorname{rank}(\Gamma_\xi \cap \Gamma_{\tilde{S}}) \geq 1$. Without loss of generality, we may assume $\xi = \infty$. Fix $T_0 > 1$. The map $\mathbf{p}_\infty$ as in (5.20) restricted to $(\Gamma_{\tilde{E}} \cap \Gamma_\infty)\backslash\mathcal{B}_{T_0}$ is proper (see (5.16) for notation). Therefore for any compact subset Ω of $\Gamma_{\tilde{E}}\backslash\tilde{E}$, we have $\mathbf{p}_\infty(\mathcal{B}_T)\cap\Omega = \emptyset$ for all sufficiently large $T > T_0$. By Proposition 5.12(2),

$$\mu_E^{\mathrm{PS}}(\mathbf{p}(\mathcal{B}_T)) \gg \sum_{\boldsymbol{k}\in\mathbb{Z}^r\smallsetminus\{0\}} |\boldsymbol{k}|^{-\delta} > 0.$$

Therefore $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ intersects $\mathbf{p}(\mathcal{B}_T)$ for all large $T \gg 1$. Since the projection of $\Gamma_{\tilde{E}}\backslash\tilde{E}$ into $\Gamma\backslash \mathrm{T}^1(\mathbb{H}^n)$ is proper, $\operatorname{supp}(\mu_E^{\mathrm{PS}})$ is noncompact. □

Theorem 6.4. pb-corank$(\Gamma_{\tilde{S}}) < \delta \Leftrightarrow |\mu_E^{\mathrm{PS}}| < \infty$.

Proof. Suppose that pb-corank$(\Gamma_{\tilde{S}}) \geq \delta > 0$. Then there exists $\xi \in \Lambda_{\mathrm{bp}}(\Gamma) \cap \tilde{S}$ such that $r := \operatorname{rank}(\Gamma_\xi) - \operatorname{rank}(\Gamma_\xi \cap \Gamma_{\tilde{S}}) \geq \max\{\delta, 1\}$. Without loss of generality, we may assume $\xi = \infty$. By the second part of the proof of Theorem 6.3, for all sufficiently large $T \gg 1$, since $r \geq \delta$,

$$|\mu_E^{\mathrm{PS}}| \geq \mu_E^{\mathrm{PS}}(\mathbf{p}(\mathcal{B}_T)) \gg \sum_{\boldsymbol{k}\in\mathbb{Z}^r\smallsetminus\{0\}} |\boldsymbol{k}|^{-\delta} = \infty.$$

Now suppose that pb-corank$(\Gamma_{\tilde{S}}) < \delta$. By the compactness of $\Lambda(\Gamma)\cap\partial(\mathcal{D}(a,\Gamma_{\tilde{S}}))$, where $\mathcal{D}(a,\Gamma_{\tilde{S}})$ is a fixed Dirichlet domain for $\Gamma_{\tilde{S}}$, to prove finiteness of μ_E^{PS}, it suffices to show that for every $\xi \in \Lambda(\Gamma) \cap \partial(\mathcal{D}(a,\Gamma_{\tilde{S}}))$, there exists a neighborhood U of ξ in $\partial\mathbb{H}^n$ such that $\mu_E^{\mathrm{PS}}(\mathbf{p}(\mathcal{E}_U)) < \infty$ with $\mathcal{E}_U$ defined as in (5.2). By Proposition 5.1 and Theorem 4.7, $\xi \in \Lambda_{\mathrm{bp}}(\Gamma)$. Let $r := \operatorname{rank}(\Gamma_\xi) - \operatorname{rank}(\Gamma_\xi \cap \Gamma_{\tilde{S}})$. If $r = 0$ then by Proposition 5.6, there exists a neighborhood U of ξ such that $\mathcal{E}_U = \emptyset$. Therefore we assume that $\delta > r \geq 1$. By Proposition 5.12(1), there exists a neighborhood U of ξ such that

$$\mu_E^{\mathrm{PS}}(\mathbf{p}(\mathcal{E}_U)) \ll \sum_{\boldsymbol{k}\in\mathbb{Z}^r\smallsetminus\{0\}} |\boldsymbol{k}|^{-\delta} < \infty.$$

□

6.2. Finiteness of $|\mu_E^{\mathrm{Leb}}|$ and $|\mu_E^{\mathrm{PS}}|$.

Theorem 6.5. *Let $\tilde{S}$ be any totally geodesic immersion in $\mathbb{H}^n$. Suppose that* $\dim(\tilde{S}) \geq (n+1)/2$ *and* $|\mu_E^{\mathrm{Leb}}| < \infty$. *Then* $|\mu_E^{\mathrm{PS}}| < \infty$.

Proof. Since $\Gamma_{\tilde{S}}$ is a lattice in $G_{\tilde{S}}$, $\Lambda(\Gamma_{\tilde{S}}) = \partial\tilde{S}$. Hence by Theorem 2.21, the natural map $\mathbf{p} : \Gamma_{\tilde{S}}\backslash\tilde{S} \to \Gamma\backslash\mathbb{H}^n$ is proper.

Let $k := \dim(\tilde{S}) \geq \lceil (n+1)/2 \rceil \geq 2$. By a property of a lattice in rank one Lie group $G_{\tilde{S}}$, $\operatorname{rank}(\Gamma_{\tilde{S}} \cap \Gamma_\xi) = k-1$ (cf. [29, §13.8]). Therefore by Lemma 6.2,

$$r := \text{pb-corank}(\Gamma_{\tilde{S}}) \leq n - k \leq n - (n+1)/2 \leq (n-1)/2.$$

Let $\xi \in \partial(\tilde{S}) \cap \Lambda_{\mathrm{bp}}(\Gamma_{\tilde{S}})$ be such that $\operatorname{rank}(\Gamma_{\tilde{S}} \cap \Gamma_\xi) = r$. Then $\operatorname{rank}(\Gamma_\xi) \geq (k-1) + r$. By a result of Dalbo, Otal and Peign [8, Proposition 2],

$$\delta > \operatorname{rank}(\Gamma_\xi)/2 \geq ((k-1)+r)/2 \geq (k-1+(n-k))/2 = (n-1)/2 \geq r.$$

Hence by Theorem 6.4(2), $|\mu_E^{\mathrm{PS}}|$ is finite. □

As an immediate corollary, we state:

Corollary 6.6. *Let $n = 2, 3$. Then $|\mu_E^{\mathrm{Leb}}| < \infty$ implies that $|\mu_E^{\mathrm{PS}}| < \infty$.*

To deduce that $\mathrm{sk}_\Gamma(w_0) > 0$, when $w_0\Gamma$ is infinite in Theorem 1.2 we need the following. Here Γ need not be geometrically finite.

Proposition 6.7. *If $[\Gamma : \Gamma_{\tilde{S}}] = \infty$, then $\Lambda(\Gamma) \not\subset \partial_\infty(\tilde{S})$, and $|\mu_E^{\mathrm{PS}}| > 0$.*

Proof. Suppose on the contrary that $\Lambda(\Gamma) \subset \partial_\infty(\tilde{S})$. Let L be a geodesic joining two distinct points say $\xi_1, \xi_2 \in \Lambda(\Gamma)$. Then $L \subset \tilde{S}$. For any $\gamma \in \Gamma$, we have γL is the geodesic joining $\gamma\xi_1$ and $\gamma\xi_2$, and hence $\gamma L \subset \tilde{S}$. Now fix $x_0 \in L$. Then $\Gamma x_0 \subset \tilde{S}$. Since $\Gamma_{\tilde{S}} \backslash \tilde{S} \to \Gamma \backslash \mathbb{H}^n$ is a proper map, we get that $\Gamma_{\tilde{S}} \backslash \Gamma$ is finite, a contradiction to our assumption. □

7. Orbital counting for discrete hyperbolic groups

As before, let $G = \mathrm{SO}(n,1)^\circ$ for $n \geq 2$ and Γ a torsion-free, non-elementary, discrete subgroup of G.

7.1. **Computation with $\tilde{m}^{\mathrm{BR}}$.** Let K be a maximal compact subgroup of G. Let $o \in \mathbb{H}^n$ be such that $K = G_o$. Then $G/K \cong \mathbb{H}^n$. Let $X_0 \in \mathrm{T}^1_o(\mathbb{H}^n)$ and $M = G_{X_0}$. Then $G/M \cong \mathrm{T}^1(\mathbb{H}^n)$, where $g[M] = gX_0$. Let $A = \{a_r : r \in \mathbb{R}\} \subset Z_G(M)$ be a one-parameter subgroup of G consisting of diagonalizable elements such that $\mathcal{G}^r(X_0) = a_r[M]$. Via the map $k \mapsto kX_0^+$, we have $K/M \cong \partial\mathbb{H}^n$.

Let $N < G$ be the expanding horospherical subgroup with respect to the right a_r-action; that is,

$$N := \{g \in G : a_r g a_r^{-1} \to e \quad \text{as } r \to \infty\}. \tag{7.1}$$

The N-leaves gNM/M correspond to unstable horospheres $\mathcal{H}^+_{gX_0}$ in $\mathrm{T}^1(G/K) = G/M$ based at gX_0^-. The map $N \ni z \mapsto zX_0^+ \in \partial\mathbb{H}^n \smallsetminus \{X_0^-\}$ is a diffeomorphism.

As before let m_o denote the G-invariant (Lebesgue) conformal density $\{m_x\}_{x\in\mathbb{H}^n}$ on $\partial\mathbb{H}^n$. We normalize it so that m_o (and hence every m_x) is a probability measure. Here m_o is K-invariant.

Lemma 7.1. *For any $g \in G$, consider the measure λ_g on N given by*

$$d\lambda_g(z) = e^{(n-1)\beta_{gzX_0^+}(o,gz(o))} dm_o(gzX_0^+), \text{ where } z \in N;$$

Then $\lambda_g = \lambda_e$. In particular λ_e is a Haar measure on N which we shall denote by the integral $dn = d\lambda_e(n)$ on N

Proof. Since $\{m_x\}$ is a G-invariant conformal density,

$$dm_o(gzX_0^+) = dm_{g^{-1}(o)}(zX_0^+) = e^{(n-1)\beta_{zX_0^+}(o,g^{-1}(o))} dm_o(zX_0^+).$$

Since $\beta_{gzX_0^+}(o, gz(o)) = \beta_{zX_0^+}(g^{-1}(o), z(o))$,

$$d\lambda_g(z) = e^{(n-1)\beta_{zX_0^+}(o,z(o))} dm_o(zX_0^+) = d\lambda_e(z). \tag{7.2}$$

For any $g \in N$, $d\lambda_e(gz) = d\lambda_g(z) = d\lambda_e(z)$. Therefore λ_e is N-invariant. $\square$

Notation 7.2. Note that $G_{X_0^-} = ANM$ and $K \cap G_{X_0^-} = M$. For $\psi \in C(K)$ and a measure λ on $\partial\mathbb{H}^n = KX_0^- \cong K/M$, we define

$$\int_{k\in K} \psi(k)\, d\lambda(kX_0^-) := \int_{K/M} \Big(\int_{m\in M} \psi(km)\, dm\Big) d\lambda(kM). \tag{7.3}$$

We also fix a Patterson-Sullivan density $\{\nu_x\}$ on $\partial\mathbb{H}^n$ and consider $\tilde{m}^{\mathrm{BR}}$ defined as in subsection 3.1 with respect to $\{m_x\}$ and $\{\nu_x\}$.

Proposition 7.3. *For any $\phi \in C_c(\mathrm{T}^1(\mathbb{H}^n)) = C_c(G)^M$,*

$$\tilde{m}^{\mathrm{BR}}(\phi) = \int_{k\in K}\int_{r\in\mathbb{R}}\int_{n\in N} \phi(ka_r n) e^{-\delta r}\, dn\, dr\, d\nu_o(kX_0^-).$$

Proof. By definition,

$$\tilde{m}^{\mathrm{BR}}(\phi) = \int \phi(u) e^{(n-1)\beta_{u^+}(o,\pi(u))} e^{\delta\beta_{u^-}(o,\pi(u))} dm_o(u^+) d\nu_o(u^-) dt,$$

where $t = \beta_{u^-}(o, \pi(u))$. Let $u = ka_r nX_0$. Then, since $G_{X_0^-} = MAN$, we have $u^- = ka_r nX_0^- = kX_0^-$ and

$$\begin{aligned} t = \;& \beta_{u^-}(o,\pi(u)) = \beta_{kX_0^-}(o, ka_r no) = \beta_{X_0^-}(o, a_r no) \\ & = \lim_{t\to\infty} d(o, a_{-t}o) - d(a_r no, a_{-t}o) \\ & = \lim_{t\to\infty} t - d(a_{t+r} n a_{-t-r}(a_{t+r}o), o) \\ & = \lim_{t\to\infty} t - d(a_{t+r}o, o) = -r. \end{aligned}$$

Therefore $e^{\delta\beta_{u^-}(o,\pi(u))}d\nu_o(u^-) = e^{-\delta r}d\nu_o(kX_0^-)$. And by Lemma 7.1 for fixed $g = ka_r$ and variable $z = n \in N$,

$$e^{(n-1)\beta_{ka_rnX_0^+}(o,ka_rn\pi(o))}dm_o(ka_rnX_0^+) = d\lambda_{ka_r}(n) = d\lambda_e(n) = dn.$$

Putting together, this proves the claim. □

Notation 7.4. (1) Let dk denote the probability Haar measure on K. Since m_o is a K-invariant probability measure on $\partial\mathbb{H}^n = K/M$, we have that $dk = dm_o(kX_0^-)$ (and similarly $dk = dm_o(kX^+)$). We fix the Haar measure dg on G given as follows: for $g = ka_rn \in KAN$, $dg = e^{-(n-1)r}\, dn\, dr\, dk$. Since G is unimodular, $dg = dg^{-1}$. Therefore if we express $g = na_rk$, then $dg = e^{(n-1)r}dn\, dr\, dk$. And if we express $g = a_rnk$, then $dg = dr\, dn\, dk$.

(2) For $\epsilon > 0$, let U_ϵ denote the ϵ-neighborhood of e in G. By an *approximate identity* on G, we mean a family of nonnegative continuous functions $\{\psi_\epsilon\}_{\epsilon>0}$ on G with $\mathrm{supp}(\psi_\epsilon) \subset U_\epsilon$ and $\int_G \psi_\epsilon(g)dg = 1$.

(3) For $\xi \in C(M\backslash K)$ and $\psi \in C_c(G)$ and a measurable $\Omega \subset K$ with $M\Omega = \Omega$, we define a function $\xi *_\Omega \psi \in C_c(G/M)$ by

$$\xi *_\Omega \psi(g) := \int_{k\in\Omega} \xi(k)\psi(gk)\, dk. \tag{7.4}$$

For $\psi \in C_c(\Gamma\backslash G)$, we define $\xi *_\Omega \psi \in C_c(\Gamma\backslash G/M)$ similarly.

Proposition 7.5. *Let $\{\psi_\epsilon\}_{\epsilon>0}$ be an approximate identity on G. Let $f \in C(M\backslash K)$ and $\Omega \subset K$ be such that $M\Omega = \Omega$ and $\nu_o(\partial(\Omega^{-1})X_0^-) = 0$. Then*

$$\lim_{\epsilon\to 0} \tilde{m}^{\mathrm{BR}}(f *_\Omega \psi_\epsilon) = \int_{k\in\Omega^{-1}} f(k^{-1})\, d\nu_o(kX_0^-).$$

Proof. Note that for some uniform constants $\ell_1, \ell_2 > 0$, we have for all $k \in K$ and for all small $\epsilon > 0$,

$$k^{-1}U_\epsilon \subset U_{\ell_1\epsilon}k^{-1} \subset (A \cap U_{\ell_2\epsilon})(N \cap U_{\ell_2\epsilon})k^{-1}(K \cap U_{\ell_2\epsilon}). \tag{7.5}$$

Set $K_\epsilon := (K \cap U_{\ell_2\epsilon})$, $\Omega_{\epsilon+} = \Omega K_\epsilon$ and $\Omega_{\epsilon-} = \cap_{k\in K_\epsilon}\Omega k$.

In view of the decomposition $G = ANK$, for a function ϕ on K, we define a function $\mathcal{R}_\phi$ on G by $\mathcal{R}_\phi(g) = \phi(k)$ for $g = ank \in ANK$. For any $\eta > 0$, there exists $\epsilon > 0$ such that for all $k \in K$ and $g \in U_\epsilon$,

$$\mathcal{R}_{f\cdot\chi_{\Omega_{\epsilon-}}}(k^{-1}) - \eta \le \mathcal{R}_{f\cdot\chi_\Omega}(k^{-1}g) \le \mathcal{R}_{f\cdot\chi_{\Omega_{\epsilon+}}}(k^{-1}) + \eta. \tag{7.6}$$

Now by Proposition 7.3,

$$\begin{aligned}
&\tilde m^{\mathrm{BR}}(f *_\Omega \psi_\epsilon)\\
&= \int_{g\in G}\int_{k'\in\Omega}\psi_\epsilon(gk')f(k')\,dk'd\tilde m^{\mathrm{BR}}(g)\\
&= \int_{(k,a_r,n)\in K\times A\times N}\int_{k'\in\Omega}\psi_\epsilon(ka_rnk')f(k')e^{-\delta r}\,dk'dndrd\nu_o(kX_0^-)\\
&= \int_{k\in K}\int_{(a_r,n,k')\in A\times N\times K}\psi_\epsilon(ka_rnk')f(k')\chi_\Omega(k')e^{-\delta r}\,drdndk'd\nu_o(kX_0^-)\\
&= \int_{k\in K}\int_{g\in G}\psi_\epsilon(kg)\mathcal{R}_{f\cdot\chi_\Omega}(g)e^{-\delta r_g}\,dgd\nu_o(kX_0^-),\quad \text{if } g=a_{r_g}nk'\\
&\le e^{\delta\ell_2\epsilon}\int_{k\in K}\int_{g\in G}\psi_\epsilon(g)\mathcal{R}_{f\cdot\chi_\Omega}(k^{-1}g)\,dgd\nu_o(kX_0^-),\quad \text{by (7.5)}\\
&\le e^{\delta\ell_2\epsilon}\int_{k\in K}\int_{g\in G}\psi_\epsilon(g)(\mathcal{R}_{f\cdot\chi_{\Omega_{\epsilon+}}}(k^{-1})+\eta)\,dgd\nu_o(kX_0^-)\\
&= e^{\delta\ell_2\epsilon}\big(\int_{k\in K}\mathcal{R}_{f\cdot\chi_{\Omega_{\epsilon+}}}(k^{-1})\,d\nu_o(kX_0^-)+\eta|\nu_o|\big),\ \text{as } \int_G\psi_\epsilon(g)\,dg=1.\\
&= e^{\delta\ell_2\epsilon}\big(\int_{k\in\Omega_{\epsilon+}^{-1}}f(k^{-1})\,d\nu_o(kX_0^-)+\eta|\nu_o|\big).
\end{aligned}$$

Since $\cap_{\epsilon>0}\Omega_{\epsilon+}=\overline{\Omega}$, and since $\eta>0$ was arbitrarily chosen,

$$\limsup_{\epsilon\to 0}\tilde m^{\mathrm{BR}}(f *_\Omega \psi_\epsilon)\le\int_{k\in\overline{\Omega^{-1}}}f(k^{-1})\,d\nu_o(kX_0^-).$$

Similarly, $\liminf_{\epsilon\to0}\tilde m^{\mathrm{BR}}(f *_\Omega \psi_\epsilon)\ge\int_{k\in\operatorname{int}\Omega^{-1}}f(k^{-1})\,d\nu_o(kX_0^-)$. Since $\nu_o(\partial(\Omega^{-1})X_0^-)=0$, we obtain (7.5). □

7.2. **Setup for counting results.** Till the end of this section, let V be a finite dimensional vector space on which G acts linearly from the right and let $w_0\in V$. We set $H:=G_{w_0}$.

7.2.1. *When H is a symmetric subgroup of G.* Let $H<G=\mathrm{SO}(n,1)^\circ$ be a symmetric subgroup, i.e., there is a non-trivial involution σ of G such that $H^\circ=(G^\sigma)^\circ$ where $G^\sigma=\{g\in G:\sigma(g)=g\}$. There exists a Cartan involution θ of G such that $\theta\circ\sigma=\sigma\circ\theta$. Let $K=G^\theta$. It turns out that H° is a subgroup of finite index in its normalizer $N_G(H^\circ)$, and up to a conjugation of G, $H^\circ=(\mathrm{SO}(k,1)\times\mathrm{SO}(n-k))^\circ$ for some $0\le k\le n-1$ and $K=\mathrm{SO}(n)$. Choose $o\in\mathbb{H}^n$ such that $G_o=K$. Then $\tilde S=H\cdot o$ is an isometric imbedding of $\mathbb{H}^k$ in $\mathbb{H}^n$. Let $\tilde E$ be the unit normal bundle over $\tilde S$.

7.2.2. *When $G_{\mathbb{R}w_0}$ is a parabolic subgroup of G.* Suppose that $G_{\mathbb{R}w_0}$ is a parabolic subgroup of G. Let θ be any Cartan involution of G and let $K=G^\theta$. Then $G=G_{\mathbb{R}w_0}K$. Let N be the unipotent radical of $G_{\mathbb{R}w_0}$. Let $o\in\mathbb{H}^n$ be such that $G_o=K$. Then $\tilde S:=N\cdot o$ is a horosphere. Let $\tilde E\subset\mathrm{T}^1(\mathbb{H}^n)$ be the unstable horosphere such that $\pi(\tilde E)=\tilde S$. Let $H=(G_{\mathbb{R}w_0}\cap K)N$.

7.2.3. *Common structure in both cases.* Let the notation be as in any of the above section 7.2.1 or 7.2.2. Let $X_0 \in \mathrm{T}^1_o(\mathbb{H}^n) \cap \tilde{E}$. Let $\tilde{E}^* = H \cdot X_0$. If H is symmetric and $\operatorname{codim}(\tilde{S}) > 1$, or in the parabolic case, then $\tilde{E}$ is connected and $\tilde{E}^* = \tilde{E}$. If H is symmetric and $\operatorname{codim}(\tilde{S}) = 1$, then $\tilde{E}$ has two connected components: $\tilde{E}^+$ containing X_0 and $\tilde{E}^-$ containing $-X_0$; and then either $\tilde{E}^* = \tilde{E}$ or $\tilde{E}^* = \tilde{E}^+$. There exists a one-parameter subgroup $A = \{a_r\} \subset G$ consisting of $\mathbb{R}$-diagonalizable elements, such that $\mathcal{G}^r(X_0) = a_r X_0$ for all $r \in \mathbb{R}$. Let $M = G_{X_0}$, which coincides with $Z_K(A)$, i.e., the centralizer of A in K, and $A^\pm = \{a_{\pm r} : r \geq 0\}$. Let N be the expanding horospherical subgroup with respect to $\{a_r\}$.

When $G_{\mathbb{R}w_0}$ is parabolic, then $G_{w_0} = MN = H$ where $M = G_{\mathbb{R}w_0} \cap K$; hence N is the unipotent radical of $G_{\mathbb{R}w_0}$ so there is no conflict of notation. In the case when H is symmetric, then $\tilde{E}^* = \tilde{E}$ if and only if $G = HA^+K$. In all cases, we have $G = HAK$. Put $E = \mathbf{p}(\tilde{E})$, $E^* = \mathbf{p}(\tilde{E}^*)$, and in the special cases when $\tilde{E}$ is not connected, we set $E^\pm = \mathbf{p}(\tilde{E}^\pm)$.

7.2.4. *HAK decomposition of Haar measure on G.* Note that $\tilde{E}^* = HX_0 \cong H/(M \cap H)$ and recall that

$$d\mu^{\mathrm{Leb}}_{\tilde{E}}(v) = e^{(n-1)\beta_{v^+}(o,\pi(v))} dm_o(v^+).$$

There is a Haar measure dh on H such that for any $\psi \in C_c(H)$ if we put $\bar{\psi}(h) = \int_{m \in M \cap H} \psi(hm)\, dm$, where dm denotes the probability Haar integral on $M \cap H$, then $\bar{\psi} \in C_c(H)^{M \cap H} = C_c(\tilde{E})$, and

$$\int_H \psi\, dh = \int_{\tilde{E}} \bar{\psi}\, d\mu^{\mathrm{Leb}}_{\tilde{E}}. \tag{7.7}$$

In view of the decompositions $G = HA^+K$ or $G = HAK$, there exists a function $\rho : \mathbb{R} \to (0, \infty)$, such that we get the following Haar measure dg on G: For any $\psi \in C_c(G)$, by [35, Theorem 8.1.1]

$$\int_G \psi\, dg = \int_{k \in K} \int_{r \in R} \int_{h \in H} \rho(r) \psi(h a_r k)\, dh dr dk, \text{ and} \tag{7.8}$$

$$\rho(r) \sim \begin{cases} e^{(n-1)|r|} & \text{if } r \to \pm\infty \text{ and } H \text{ is symmetric,} \\ e^{(n-1)r} & \text{if } r \to \pm\infty \text{ and } G_{\mathbb{R}w_0} \text{ is parabolic.} \end{cases} \tag{7.9}$$

where $R = \{r \geq 0\}$ if $G = HA^+K$, otherwise $R = \mathbb{R}$. In fact the Haar measure dg described in Notation 7.4(1) and the Haar measure dg defined in (7.8) are identical, see §8.

7.3. **Extension of Theorem 1.8 to $\Gamma\backslash G$ for Zariski dense Γ.** The result in this subsection will enable us to state our counting theorems for general norms, provided Γ is Zariski dense.

Let $\bar{m}^{\mathrm{BR}}$ be the measure on $\Gamma\backslash G$ which is the M-invariant extension of m^{BR}, that is, for $\psi \in C_c(\Gamma\backslash G)$,

$$\bar{m}^{\mathrm{BR}}(\psi) := m^{\mathrm{BR}}(\bar{\psi})$$

where $\bar{\psi}(\mathbf{p}(gX_0)) = \int_{m\in M} \psi(\Gamma g m)\, dm$ and dm denotes the Haar probability measure on M.

As M normalizes N, $\bar{m}^{\mathrm{BR}}$ is invariant for the right-translation action of N on $\Gamma\backslash G$.

Theorem 7.6 (Flaminio-Spatzier [11, Cor. 1.6])**.** *If Γ is Zariski dense and $|m^{\mathrm{BMS}}| < \infty$, then $\bar{m}^{\mathrm{BR}}$ is N-ergodic.*

Let H and $\tilde{E}$ be as in subsection 7.2.1 or 7.2.2 so that $H = G_{\tilde{E}}$. Let dh be the Haar measure on H defined as in (7.7); by abuse of notation, we also denote by dh the measure on $\Gamma_H\backslash H$ induced by dh.

We recall that for Γ Zariski dense, $|\mu_E^{\mathrm{PS}}| < \infty$ implies that the canonical map $\Gamma_H\backslash H \to \Gamma\backslash G$ is proper by Theorem 2.21.

Theorem 7.7. *Let Γ be a Zariski dense discrete subgroup of G such that $|m^{\mathrm{BMS}}| < \infty$ and $|\mu_E^{\mathrm{PS}}| < \infty$. Then for any $\psi \in C_c(\Gamma\backslash G)$,*

$$\lim_{r\to\infty} e^{(n-1-\delta)r} \int_{h\in\Gamma_H\backslash H} \psi(\Gamma h a_r)\, dh = \frac{|\mu_E^{\mathrm{PS}}|}{|m^{\mathrm{BMS}}|} \bar{m}^{\mathrm{BR}}(\psi).$$

Proof. Define a measure λ_r on $\Gamma\backslash G$ as follows: for any $\psi \in C_c(\Gamma\backslash G)$,

$$\lambda_r(\psi) = e^{(n-1-\delta)r} \int_{h\in\Gamma_H\backslash H} \psi(\Gamma h a_r)\, dh.$$

Let $\mathbf{q} : \Gamma\backslash G \to \mathrm{T}^1(\Gamma\backslash\mathbb{H}^n) \cong \Gamma\backslash G/M$ be the natural quotient map. Then for any $\bar{\psi} \in C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$, we have $\bar{\psi}(\mathbf{q}(xma_r)) = \bar{\psi}(\mathbf{q}(xa_r))$ for any $m \in M$ and $x \in \Gamma\backslash G$, as M and A commute with each other, and hence

$$\mathbf{q}_*(\lambda_r)(\psi) = \lambda_r(\bar{\psi}\circ\mathbf{q}) = e^{(n-1-\delta)r} \int_E \bar{\psi}(va_r)\, d\mu_E^{\mathrm{Leb}}(v).$$

Therefore by Theorem 1.8, $\mathbf{q}_*(\lambda_r) \to C\cdot m^{\mathrm{BR}}$, where $C = \frac{|\mu_E^{\mathrm{PS}}|}{|m^{\mathrm{BMS}}|}$.

In order to show that λ_r weakly converges to $C\bar{m}^{\mathrm{BR}}$, it suffices to show that every sequence λ_{r_k} has a subsequence converging to $C\bar{m}^{\mathrm{BR}}$.

For any sequence $r_k \to \infty$, since $\mathbf{q}$ is a proper map, after passing to a subsequence of $\{r_k\}$ there exists a measure λ on $\Gamma\backslash G$ such that $\lambda_{r_k}(\phi) \to \lambda(\phi)$ for every $\phi \in C_c(\Gamma\backslash G)$. Therefore

$$\mathbf{q}_*(\lambda) = Cm^{\mathrm{BR}}.$$

For any $g \in G$, define a measure $g\lambda$ on $\Gamma\backslash G$ by $g\lambda(A) = \lambda(Ag)$ for any measurable $A \subset \Gamma\backslash G$. Now for any $\psi \in C_c(\Gamma\backslash G)$,

$$\begin{aligned} \int_{m\in M}(m\lambda)(\psi)\,dm &= \int_{m\in M}\int_{\Gamma\backslash G}\psi(xm)\,d\lambda(x)\,dm \\ &= \bar{\mathbf{q}}_*(\lambda)(\bar\psi) = Cm^{\mathrm{BR}}(\bar\psi) = C\bar m^{\mathrm{BR}}(\psi). \end{aligned} \tag{7.10}$$

Claim 1. λ is N-invariant.

Proof of Claim 1. Due to Lemma 2.1, the map $h \mapsto hX_0^+$ is a submersion and hence there exists a neighborhood Ω of e in N and a continuous injective map $\sigma : \Omega \to H$ such that $\sigma(e) = e$ and $\sigma(z)X_0^+ = zX_0^+$ for all $z \in \Omega$.

Fix $z \in \Omega$, let $z_k := a_{r_k} z a_{-r_k}$, and $h_k = \sigma(z_k)$ for all large k. Then $b_k = z_k^{-1}h_k \in G_{X_0^+} = MAN^-$. Therefore $b_k \to e$ and $a_{-r_k}b_k a_{r_k} \to e$ as $k \to \infty$.

Let $\psi \in C_c(\Gamma\backslash G)$. Given $\epsilon > 0$ and $x \in \Gamma\backslash G$, set

$$\psi_{\epsilon+}(x) = \sup_{g\in U_\epsilon}\psi(xg) \quad \text{and} \quad \psi_{\epsilon-} = \inf_{g\in U_\epsilon}\psi(xg).$$

Since ψ is uniformly continuous and $a_{r_k}z = h_k b_k^{-1} a_{r_k} = h_k a_{r_k}(a_{-r_k} b_k^{-1} a_{r_k})$, we have for all large k and for all $x \in \Gamma\backslash G$,

$$\psi_{\epsilon-}(xhh_k a_{r_k}) \le \psi(xa_{r_k}z) \le \psi_{\epsilon+}(xh_k a_{r_k}).$$

Since the measure dh is H-invariant,

$$\int_{h\in\Gamma_H\backslash H}\psi(\Gamma h a_{r_k} z)\,dh \le \int_{\Gamma_H\backslash H}\psi_{\epsilon+}(\Gamma h h_k a_{r_k})\,dh = \int_{\Gamma_H\backslash H}\psi_{\epsilon+}(\Gamma h a_{r_k})\,dh.$$

Similarly we get a lower bound in terms of $\psi_{\epsilon-}$. Since $\lambda_{r_k} \to \lambda$ as $k \to \infty$,

$$\lambda(\psi_{\epsilon-}) \le \int_{\Gamma\backslash G}\psi(xz)\,d\lambda(x) \le \lambda(\psi_{\epsilon+}).$$

Since $\psi \in C_c(\Gamma\backslash G)$, we have that $\lambda(\psi_{\epsilon\pm}) \to \lambda(\psi)$ as $\epsilon \to 0$. Therefore the z-action preserves λ. □

Claim 2. $\lambda = C\bar m^{\mathrm{BR}}$.

Proof of Claim 2. By (7.10), it is enough to show that λ is M-invariant. For any $\epsilon > 0$, define a measure η_ϵ on $\Gamma\backslash G$ by

$$\eta_\epsilon := \frac{1}{|M_\epsilon|}\int_{m\in M_\epsilon} m\lambda\,dm,$$

where $|M_\epsilon| = \int_{M_\epsilon} dm$. Then since M normalizes N, η_ϵ is N-invariant. By (7.10)

$$\eta_\epsilon \ll \bar m^{\mathrm{BR}}.$$

almost all $m \in M$, and hence

Therefore, since $\bar{m}^{\mathrm{BR}}$ is N-ergodic by Theorem 7.6, there exists $c_\epsilon > 0$ such that $\eta_\epsilon = c_\epsilon \bar{m}^{\mathrm{BR}}$. Thus η_ϵ is M-invariant, as $\bar{m}^{\mathrm{BR}}$ is M-invariant.

If λ is not M-invariant, there exist $\psi \in C_c(\Gamma\backslash G)$, $m_0 \in M$ and $\beta > 0$ such that $\lambda(m_0.\psi) \geq \lambda(\psi) + \beta$. There exists $\epsilon > 0$ such that for all $m \in M_\epsilon$, $\lambda((mm_0)\psi) \geq \lambda(m.\psi) + \beta/2$. This implies that $\eta_\epsilon(m_0\psi) \geq \eta_\epsilon(\psi) + \beta/2$, which is a contradiction to the M-invariance of η_ϵ. □

As noted before this completes the proof of Theorem 7.7. □

7.4. **Statements of Counting theorems.** Now we describe the main counting results of this section. In the next two theorems 7.8, and 7.10, we suppose that the following conditions hold for $w_0 \in V$ and Γ a non-elementary discrete torsion-free subgroup of G:

(1) $w_0\Gamma$ is discrete.
(2) H is a symmetric subgroup of G, or $G_{\mathbb{R}w_0}$ is a parabolic subgroup of G.
(3) $|m^{\mathrm{BMS}}| < \infty$ and $|\mu_E^{\mathrm{PS}}| < \infty$.

Let $\lambda \in \mathbb{N}$ be the log of the largest eigenvalue of a_1 on $\mathbb{R}$-span(w_0G) and set

$$w_0^{\lambda} := \lim_{r\to\infty} \frac{w_0 a_r}{e^{\lambda r}} \quad \text{and} \quad w_0^{-\lambda} := \lim_{r\to\infty} \frac{w_0 a_{-r}}{e^{\lambda r}}. \tag{7.11}$$

Theorem 7.8 (Counting in sectors)**.** *Let $\|\cdot\|$ be a norm on V satisfying*

$$\|w_0^{\pm\lambda} mk\| = \|w_0^{\pm\lambda} k\|, \text{ for all } m \in M \text{ and } k \in K, \tag{7.12}$$

and set $B_T := \{v \in V : \|v\| < T\}$.

(1) *For any Borel measurable $\Omega \subset K$ such that $M\Omega = \Omega$ and $\nu_o(\partial(\Omega^{-1}X_0^-)) = 0$,*

$$\lim_{T\to\infty} \frac{\#(w_0\Gamma \cap B_T \cap (w_0 A^+ \Omega))}{T^{\delta/\lambda}} = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta \cdot |m^{\mathrm{BMS}}|} \int_{k\in\Omega^{-1}} \|w_0^{\lambda} k^{-1}\|^{-\delta/\lambda}\, d\nu_o(kX_0^-).$$

(2) *For the full count in a ball, we get*

$$\lim_{T\to\infty} \frac{\#(w_0\Gamma \cap B_T)}{T^{\delta/\lambda}} \tag{7.13}$$
$$= \begin{cases} \frac{\mu_E^{\mathrm{PS}}(E)}{\delta\cdot|m^{\mathrm{BMS}}|} \int_{k\in K} \|w_0^{\lambda} k^{-1}\|^{-\delta/\lambda}\, d\nu_o(kX_0^-) > 0, & \text{if } \tilde{E} = G_{w_0}\cdot X_0 \\ \sum_{\pm} \frac{\mu_E^{\mathrm{PS}}(E^{\pm})}{\delta\cdot|m^{\mathrm{BMS}}|} \int_{k\in K} \|w_0^{\pm\lambda} k^{-1}\|^{-\delta/\lambda}\, d\nu_o(kX_0^-) > 0, & \text{otherwise.} \end{cases}$$

Remark 7.9. (1) By [13, Lemma 4.2], we have $w_0^{\lambda} \neq 0$. And if H is symmetric, then $w_0^{-\lambda} \neq 0$.

(2) Since $w_0\Gamma$ is discrete, $H\Gamma$ is closed in G, and hence ΓH is closed in G. It follows that the canonical imbedding $(\Gamma\cap H)\backslash H \to \Gamma\backslash G$ is a proper injective map; the properness follows from a suitable open mapping theorem in the category of locally compact Hausdorff second countable topological group actions. Therefore the map $(\Gamma \cap G_{\tilde S})\backslash \tilde S \to \Gamma\backslash \mathbb{H}^n$ is a proper map. In particular, E and $E^{\pm}$ are closed subsets of $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$.

(3) The condition (7.12) holds if $\|\cdot\|$ is K-invariant as in Theorem 1.2. There exists a Weyl group element $k_0 \in K$ such that $k_0^{-1} a_r k_0 = a_{-r}$ for all $r \in \mathbb{R}$. Then $w_0^{-\lambda} = w_0^{\lambda} k_0$. Therefore if $\|\cdot\|$ is K-invariant, then $\|w_0^{\pm\lambda} k\| = \|w_0^{\lambda}\|$ for all $k \in K$. Then the limit (7.13) becomes (1.1). Thus Theorem 7.8 implies Theorem 1.2.

(4) When Γ is Zariski dense in G, Theorem 7.8 holds for any norm on V without the condition (7.12) and for the Ω without the M-invariance condition. See §7.7 for details.

(5) Since $w_0^{\pm\lambda}$ is fixed by $H \cap Z_K(A)$, if $M = Z_K(A) \subset H$, then the condition (7.12) holds for any norm on V. We have $M \subset H$ in the parabolic case. In the case when H is symmetric, if $\tilde S$ is a single point or $\tilde S$ is of codimension one, then $M \subset H$.

Theorem 7.10 (Counting in cones)**.** *Suppose further that Γ is Zariski dense in G. Let Θ be a measurable subset of V. Let*

$$\Omega_{\pm} = \{k \in K : w_0^{\pm\lambda} k \in \mathbb{R}^+\Theta\}.$$

If $\nu_o(\partial(\Omega_{\pm}^{-1} X_0^-)) = 0$, then for any norm $\|\cdot\|$ on V,

$$\lim_{T\to\infty} \frac{\#(w_0\Gamma \cap B_T \cap \mathbb{R}^+\Theta)}{T^{\delta/\lambda}} = \frac{1}{\delta\cdot|m^{\mathrm{BMS}}|}\times$$
$$\begin{cases} \mu_E^{\mathrm{PS}}(E) \int_{k\in\Omega_+^{-1}} \|w_0^{\lambda} k^{-1}\|^{-\delta/\lambda}\, d\nu_o(kX_0^-), & \text{if } \tilde E = HX_0 \\ \sum \mu_E^{\mathrm{PS}}(E^{\pm}) \int_{k\in\Omega_{\pm}^{-1}} \|w_0^{\pm\lambda} k^{-1}\|^{-\delta/\lambda}\, d\bar\nu_o(kX_0^-), & \text{otherwise}; \end{cases} \tag{7.14}$$

Note that if Γ is Zariski dense in G, and if $\partial(\Omega_{\pm})$ is contained in a countable union of proper real algebraic subvarieties of $\partial\mathbb{H}^n$ then $\nu_o(\partial(\Omega_{\pm})) = 0$ (see [11, Corollary 1.4] and [23, Remark 1.7(2)]).

7.5. **Proof of the counting statements.** We follow the counting technique of [9] and [10]. For a Borel subset $\Omega \subset K$ satisfying the condition of Theorem 7.8, we set

$$B_T(\Omega) = B_T \cap w_0 A^+ \Omega,$$

and define the following counting function on $\Gamma\backslash G$:

$$F_{B_T(\Omega)}(g) := \sum_{\gamma\in\Gamma_{w_0}\backslash\Gamma} \chi_{B_T(\Omega)}(w_0\gamma g).$$

We note that

$$F_{B_T(\Omega)}(e) = \#(w_0\Gamma\cap B_T(\Omega)) = \#(w_0\Gamma\cap B_T\cap(w_0A^+\Omega)). \tag{7.15}$$

For $\psi_1,\psi_2\in C_c(\Gamma\backslash G)$, we set $\langle\psi_1,\psi_2\rangle := \int_{\Gamma\backslash G}\psi_1(g)\psi_2(g)\,dg$.

Let $\psi\in C_c(\Gamma\backslash G)$. Then by (7.8),

$$\begin{aligned}\langle F_{B_T(\Omega)},\psi\rangle &= \int_{\Gamma_{w_0}\backslash G}\chi_{B_T(\Omega)}(w_0g)\psi(g)\,dg\\ &= \int_{k\in\Omega}\int_{\{r\geq 0:\|w_0a_rk\|<T\}}\Big(\int_{[h]\in\Gamma_{w_0}\backslash H}\psi(ha_rk)\,dh\Big)\rho(r)\,drdk \end{aligned}\tag{7.16}$$

For any $k\in K$ and $T>0$, define

$$r(k,T) = \sup\{r>0 : \|w_0a_rk\|<T\}. \tag{7.17}$$

Let λ_1 be the log of the largest eigenvalue of a_1 on V strictly less than e^{λ}. Then by (7.11) there exist $C_1\geq 1$ and $r_1\geq 0$ such that

$$\|w_0a_rk - e^{\lambda r}w_0^{\lambda}k\| \leq C_1e^{\lambda_1 r},\ \text{for all } k\in K \text{ and } r\geq r_1. \tag{7.18}$$

Put $\epsilon_0 = (\lambda-\lambda_1)/\lambda > 0$ and $C_2 = 2C_1/\inf_{k\in K}\|w_0^{\lambda}k\|$. Let $T_1\geq 1$ be such that $C_2T_1^{-\epsilon_0}\leq 1/2$ and $(1/2)(T_1/\sup_{k\in K}\|w_0^{\lambda}k\|)^{1/\lambda}\geq e^{r_1}$. For $T\geq T_1$, we define functions $r_{\pm}(k,T)$ via

$$e^{r_{\pm}(k,T)} = (T/\|w_0^{\lambda}k\|)^{1/\lambda}(1\pm C_2T^{-\epsilon_0}). \tag{7.19}$$

Then by elementary calculation using (7.18)

$$r_-(k,T)\leq r(k,T)\leq r_+(k,T),\ \text{for all } T\geq T_1 \text{ and } k\in K. \tag{7.20}$$

By (7.12),

$$r_{\pm}(mk,T) = r_{\pm}(k,T),\ \text{for all } m\in M \text{ and } k\in K. \tag{7.21}$$

We note that by (7.19), given $\epsilon>0$, for $T_1(\epsilon)$ sufficiently large,

$$e^{\delta r_{\pm}(k,T)} = (1+O(\epsilon))(T/\|w_0^{\lambda}k\|)^{\delta/\lambda} \text{ for all } T\geq T_1(\epsilon). \tag{7.22}$$

Proposition 7.11. *For any non-negative $\psi\in C_c(\Gamma\backslash G)$,*

$$\begin{aligned}&\int_{k\in\Omega}\int_0^{r_-(k,T)}\rho(r)\Big(\int_{E^*}\psi_k(\mathcal{G}^r(v))d\mu_E^{\mathrm{Leb}}(v)\Big)\,drdk \leq \langle F_{B_T(\Omega)},\psi\rangle\\ &\leq \int_{k\in\Omega}\int_0^{r_+(k,T)}\rho(r)\Big(\int_{E^*}\psi_k(\mathcal{G}^r(v))d\mu_E^{\mathrm{Leb}}(v)\Big)\,drdk,\end{aligned}$$

where $\psi_k\in C_c(\Gamma\backslash G)^M\cong C_c(\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n))$ is given by

$$\psi_k(g) = \int_{m\in M}\psi(gmk)dm.$$

Proof. By (7.7), (7.8), (7.16), (7.20), (7.21) and Lemma 7.1, we get

$$\begin{aligned}
&\langle F_{B_T(\Omega)}, \psi\rangle \\
&= \int_{k\in\Omega}\int_{\{r\geq 0: \|w_0 a_r k\|<T\}} \big(\int_{[h]\in\Gamma_{w_0}\backslash H} \psi(ha_rk)\, dh\big)\rho(r)\, drdk \\
&\leq \int_{k\in\Omega}\int_0^{r_+(k,T)} \big(\int_{[h]\in\Gamma_{w_0}\backslash H} \psi(ha_rk)\, dh\big)\rho(r)\, drdk \\
&= \int_{k\in\Omega}\int_0^{r_+(k,T)} \big(\int_{[h]\in\Gamma_{w_0}\backslash H}\int_{m\in M} \psi(ha_rmk)\, dmdh\big)\rho(r)\, dk, \text{ as } M\Omega=\Omega \\
&= \int_{k\in\Omega}\int_0^{r_+(k,T)} \big(\int_{[h]\in\Gamma_{w_0}\backslash H} \psi_k(ha_r)\, dh\big)\rho(r)\, drdk \\
&= \int_{k\in\Omega}\int_0^{r_+(k,T)} \rho(r)\big(\int_{E^*} \psi_k(\mathcal{G}^r(v))d\mu_E^{\mathrm{Leb}}(v)\big)\, drdk.
\end{aligned}$$

The other inequality is proved similarly. □

Proposition 7.12. *For any $\psi\in C_c(\Gamma\backslash G)$, we have*

$$\lim_{T\to\infty} T^{-\delta/\lambda}\langle F_{B_T(\Omega)},\psi\rangle = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta\cdot|m^{\mathrm{BMS}}|}\cdot m^{\mathrm{BR}}(\xi_{w_0} *_\Omega \psi),$$

where $\xi_{w_0}(k) = \|w_0^\lambda k\|^{-\delta/\lambda}$.

Proof. Without loss of generality, we may assume that ψ is non-negative. For any $\epsilon>0$ and $k\in K$, by Theorem 1.8 and (7.9), there exists $r_0>0$ such that for any $r>r_0$:

$$e^{(n-1-\delta)r}\int_{v\in E^*}\psi_k(\mathcal{G}^r(v))\, d\mu_E^{\mathrm{Leb}}(v) = \frac{\mu_E^{\mathrm{PS}}(E^*)\cdot m^{\mathrm{BR}}(\psi_k)}{|m^{\mathrm{BMS}}|} + O(\epsilon); \tag{7.23}$$

$$\rho(r) = (1+O(\epsilon))e^{(n-1)r}. \tag{7.24}$$

Since $\psi\in C_c(\Gamma\backslash G)$, the map $K\ni k\mapsto\psi_k$ is continuous with respect to the sup-norm on $C_c(\mathrm{T}^1(\mathbb{H}^n))$. Therefore since K is compact, we can choose $r_0>0$ independent of $k\in K$. Now for sufficiently large $T>1$,

$$\begin{aligned}
&\int_{r_0}^{r^\pm(k,T)}\rho(r)\int_{E^*}\psi_k(\mathcal{G}^r(v))d\mu_E^{\mathrm{Leb}}(v)dr \\
&= \int_{r_0}^{r_\pm(k,T)}\rho(r)e^{(-n+1+\delta)r}\big(e^{(n-1-\delta)r}\int_{E^*}\psi_k(\mathcal{G}^r(v))\, d\mu_E^{\mathrm{Leb}}(v)\big)\, dr \\
&= \big(\tfrac{\mu_E^{\mathrm{PS}}(E^*)\cdot m^{\mathrm{BR}}(\psi_k)}{|m^{\mathrm{BMS}}|} + O(\epsilon)\big)(1+O(\epsilon))\int_{r_0}^{r_\pm(k,T)} e^{\delta r}\, dr \\
&= \tfrac{\mu_E^{\mathrm{PS}}(E^*)\cdot m^{\mathrm{BR}}(\psi_k)}{|m^{\mathrm{BMS}}|}\cdot\tfrac{T^{\delta/\lambda}\|w_0^\lambda k\|^{-\delta/\lambda}}{\delta} + O(\epsilon)T^{\delta/\lambda} + O(e^{\delta r_0}),
\end{aligned}\tag{7.25}$$

where the last equation follows from (7.22) for sufficiently large T.

Since $E\subset\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is a closed subset, $\psi\in C_c(\Gamma\backslash G)$ and K is compact, it follows that for fixed $r_0>1$, we have

$$\sup_{|r|\leq r_0, k\in K}\int_E \psi_k(va_r)\, d\mu_E^{\mathrm{Leb}}(v) = O(1).$$

Hence

$$\int_{\{r:\|w_0 a_r k\|<T,|r|\le r_0\}} \rho(r) \int_E \psi_k(\mathcal{G}^r(v)) d\mu_E^{\mathrm{Leb}}(v) dr = O(e^{(n-1)r_0}). \quad (7.26)$$

By Proposition 7.11, (7.25) and (7.26),

$$\lim_{T\to\infty} \frac{\langle F_{B_T(\Omega)}, \psi\rangle}{T^{\delta/\lambda}} = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta \cdot |m^{\mathrm{BMS}}|} \cdot \int_{k\in\Omega} \|w_0^\lambda k\|^{-\delta/\lambda} m^{\mathrm{BR}}(\psi_k)\, dk + O(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, we finish the proof. □

Lemma 7.13 (Strong wavefront lemma)**.** *There exist $\ell > 1$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and $g = hak \in HA^+K$ with $\|a\| \ge 2$,*

$$gU_\epsilon \subset h(H\cap U_{\ell\epsilon})a(A\cap U_{\ell\epsilon})k(K\cap U_{\ell\epsilon}),$$

where $\|g\|$ denotes the distance of g from e in G which is K-invariant.

Proof. If H is symmetric, the result follows from [14, Theorem 4.1].

Now suppose that $H = N$ is horospherical. We may assume that the distance from e in G is invariant under conjugation by elements of K. Let $u \in U_\epsilon$. Then $kuk^{-1} \in U_\epsilon$. Write $kuk^{-1} = h_1a_1k_1$, where $h_1 \in H\cap U_{\ell\epsilon}$, $a_1 \in A\cap U_{\ell\epsilon}$ and $k_1 \in K\cap U_{\ell\epsilon}$ for some $\ell \ge 1$ independent of ϵ. Now

$$gu = haku = ha(kuk^{-1})k = (h(ah_1a^{-1}))(aa_1)k(k^{-1}k_1k).$$

Since $a \in A^+$ and $h_1 \in H = N$, by (7.1), $\|ah_1a^{-1}\| \le \|h_1\|$. Also $\|k^{-1}k_1k\| = \|k_1\|$. Hence gu has the required form. □

Proof of Theorem 7.8(1). By the assumption that $\nu_o(\partial(\Omega^{-1})) = 0$, for all sufficiently small $\epsilon > 0$, there exists an ϵ-neighborhood K_ϵ of e in K such that for $\Omega_{\epsilon+} = \Omega K_\epsilon$ and $\Omega_{\epsilon-} = \cap_{k\in K_\epsilon}\Omega k$,

$$\lim_{\epsilon\to 0} \nu_o(\Omega_{\epsilon+}^{-1} - \Omega_{\epsilon-}^{-1}) = 0. \quad (7.27)$$

Let $\ell > 1$ as in Lemma 7.13. Then for $T \gg 1$,

$$B_T(\Omega)U_{\ell^{-1}\epsilon} \subset B_{(1+\epsilon)T}(\Omega_{\epsilon+}) \quad \text{and} \quad B_{(1-\epsilon)T}(\Omega_{\epsilon-}) \subset \cap_{u\in U_{\ell^{-1}\epsilon}} B_T(\Omega)u.$$

Let $\psi_\epsilon \in C_c(G)$ be a non-negative function supported on $U_{\ell^{-1}\epsilon}$ and $\int \psi_\epsilon dg = 1$, and let $\Psi_\epsilon \in C_c(\Gamma\backslash G)$ the Γ-average of ψ_ϵ:

$$\Psi_\epsilon(g) := \sum_{\gamma\in\Gamma} \psi_\epsilon(\gamma g). \quad (7.28)$$

Then $F_{B_{(1-\epsilon)T}(\Omega_{\epsilon-})}(g) \le F_{B_T(\Omega)}(e) \le F_{B_{(1+\epsilon)T}(\Omega_{\epsilon+})}(g)$ for all $g \in U_{\ell^{-1}\epsilon}$. Therefore, by integrating against Ψ_ϵ, we have

$$\langle F_{B_{(1-\epsilon)T}(\Omega_{\epsilon-})}, \Psi_\epsilon\rangle \le F_{B_T(\Omega)}(e) \le \langle F_{B_{(1+\epsilon)T}(\Omega_{\epsilon+})}, \Psi_\epsilon\rangle.$$

Let ξ_{w_0} be as defined in Proposition 7.12. By Proposition 7.5, for any $\eta > 0$, there exists $\epsilon > 0$ such that

$$m^{\mathrm{BR}}(\xi_{w_0} *_\Omega \Psi_\epsilon) = \tilde{m}^{\mathrm{BR}}(\xi_{w_0} *_\Omega \psi_\epsilon) = \int_{k\in\Omega^{-1}} \xi_{w_0}(k^{-1})\, d\nu_o(kX_0^-) + O(\eta).$$

Therefore by Proposition 7.12,

$$\begin{aligned} &\lim_{T\to\infty} T^{-\delta/\lambda} \cdot \langle F_{B_{(1\pm\epsilon)T}(\Omega_{\epsilon\pm})}, \Psi_\epsilon\rangle \\ &= \tfrac{\mu_E^{\mathrm{PS}}(E^*)}{\delta\cdot|m^{\mathrm{BMS}}|} \cdot \textstyle\int_{k\in\Omega_{\epsilon\pm}^{-1}} \xi_{w_0}(k^{-1})\, d\nu_o(kX_0^-) + O(\eta); \end{aligned} \tag{7.29}$$

In view of (7.27), we get

$$\lim_{T\to\infty} \frac{F_{B_T(\Omega)}(e)}{T^{\delta/\lambda}} = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta\cdot|m^{\mathrm{BMS}}|} \cdot \int_{k\in\Omega^{-1}} \xi_{w_0}(k^{-1})\, d\nu_o(kX_0^-) + O(\eta).$$

Since $\eta > 0$ is arbitrarily chosen, we finish the proof of (1). □

Proposition 7.14. *Suppose that $H = G_{w_0}$ is symmetric and that $G \neq HA^+K$. Let $\Omega \subset M\backslash K$ such that $\nu_o(\partial(\Omega^{-1}X_0^-)) = 0$. Then*

$$\begin{aligned} &\lim_{T\to\infty} \tfrac{\#(w_0\Gamma\cap B_T\cap w_0A^-\Omega)}{T^{\delta/\lambda}} \\ &= \tfrac{\mu_E^{\mathrm{PS}}(E^-)}{\delta\cdot|m^{\mathrm{BMS}}|} \textstyle\int_{k\in\Omega^{-1}} \|w_0^{-\lambda}k^{-1}\|^{-\delta/\lambda}\, d\nu_o(kX_0^-). \end{aligned} \tag{7.30}$$

Proof. For $k \in K$ and $T > 0$, let $s(k,T) = \sup\{r > 0 : \|w_0a_{-r}k\| < T\}$. Then there exist $A_0 > 0$ and $T_0 > 0$ such that if we define $s_\pm(k,T)$ via

$$e^{s_\pm(k,T)} = (1 \pm A_0T^{-\epsilon_0})(T/\|w_0^{-\lambda}k\|)^{1/\lambda},$$

then for all $T \geq T_0$, we have $s_-(k,T) \leq s(k,T) \leq s_+(k,T)$.

By Theorem 1.8, for any $\phi \in C_c(\Gamma\backslash \mathrm{T}^1(\mathbb{H}^n))$, we have

$$\begin{aligned} &\lim_{r\to\infty} e^{(n-1-\delta)r} \textstyle\int_{E^+} \phi(\mathcal{G}^{-r}(v))\, d\mu_E^{\mathrm{Leb}}(v) \\ &= \lim_{r\to\infty} e^{(n-1-\delta)r} \textstyle\int_{E^-} \phi(\mathcal{G}^{r}(v))\, d\mu_E^{\mathrm{Leb}}(v) = \tfrac{\mu_E^{\mathrm{PS}}(E^-)}{\delta\cdot|m^{\mathrm{BMS}}|} \cdot m^{\mathrm{BR}}(\phi). \end{aligned}$$

Let $B_T^-(\Omega) = B_T \cap w_0A^-\Omega$ and

$$F_{B_T^-(\Omega)}(g) := \sum_{\gamma\in\Gamma_{w_0}\backslash\Gamma} \chi_{B_T^-(\Omega)}(w_0\gamma g).$$

In view of these observations, by arguing as in the proof of Proposition 7.12, we get that for any $\psi \in C_c(\Gamma\backslash G)$,

$$\begin{aligned} &\lim_{T\to\infty} T^{-\delta/\lambda}\langle F_{B_T^-(\Omega)}, \psi\rangle \\ &= \lim_{T\to\infty} T^{-\delta/\lambda} \textstyle\int_{k\in\Omega} \int_{\{r>0:\|w_0a_{-r}k\|<T\}} \left[\int_{[h]\in\Gamma_{w_0}\backslash H} \psi(ha_{-r}k)\, dh\right] \rho(r)\, drdk \\ &= \tfrac{\mu_E^{\mathrm{PS}}(E^-)}{\delta\cdot|m^{\mathrm{BMS}}|} \cdot \textstyle\int_{k\in\Omega} \|w_0^{-\lambda}k\|^{-\delta/\lambda} m^{\mathrm{BR}}(\psi_k)\, dk. \end{aligned}$$

Now (7.30) follows from the arguments as in the proof of Theorem 7.8 (1). □

Remark 7.15. If $G_{\mathbb{R}w_0}$ is parabolic, then $w_0 a_r \to 0$ as $r \to -\infty$. Since $w_0\Gamma$ is discrete,

$$\#(w_0\Gamma \cap w_0 A^- K) < \infty. \tag{7.31}$$

Proof of Theorem 7.8(2). If $G = HA^+K$, then (2) follows from (1) by putting $\Omega = K$.

If H is symmetric and $G \neq HA^+K$, $G = HA^+K \sqcup HA^-K$, and then (2) follows by combining (1) and Proposition 7.14 and putting $\Omega = K$.

If $G_{\mathbb{R}w_0}$ is parabolic, (7.13) follows from Theorem 7.8 and (7.31). □

7.6. **Counting in bisectors of HA^+K coordinates.** 7.4. K a maximal compact subgroup. We state a counting result for bisectors in HA^+K coordinates. For any $g \in HA^+K$, we set $a(g)$ to be the A^+-component of g, which is unique. Consider bounded Borel subsets $\Omega_1 \subset H$ and $\Omega_2 \subset K$ with $\Omega_1(H \cap M) = \Omega_1$ and $M\Omega_2 = \Omega_2$. Set

$$N_T(\Omega_1, \Omega_2) = \#(\Gamma \cap \Omega_1 A_T^+ \Omega_2)$$

where $A_T^+ = \{a_r \in A^+ : e^r < T\}$. For the sake of simplicity, we assume that the projection map $\Omega_1 \to \Gamma\backslash G$ is injective.

Theorem 7.16. *If $\mu_E^{\mathrm{PS}}(\partial(\Omega_1(X_0))) = \nu_o(\partial(\Omega_2^{-1}(X_0^-))) = 0$, then*

$$\lim_{T\to\infty} \frac{N_T(\Omega_1,\Omega_2)}{T^\delta} = \frac{1}{\delta \cdot |m^{\mathrm{BMS}}|} \mu_E^{\mathrm{PS}}(\Omega_1(X_0)) \cdot \nu_o(\Omega_2^{-1}(X_0^-)).$$

This result for $H = K$ was also obtained by Roblin [31] by a different approach. When Γ is a lattice in a semisimple Lie group G and $H = K$, the analogue of Theorem 7.16 was obtained in [12].

Proof. We define the following function on $\Gamma\backslash G$:

$$F_{T,\Omega_1,\Omega_2}(g) := \sum_{\gamma\in\Gamma} \chi_{\Omega_1 A_T^+ \Omega_2}(\gamma g).$$

For $\psi \in C_c(\Gamma\backslash G)$, given $\epsilon > 0$, by Theorem 3.6 for sufficiently large $T > 1$,

$$\begin{aligned}
&\langle F_{T,\Omega_1,\Omega_2}, \psi\rangle = \textstyle\int_{g\in\Omega_1 A_T^+\Omega_2} \psi(g)dg \\
&= \textstyle\int_{k\in\Omega_2}\int_{1\le e^r<T}\int_{h\in\Omega_1} \psi(ha_r k)\rho(a_r)dhdrdk \\
&= \textstyle\int_{k\in\Omega_2}\int_{T_0\le e^r<T} \rho(a_r)\big(\int_{h\in\Omega_1.X_0}\psi_k(ha_r)dh\big)drdk + O_{T_0}(1) \\
&= \Big(\tfrac{1}{\delta\cdot|m^{\mathrm{BMS}}|}\mu_E^{\mathrm{PS}}(\Omega_1 X_0^+)\textstyle\int_{k\in\Omega_2} m^{\mathrm{BR}}(\psi_k)dk + O(\epsilon)\Big) \\
&\qquad \times\big(\textstyle\int_0^{\log T} e^{(r-n-1)\delta}\rho(r)dr\big) + O_{T_0}(1) \\
&= \tfrac{T^\delta}{\delta\cdot|m^{\mathrm{BMS}}|}\mu_E^{\mathrm{PS}}(\Omega_1 X_0)\cdot m^{\mathrm{BR}}(\chi_K *_{\Omega_2}\psi) + O(\epsilon)T^\delta + O_{T_0}(1),
\end{aligned} \tag{7.32}$$

where $\chi_K *_{\Omega_2} \psi(g) = \int_{k\in\Omega_2}\psi(gk)dk$.

By the assumptions on Ω_1 and Ω_2, for every $\epsilon > 0$ there exist ϵ-neighborhoods H_ϵ and K_ϵ of e in H and K respectively such that for $\Omega_{1,\epsilon^-} := \cap_{h\in H_\epsilon(H\cap M)}\Omega_1 h$, $\Omega_{1,\epsilon^+} := \Omega_1 H_\epsilon(H\cap M)$, $\Omega_{2,\epsilon^-} := \cap_{k\in K_\epsilon}\Omega_2 k$ and $\Omega_{2,\epsilon^+} := \Omega_2 K_\epsilon$, as $\epsilon \to 0$,

$$\mu_E^{\mathrm{PS}}(\Omega_{1,\epsilon^+}(X_0) \smallsetminus \Omega_{1,\epsilon^-}(X_0)) \to 0,\ \nu_o(\Omega_{2,\epsilon^+}^{-1}(X_0^-) \smallsetminus \Omega_{2,\epsilon^-}^{-1}(X_0^-)) \to 0.$$

By Lemma 7.13, for $\ell > 1$ as therein, there exists an ϵ-neighborhood U_ϵ of G such that for all $T \gg 1$,

$$\Omega_1 A_T^+ \Omega_2 U_{\ell^{-1}\epsilon} \subset \Omega_{1,\epsilon^+} A_{(1+\epsilon)T}^+ \Omega_{2,\epsilon^+}$$
$$\Omega_{1,\epsilon^-} A_{(1-\epsilon)T}^+ \Omega_{2,\epsilon^-} \subset \cap_{g\in U_{\ell^{-1}\epsilon}} \Omega_1 A_T^+ \Omega_2 g.$$

Let $\psi_\epsilon \in C_c(G)$ be a non-negative function supported on $U_{\ell^{-1}\epsilon}$ and $\int \psi_\epsilon dg = 1$, and let $\Psi_\epsilon \in C_c(\Gamma\backslash G)$ the Γ-average of ψ_ϵ:

$$\Psi_\epsilon(g) = \sum_{\gamma\in\Gamma} \psi_\epsilon(\gamma g).$$

It follows that

$$\langle F_{(1-\epsilon)T,\Omega_{1,\epsilon^-},\Omega_{2,\epsilon^-}}, \Psi_\epsilon\rangle \le F_{T,\Omega_1,\Omega_2}(e) \le \langle F_{(1+\epsilon)T,\Omega_{1,\epsilon^+},\Omega_{2,\epsilon^+}}, \Psi_\epsilon\rangle. \quad (7.33)$$

On the other hand, by Proposition 7.5,

$$\lim_{\epsilon\to 0} m^{\mathrm{BR}}(\chi_K *_{\Omega_{2,\epsilon^\pm}} \Psi_\epsilon) = \nu_o(\Omega_{2,\epsilon^\pm}^{-1}(X_0^-)).$$

Therefore by (7.32),

$$\lim_{T\to\infty} T^{-\delta}\langle F_{(1\pm\epsilon)T,\Omega_{1,\epsilon^\pm},\Omega_{2,\epsilon^\pm}}, \Psi_\epsilon\rangle = \frac{\mu_E^{\mathrm{PS}}(\Omega_{1,\epsilon^\pm}(X_0))\nu_o(\Omega_{2,\epsilon^\pm}^{-1}(X_0^-))}{\delta\cdot|m^{\mathrm{BMS}}|}.$$

By (7.33) we get

$$\lim_{T\to\infty} \frac{F_{T,\Omega_1,\Omega_2}(e)}{T^\delta} = \frac{1}{\delta\cdot|m^{\mathrm{BMS}}|}\mu_E^{\mathrm{PS}}(\Omega_{1,\epsilon^+}(X_0))\nu_o(\Omega_{2,\epsilon^+}^{-1}(X_0^-)).$$

□

7.7. **Counting theorems for Γ Zariski dense.** In the case when Γ Zariski dense, Theorem 7.8 holds for any norm on V and for any Ω without the M-invariance condition. Similarly, Theorem 7.16 holds without the M-invariance assumption on Ω_1 and Ω_2.

The reason that this generalization is possible is because for Γ Zariski dense, we use Theorem 7.7 instead of Theorem 1.8. In proving Theorem 7.8, the place where we needed the M-invariance of Ω is Proposition 7.11; for general Ω, we replace this proposition by:

$$\int_{k\in\Omega}\int_0^{r_-(k,T)} \rho(r)\Big(\int_{\Gamma_{w_0}\backslash H}\psi_k(ha_r)dh\Big)\,drdk$$
$$\le \langle F_{B_T(\Omega)},\psi\rangle \le \int_{k\in\Omega}\int_0^{r_+(k,T)} \rho(r)\Big(\int_{\Gamma_{w_0}\backslash H}\psi_k(ha_r)dh\Big)\,drdk,$$

where $\psi_k(g) := \psi(gk) \in C_c(\Gamma\backslash G)$ is simply the translation of ψ by k.

Applying Theorem 7.7 to the inner integral in the above, we deduce in the same way as in the proof of Proposition 7.12 that for any $\psi \in C_c(\Gamma\backslash G)$, we have

$$\lim_{T\to\infty} T^{-\delta/\lambda}\langle F_{B_T(\Omega)}, \psi\rangle = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta\cdot|m^{\mathrm{BMS}}|}\cdot \bar{m}^{\mathrm{BR}}(\xi_{w_0} *_\Omega \psi), \tag{7.34}$$

where $\bar{m}^{\mathrm{BR}}$ defined as in subsection 7.3 and $\xi_{w_0}(k) := \|w_0^\lambda k\|^{-\delta/\lambda}$. Now for a general norm $\|\cdot\|$ on V, note that the function $\xi_{w_0}(k)$ is not necessarily M-invariant. However, for an approximate identity $\{\psi_\epsilon\}_{\epsilon>0}$ on G and any $f \in C(K)$, the proof of Proposition 7.5 can be easily modified to prove

$$\lim_{\epsilon\to 0}\bar{m}^{\mathrm{BR}}(f *_\Omega \psi_\epsilon) = \int_{k\in\Omega^{-1}} f(k^{-1})\, d\nu_o(kX_0^-). \tag{7.35}$$

Hence applying (7.34) to $\psi = \psi_\epsilon$ and (7.35) to $f = \xi_{w_0}$ and by sending $\epsilon \to 0$, we obtain

$$\lim_{T\to\infty} T^{-\delta/\lambda}\cdot F_{B_T(\Omega)}(e) = \frac{\mu_E^{\mathrm{PS}}(E^*)}{\delta\cdot|m^{\mathrm{BMS}}|}\cdot \int_{k\in\Omega^{-1}} \|w_0^\lambda k^{-1}\|^{-\delta/\lambda} d\nu_o(kX_0^-). \tag{7.36}$$

This explains the generalization of Theorem 7.8 (1). The generalization for Theorem 7.8 (2) and Theorem 7.16 can be done similarly.

Proof of Theorem 7.10. In view of the above explanation, the result can be deduced from Theorem 7.8 (or its combination with Proposition 7.14 or Remark 7.15) via elementary arguments; see [13]. □

8. Appendix: Equality of two Haar measures

Let H be a symmetric group as in §7.2.1. As in Notation 7.4(1), consider the Haar measure on G corresponding to the Iwasawa decomposition $G = NAK$ given by

$$dg = e^{(n-1)t}\, dn\, dt\, dq, \text{ for } g = na_t q,\ n \in N,\ a_t \in A,\ q \in K.$$

Corresponding to the generalized Cartan decomposition $G = HAK$, by (7.8) the Haar measure on G can be expressed as

$$dg = c_0 \cdot \rho(r)\, dh\, dr\, dk, \text{ for } g = ha_r k \in HAK,$$

where $c_0 > 0$ is a constant. We note that dn is defined by Lemma 7.1 and dh is determined by (7.7).

Theorem 8.1. $c_0 = 1$.

Proof. Let the notation be as in §7.1. Let $N^- = \{g \in G : a_{-r} g a_r \to e \text{ as } r \to \infty\}$. Then for $y \in \mathrm{Lie}(N^-)$ we have $a_{-r}\exp(y)a_r = \exp(e^{-r}y)$. In view of NAN^-M-decomposition of a small neighborhood of e in G, for h in such a neighborhood we write

$$h = n(x(h))a_{b(h)}v(y(h))m(h)$$

where $x(h) \in \mathrm{Lie}(N) \cong \mathbb{R}^{n-1}$ and $n(x(h)) = \exp(x(g))$, $y(h) \in \mathrm{Lie}(N^+) \cong \mathbb{R}^{n-1}$ and $v(y(h)) = \exp(y(h))$, $b(h) \in \mathbb{R}$ and $m(h) \in M$. In particular,

$$hX_0^+ = n(x(h))a_{b(h)}v(y(h))m(h)X_0^+ = n(x(h))X_0^+ \tag{8.1}$$

In view of the decompositions $G = HAK$ and $G = NAK$, For $h \in H$, $r > 0$ and $k \in K$, we express

$$ha_rk = n(z(h,r,k))a_{t(h,r,k)}q(h,r,k), \text{ where } q(h,r,k) \in K.$$

Now for h in a small neighborhood of e in H, we have

$$ha_rk = n(x(h))a_{b(h)}v(y(h))m(h)a_rk = n(x(h))a_{r+b(h)}v(e^{-r}y(h))(m(h)k).$$

In view of $G = NAK$ decomposition,

$$v(e^{-r}y(h)) = n(x_1(h,r))a_{b_1(h,r)}k_1(h,r), \text{ with}$$
$$\max(\|x_1(h,r)\|, \|b_1(h,r)\|, \|k_1(h,r)\|) = O(e^{-r}\|x(h)\|).$$

Therefore

$$\begin{aligned} ha_rk &= n(x(h))a_{r+b(h)}n(x_1(h,r))a_{b_1(h,r)}(k_1(h,r)m(h)k) \\ &= n(x(h) + x_2(h,r))a_{r+b(h)+b_1(h,r)}(k_1(h,r)m(h)k), \end{aligned}$$

where $x_2(h,r) = e^{-r-b(h)}x_1(h,r)$. So

$$\|x_2(h,r)\| = e^{-2r}O(\|x(h)\|). \tag{8.2}$$

Therefore

$$\begin{aligned} &z(h,r,k) = n(x(h) + x_2(h,r)), \quad t(h,r,k) = r + b(h) + b_1(h,r), \\ &q(h,r,k) = k_1(h,r)m(h)k. \end{aligned} \tag{8.3}$$

Since $z(h,r,k) = z(hm,r,e)$ and $t(h,r,k) = t(hm,r,e)$ for any $k \in K$ and $m \in M \cap H = G_{X_0^+} \cap H$, we can write $z(h,r,k) = z([h],r)$ and $t([h],r,k) = t([h],r)$, where $[h] = h(M \cap H) = hX_0^+$. Moreover, for any fixed h and r, since dk is K-invariant, we have that $dq(h,r,k) = dk$.

For h in a small neighborhood of e in H, $r > 0$ and $k \in K$,

$$\begin{aligned} c_0 &= \frac{e^{(n-1)t(h,r,k)}\,dn(z(h,r,k))\,dt(h,r,k)\,dq(h,r,k)}{\rho(r)\,dh\,dr\,dk} \\ &= \frac{e^{(n-1)t([h],r)}\,dn(z([h],r))\,dt([h],r)}{\rho(r)\,dh\,dr} \cdot \frac{dq(h,r,k)}{dk} \\ &= \frac{e^{(n-1)t([h],r)}\,dn(z([h],r))\,dt([h],r)}{\rho(r)\,dh\,dr}, \end{aligned}$$

because z and t do not depend on k and for fixed (h,r) we have $d(q(h,r,k)) = dk$. Now the numerator depends only on $[h] = hM$ and $\int_{m\in H\cap M} 1\, dm = 1$. Therefore

$$c_0 = \frac{e^{(n-1)t([h],r)} e^{(n-1)\beta_{n(z([h],r))X_0^+}(o,n(z(h,r,k))o)}}{\rho(r)e^{(n-1)\beta_{[h]}(o,[h]o)}} \times \frac{dm_o(n(z([h],r))X_0^+)\, dt([h],r)}{dm_o([h])\, dr}. \quad (8.4)$$

To compute c_0, we evaluate the Radon-Nikodym derivative at the point $([h],r) = ([e],s) = (X_0^+, s)$ for any fixed $s > 0$. Then we consider the upper half space model $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ for $\mathbb{H}^n$ with $o = (0,1)$ and $X_0^- = \infty$. Then $X_0^+ = 0 \in \mathbb{R}^{n-1} = \partial\mathbb{H}^n \smallsetminus \{\infty\}$. Since m_o is equivalent to the Lebesgue measure, let

$$0 < C := \frac{dm_o(x)}{dx}\Big|_{x=0}; \text{ also } n(x)X_0^+ = x \text{ for all } x \in \mathbb{R}^{n-1}. \quad (8.5)$$

We define a map Φ from a small neighborhood of $(0,s)$ in $\mathbb{R}^{n-1}\times\mathbb{R}$ to $\mathbb{R}^{n-1}\times\mathbb{R}$ by

$$\Phi([h],r) = (n(z(h,r,k)X_0^+, t([h],r))).$$

To compute the Jacobian of Φ at the point $(X_0^+, s) = (0,s)$, we write $\Phi = (\Phi_1, \Phi_2)$ and $([h],r) = (z_1, z_2)$.

Fixing $[h] = [e]$, we get $z([e],r) = 0$, $t([e],r) = r$. Therefore $\partial_{z_2}(\Phi_1,\Phi_2) = (0,1)$. Hence the Jacobian of Φ at $([h],r) = (0,s)$ is

$$\begin{aligned} J(\Phi)(0,s) &= |\partial_{z_1}\Phi_1(0,s)| \\ &= \tfrac{dm_o(n(x_2([h],s)+x(h))X_0^+)}{dm_o([h])} \text{ at } [h]=0, \text{ by (8.3)} \\ &= \tfrac{dm_o(n(x_2([h],s)+x([h]))X_0^+)}{dm_o(n(x([h]))X_0^+)}, \text{ by (8.1)} \\ &= \tfrac{d(x_2([h],s)+x([h]))}{d(x([h]))} \text{ at } [h]=0=x([h]), \text{ by (8.5)} \\ &= 1 + \tfrac{d(x_2([h],s))}{d(x([h]))} \text{ at } [h]=0=x([h]) \\ &= 1 + O(e^{-2s(n-1)}), \text{ by (8.2);} \end{aligned}$$

note that for a fixed s, due to (8.1) and (8.5), $x_2([h],s)$ is a smooth function of $x([h])$. By (8.4), the Radon-Nikodym derivative at $([h],r) = ([e],s)$ is

$$\begin{aligned} c_0 &= \frac{e^{(n-1)t([e],s)} e^{(n-1)\beta_{n(z([e],s))X_0^+}(o,n(z([e],s))o)}}{\rho(s)e^{(n-1)\beta_{[e]}(o,[e]o)}} \cdot J(\Phi)(0,s) \\ &= (e^{(n-1)s}/\rho(s))(1+O(e^{-2s(n-1)})). \end{aligned}$$

Since $\rho(s)/e^{(n-1)s} \to 1$ as $s\to\infty$, we have $c_0 = 1$. ☐

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